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ON $\mathcal{I}$-ASYMPTOTICALLY LACUNARY STATISTICAL EQUIVALENCE OF FUNCTIONS ON AMENABLE SEMIGROUPS

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Abstract. In this study we define the notions of asymptotically paper, we introduce the concept of $\mathcal{I}$-asymptotically statistical equivalent and $\mathcal{I}$-asymptotically lacunary statistical equivalent functions defined on discrete countable amenable semigroups. In addition to these definitions, we give some inclusion theorems.

1. Introduction

Fast [5] presented an interesting generalization of the usual sequential limit which he called statistical convergence for number sequences. Schoenberg [24] established some basic properties of statistical convergence and also studied the concept as a summability method.

Using lacunary sequences Fridy and Orhan defined lacunary statistical convergence in [6]. Also, in another study, they gave the relationships between the lacunary statistical convergence and the Cesàro summability. After their definition, Freedman et al. [7] established the connection between the strongly Cesàro summable sequences and the strongly lacunary summable sequences.

The concept of $\mathcal{I}$-convergence of real sequences is a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subsets of the set of natural numbers. P. Kostyrko et al. [8] introduced the concept of $\mathcal{I}$-convergence of sequences in a metric space and studied some properties of this convergence.

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Recently, the idea of statistical convergence and lacunary convergence was further extended by Das et al. [2] to $\mathcal{I}$-statistical convergence, $\mathcal{I}$-lacunary statistical convergence.

In 1993, Mursaleen [15] defined λ-statistical convergence by using the λ sequence. Let Λ denote the set of all non-decreasing sequences $\lambda = (\lambda_n)$ of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n$ +1 and $\lambda_1 = 1$.

Asymptotic equivalence of sequences was introduced by Pobyvanets [18]. Marouf's work [9] was extension of Pobyvanets's work. In 2003, Patterson [16] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices.

In [17] asymptotically lacunary statistical equivalent which is a natural combination of the definitions for asymptotically equivalent, statistical convergence and lacunary sequences was studied. Also in [20], $\mathcal{I}$-asymptotically statistical equivalent and $\mathcal{I}$-asymptotically lacunary statistical equivalent sequences were examined.

Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold, and $w(G)$ and $m(G)$ denote the spaces of all real valued functions and all bounded real functions f on G respectively. $m(G)$ is a Banach space with the supremum norm $\|f\|_\infty = sup\{|f(g)| : g \in G\}$.

Nomika [23] showed that, if G is countable amenable group, there exists a sequence $\{S_n\}$ of finite subsets of G such that (i) $G = \cup_{i=1}^{\infty} S_n$, (ii) $S_n \subset S_{n+1}$, $n = 1, 2, 3, ...$, (iii) $\lim_{n\to\infty} \frac{|S_n g - \cap S_n|}{|S_n|} = 1$, $\lim_{n\to\infty} \frac{|g S_n - \cap S_n|}{|S_n|} = 1$ for all $g \in G$. Here $|A|$ denotes the number of elements in the finite set A. Any sequence of finite subsets of G satisfying (i), (ii) and (iii) is called a Folner sequence for G.

Amenable semigroups were studied by [1]. The concept of summability in amenable semigroups was introduced in [13], [14]. In [3], Douglas extended the notion of arithmetic mean to amenable semigroups and obtained a characterization for almost convergence in amenable semigroups.

In [21], the notions of convergence and statistical convergence, statistical limit point and statistical cluster point of functions on discrete countable amenable semigroups were introduced.

Nuray F. Rhoades B.E. [22] defined the notions of asymptotically, statistically, almost statistically and strong almost asymptotically equivalent functions defined on discrete countable amenable semigroups. In addition to these definitions, they gave some inclusion theorems. Also, they proved that the strong almost asymptotically equivalence of the functions $f(g)$ and $h(g)$ defined on discrete countable amenable semigroups does not depend on the particular choice of the Folner sequence.

In [12], the concepts of σ-uniform density of subsets A of the set $\mathbb{N}$ of positive integers and corresponding $\mathcal{I}_\sigma$-convergence of functions defined on discrete countable amenable semigroups were introduced. Furthermore, for any Folner sequence inclusion relations between $\mathcal{I}_\sigma$-convergence and invariant convergence also $\mathcal{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence were given. They introduced the concept of $\mathcal{I}_\sigma$-statistical convergence and $\mathcal{I}_\sigma$-lacunary statistical convergence of functions defined on discrete countable amenable semigroups. In addition to these definitions, they gave some inclusion theorems. Also, they made a new approach to the notions of $[V,\lambda]$-summability, σ-convergence and λ-statistical convergence of Folner sequences by using ideals and introduced new notions, namely, $\mathcal{I}_\sigma$-$[V,\lambda]$-summability, $\mathcal{I}_\sigma$-λ-statistical convergence of Folner sequences.

Kişi and Güler [11] introduced the concepts of S_σ-asymptotically equivalent, $S_{\sigma,\lambda}$-asymptotically equivalent, σ-asymptotically lacunary statistical equivalent and strong (σ,θ)-asymptotically equivalent functions defined on discrete countable amenable semigroups.

The purpose of the study [10] was to extend the notions of $\mathcal{I}$-convergence, $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior, $\mathcal{I}$-cluster point and $\mathcal{I}$-limit point to functions defined on discrete countable amenable semigroups. Also, a new approach to the notions of $[V,\lambda]$-summability and λ-statistical convergence by using ideals and introduce new notions, namely, $\mathcal{I}$-$[V,\lambda]$-summability and $\mathcal{I}$-λ-statistical convergence to functions was defined on discrete countable amenable semigroups.

This study presents the notion of $\mathcal{I}$-asymptotically lacunary statistical equivalence which is a natural combination of $\mathcal{I}$-asymptotically equivalence, lacunary statistical equivalence for functions defined on discrete countable amenable semigroups. We introduce new concepts, and establish certain inclusion theorems.

2. Definitions and Notations

Definition 2.1. *[21] Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f\in w(G)$ is said to be convergent to s, for any Folner sequence $\{S_n\}$ for G, if for each $\varepsilon>0$ there exists $k_0\in\mathbb{N}$ such that $|f(g)-s|<\varepsilon$ for all $m>k_0$ and $g\in G\setminus S_m$.*

Definition 2.2. *[21] Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f\in w(G)$ is said to be strongly summable to s, for any Folner sequence $\{S_n\}$ for G, if*

$$\lim_{n\to\infty}\frac{1}{|S_n|}\sum_{g\in S_n}|f(g)-s|=0,$$

where $|S_n|$ denotes the cardinality of the set S_n.

Definition 2.3. *[21] Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f\in w(G)$ is said to be statistically convergent to s, for any Folner sequence $\{S_n\}$ for G, if for every $\varepsilon>0$,*

$$\lim_{n\to\infty}\frac{1}{|S_n|}|\{g\in S_n:|f(g)-s|\geq\varepsilon\}|=0$$

The set of all statistically convergent functions will be denoted $S(G)$.

Definition 2.4. *[22] Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. Two nonnegative functions $f, h \in w(G)$ are said to be asymptotically equivalent, for any Folner sequence $\{S_n\}$ for G, if for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that*

$$\left|\frac{f(g)}{h(g)} - 1\right| < \varepsilon$$

for all $m > k_0$ and $g \in G \backslash S_m$. It will be denoted by $f \sim h$.

Definition 2.5. *[22] Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. Two nonnegative functions $f, h \in w(G)$ are said to be strong asymptotically equivalent, for any Folner sequence $\{S_n\}$ for G, if*

$$\lim_{n\to\infty} \frac{1}{|S_n|} \sum_{g\in S_n} \left|\frac{f(g)}{h(g)} - 1\right| = 0$$

It will be denoted by $f \sim^w h$.

Definition 2.6. *[22] Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. Two nonnegative functions $f, h \in w(G)$ are said to be statistically equivalent, for any Folner sequence $\{S_n\}$ for G, if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{|S_n|} \left|\left\{g \in S_n : \left|\frac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| = 0$$

Definition 2.7. *[10] Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be $\mathcal{I}$-convergent to s for any Folner sequence $\{S_n\}$ for G, if for every $\varepsilon > 0$;*

$$\{g \in S_n : |f(g) - s| \geq \varepsilon\} \in \mathcal{I};$$

i.e., $|f(g) - s| < \varepsilon$ a.a.g. The set of all $\mathcal{I}$-convergent sequences will be denoted by $\mathcal{I}(G)$.

3. Main Results

Definition 3.1. *Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be $\mathcal{I}$-statistically convergent to s, for any Folner sequence $\{S_n\}$ for G, if for each $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{|S_n|} |\{g \in S_n : |f(g) - s| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

The set of all $\mathcal{I}$-statistically convergent Folner sequences will be denoted by $S_{\mathcal{I}}(G)$.

Definition 3.2. *Let θ be lacunary sequence and G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be lacunary statistically convergent to s, for any Folner sequence $\{S_n\}$ for G, if for every $\varepsilon > 0$,*

$$\lim_{r\to\infty} \frac{1}{h_r} |\{g \in S_n : |f(g) - s| \geq \varepsilon\}| = 0.$$

The set of all lacunary statistically convergent Folner sequences will be denoted $S_\theta(G)$.

Definition 3.3. *Let θ be lacunary sequence, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$ and G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be $\mathcal{I}$-lacunary statistically convergent to s, for any Folner sequence $\{S_n\}$ for G, if for each $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \{ g \in S_n : |f(g) - s| \geq \varepsilon \} \right| \geq \delta \right\} \in \mathcal{I}.$$

The set of all $\mathcal{I}$-lacunary statistically convergent sequences will be denoted by $S_{\mathcal{I}}^{\theta}(G)$.

Definition 3.4. *Let θ be lacunary sequence, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$ and G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be strongly $\mathcal{I}$-lacunary convergent to s or $N_{\mathcal{I}}^{\theta}(G)$-convergent to s, for any Folner sequence $\{S_n\}$ for G, if for each $\varepsilon > 0$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{g \in S_n} |f(g) - s| \geq \varepsilon \right\} \in \mathcal{I}.$$

The set of all strongly $\mathcal{I}$-lacunary convergent sequences will be denoted by $N_{\mathcal{I}}^{\theta}(G)$.

Definition 3.5. *Two nonnegative functions $f, h \in w(G)$ are said to be asymptotically $\mathcal{I}$-equivalent, for any Folner sequence $\{S_n\}$ for G, if every $\varepsilon > 0$,*

$$\left\{ g \in S_n : \left| \frac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \in \mathcal{I}.$$

It will be denoted by $f \sim^{\mathcal{I}(G)} h$.

Definition 3.6. *Two nonnegative functions $f, h \in w(G)$ are said to be $\mathcal{I}$-asymptotically statistical equivalent, for any Folner sequence $\{S_n\}$ for G, if for each $\varepsilon > 0$ and $\delta > 0$, the following set*

$$\left\{ n \in \mathbb{N} : \frac{1}{|S_n|} \left| \left\{ g \in S_n : \left| \frac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \right| \geq \delta \right\}$$

belongs to $\mathcal{I}$. It will be denoted by $f \sim^{S_{\mathcal{I}}(G)} h$.

For $\mathcal{I} = \mathcal{I}_{fin}$, $\mathcal{I}$-asymptotically statistical equivalence coincides with asymptotically statistical equivalence for any Folner sequence $\{S_n\}$ for G.

Definition 3.7. *Two nonnegative functions $f, h \in w(G)$ are said to be $\mathcal{I}$-asymptotically lacunary statistical equivalent, for any Folner sequence $\{S_n\}$ for G, if for every $\varepsilon > 0$ and $\delta > 0$, the following set*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ g \in S_n : \left| \frac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \right| \geq \delta \right\}$$

belongs to $\mathcal{I}$. It will be denoted by $f \sim^{S_{\mathcal{I}}^{\theta}(G)} h$.

For $\mathcal{I} = \mathcal{I}_{fin}$, $\mathcal{I}$-asymptotically lacunary statistical equivalence coincides with asymptotically lacunary statistical equivalence for any Folner sequence $\{S_n\}$ for G.

Definition 3.8. *Two nonnegative functions $f, h \in w(G)$ are said to be strongly $\mathcal{I}$-asymptotically lacunary equivalent, for any Folner sequence $\{S_n\}$ for G, if for every $\varepsilon > 0$*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{g \in S_n} \left| \frac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \in \mathcal{I}.$$

It will be denoted by $f \sim^{N_{\mathcal{I}}^{\theta}(G)} h$.

Definition 3.9. *Two nonnegative functions $f, h \in w(G)$ are said to be strongly $\lambda_{\mathcal{I}}(G)$-asymptotically equivalent, for any Folner sequence $\{S_n\}$ for G, if for every $\varepsilon > 0$*

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{g \in S_n} \left| \frac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \in \mathcal{I}.$$

It will be denoted by $f \sim^{V_{\lambda_{\mathcal{I}}}(G)} h$.

Definition 3.10. *Two nonnegative functions $f, h \in w(G)$ are said to be $\mathcal{I}$-asymptotically λ-statistical equivalent provided that for every ε, $\delta > 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \left| \left\{ g \in S_n : \left| \frac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}.$$

It will be denoted by $f \sim^{S_{\lambda_{\mathcal{I}}}(G)} h$.

Theorem 3.1. *Let $\lambda \in \Lambda$ and $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$. If $f \sim^{V_{\lambda_{\mathcal{I}}}(G)} h$ then $f \sim^{S_{\lambda_{\mathcal{I}}}(G)} h$.*

Proof. Assume that $f \sim^{V_{\lambda_{\mathcal{I}}}(G)} h$ and $\varepsilon > 0$. Then,

$$\begin{aligned} \sum_{g \in S_n} \left| \frac{f(g)}{h(g)} - 1 \right| &\geq \sum_{g \in S_n \& \left| \frac{f(g)}{h(g)} - 1 \right| \geq \varepsilon} \left| \frac{f(g)}{h(g)} - 1 \right| \\ &\geq \varepsilon. \left| \left\{ g \in S_n : \left| \frac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \right| \end{aligned}$$

and so,

$$\frac{1}{\varepsilon.\lambda_n} \sum_{g \in S_n} \left| \frac{f(g)}{h(g)} - 1 \right| \geq \frac{1}{\lambda_n} \left| \left\{ g \in S_n : \left| \frac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \right|.$$

Then for any $\delta > 0$,

$$\begin{aligned} &\left\{ n \in \mathbb{N} : \tfrac{1}{\lambda_n} \left| \left\{ g \in S_n : \left| \tfrac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \\ &\qquad \subseteq \left\{ n \in \mathbb{N} : \tfrac{1}{\lambda_n} \sum_{g \in S_n} \left| \tfrac{f(g)}{h(g)} - 1 \right| \geq \varepsilon\delta \right\}. \end{aligned}$$

Since right hand belongs to $\mathcal{I}$ then left hand also belongs to $\mathcal{I}$ and this completes the proof. □

Theorem 3.2. *Let $\lambda \in \Lambda$ and $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$. If $f, h \in m(G)$ are bounded functions and $f \sim^{S_{\lambda_{\mathcal{I}}}(G)} h$ then $f \sim^{V_{\lambda_{\mathcal{I}}}(G)} h$.*

Proof. Let $f, h \in m(G)$ are bounded functions and $f \sim^{S_{\lambda_{\mathcal{I}}}(G)} h$. Then, there is an M such that

$$\left|\frac{f(g)}{h(g)} - 1\right| \leq M$$

for all k. For each $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{g \in S_n} \left|\tfrac{f(g)}{h(g)} - 1\right| &= \frac{1}{\lambda_n} \sum_{g \in S_n \& \left|\frac{f(g)}{h(g)} - 1\right| \geq \varepsilon} \left|\tfrac{f(g)}{h(g)} - 1\right| \\ &\quad + \frac{1}{\lambda_n} \sum_{g \in S_n \& \left|\frac{f(g)}{h(g)} - 1\right| < \varepsilon} \left|\tfrac{f(g)}{h(g)} - 1\right| \\ &\leq M \frac{1}{\lambda_n} \left|\left\{g \in S_n : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2} \end{aligned}$$

Then,

$$\begin{aligned} &\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{g \in S_n} \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\} \\ &\quad \subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \left|\left\{g \in S_n : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{\varepsilon}{2M}\right\} \in \mathcal{I}. \end{aligned}$$

Therefore $f \sim^{V_{\lambda_{\mathcal{I}}}(G)} h$. □

Theorem 3.3. *If $\liminf \dfrac{\lambda_n}{|S_n|} > 0$ then $f \sim^{S_{\mathcal{I}}(G)} h$ implies $f \sim^{S_{\lambda_{\mathcal{I}}}(G)} h$.*

Proof. Assume that $\liminf \dfrac{\lambda_n}{|S_n|} > 0$ there exists a $\delta > 0$ such that $\dfrac{\lambda_n}{|S_n|} \geq \delta$ for sufficiently large n. For given $\varepsilon > 0$ we have,

$$\frac{1}{|S_n|} \left\{k \leq |S_n| : \left|\frac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\} \supseteq \frac{1}{|S_n|} \left\{k \in I_n : \left|\frac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\},$$

where $I_n = [n - \lambda_n + 1, n]$. Therefore,

$$\begin{aligned} \tfrac{1}{|S_n|} \left|\left\{k \leq |S_n| : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| &\geq \tfrac{1}{|S_n|} \left|\left\{k \in I_n : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| \\ &\geq \frac{\lambda_n}{|S_n|} . \frac{1}{\lambda_n} \left|\left\{k \in I_n : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| \\ &\geq \delta . \frac{1}{\lambda_n} \left|\left\{k \in I_n : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| \end{aligned}$$

then for any $\eta > 0$ we get

$$\begin{aligned} &\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \left|\left\{k \in I_n : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| \geq \eta\right\} \\ &\quad \subseteq \left\{n \in \mathbb{N} : \tfrac{1}{|S_n|} \left|\left\{k \leq |S_n| : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| \geq \eta\delta\right\} \in \mathcal{I}, \end{aligned}$$

and this completes the proof. □

Theorem 3.4. *If $\lambda = (\lambda_n) \in \Delta$ be such that $\lim_{n \to \infty} \dfrac{\lambda_n}{|S_n|} = 1$, then $f \sim^{S_{\lambda_{\mathcal{I}}}(G)} h$ is subset of $f \sim^{S_{\mathcal{I}}(G)} h$.*

Proof. Let $\delta > 0$ be given. Since $\lim_{n\to\infty} \frac{\lambda_n}{|S_n|} = 1$, we can choose $m \in N$ such that $\left|\frac{\lambda_n}{|S_n|} - 1\right| < \frac{\delta}{2}$, for all $n \geq m$. Now observe that, for $\varepsilon > 0$

$$\begin{aligned}
\frac{1}{|S_n|}\left|\left\{k \leq |S_n| : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| &= \frac{1}{|S_n|}\left|\left\{k \leq |S_n| - \lambda_n : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| \\
&+ \frac{1}{|S_n|}\left|\left\{k \in I_n : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| \\
&\leq \frac{|S_n| - \lambda_n}{|S_n|} + \frac{1}{|S_n|}\left|\left\{k \in I_n : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| \\
&\leq 1 - \left(1 - \frac{\delta}{2}\right) + \frac{1}{|S_n|}\left|\left\{k \in I_n : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| \\
&= \frac{\delta}{2} + \frac{1}{|S_n|}\left|\left\{k \in I_n : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right|,
\end{aligned}$$

for all $n \geq m$. Hence,

$$\left\{g \in S_n : \frac{1}{|S_n|}\left|\left\{k \leq |S_n| : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| \geq \delta\right\}$$

$$\subset \left\{g \in S_n : \frac{1}{|S_n|}\left|\left\{k \in I_n : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| \geq \tfrac{\delta}{2}\right\} \cup \{1, 2, ..., m\}$$

If $f \sim^{S_{\lambda_{\mathcal{I}}}(G)} h$, then the set on the right hand side belongs to $\mathcal{I}$ and so the set on the left hand side also belongs to $\mathcal{I}$. This shows that $f \sim^{S_{\mathcal{I}}(G)} h$. □

Definition 3.11. *Two nonnegative functions $f, h \in w(G)$ are said to be strongly Cesáro $\mathcal{I}$-asymptotically equivalent, provided that for every $\varepsilon > 0$,*

$$\left\{g \in S_n : \frac{1}{|S_n|}\sum_{1\leq k\leq |S_n|}\left|\frac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\} \in \mathcal{I}.$$

It will be denoted by $f \sim^{[C_1(\mathcal{I})](G)} h$.

Theorem 3.5. *If $f \sim^{V_{\lambda_{\mathcal{I}}}(G)} h$, then $f \sim^{[C_1(\mathcal{I})](G)} h$.*

Proof. Assume that $f \sim^{V_{\lambda_{\mathcal{I}}}(G)} h$ and $\varepsilon > 0$. Then,

$$\begin{aligned}
\tfrac{1}{|S_n|}\sum_{g\in S_n}\left|\tfrac{f(g)}{h(g)} - 1\right| &= \tfrac{1}{|S_n|}\sum_{k=1}^{|S_n|-\lambda_n}\left|\tfrac{f(g)}{h(g)} - 1\right| + \tfrac{1}{|S_n|}\sum_{k\in I_n, g\in S_n}\left|\tfrac{f(g)}{h(g)} - 1\right| \\
&\leq \frac{1}{\lambda_n}\sum_{k=1}^{|S_n|-\lambda_n}\left|\tfrac{f(g)}{h(g)} - 1\right| + \frac{1}{\lambda_n}\sum_{k\in I_n, g\in S_n}\left|\tfrac{f(g)}{h(g)} - 1\right| \\
&\leq \frac{2}{\lambda_n}\sum_{k\in I_n, g\in S_n}\left|\tfrac{f(g)}{h(g)} - 1\right|
\end{aligned}$$

and so,

$$\left\{g \in S_n : \frac{1}{|S_n|} \sum_{g \in S_n} \left| \frac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \subseteq \left\{g \in S_n : \frac{1}{\lambda_n} \sum_{k \in I_n, g \in S_n} \left| \frac{f(g)}{h(g)} - 1 \right| \geq \frac{\varepsilon}{2} \right\}$$

belongs to $\mathcal{I}$. Hence $f \sim^{[C_1(\mathcal{I})](G)} h$. □

Theorem 3.6. *Let $\lambda \in \Lambda$ and $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$. If $f, h \in m(G)$ are bounded and $f \sim^{S_{\lambda_{\mathcal{I}}}(G)} h$ then $f \sim^{C_1(\mathcal{I})(G)} h$.*

Proof. Suppose that $f, h \in m(G)$ and $f \sim^{S_{\lambda_{\mathcal{I}}}(G)} h$. Since $f, h \in m(G)$, they are bounded, assume that $\left| \frac{f(g)}{h(g)} - 1 \right| < M$ for all g. Let $\varepsilon > 0$ be given and set

$$L_n = \left\{ g \in S_n : \left| \frac{f(g)}{h(g)} - 1 \right| \geq \frac{\varepsilon}{2} \right\}.$$

For each $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{g \in S_n} \left| \frac{f(g)}{h(g)} - 1 \right| &= \frac{1}{\lambda_n} \sum_{g \in S_n} \left| \frac{f(g)}{h(g)} - 1 \right| + \frac{1}{\lambda_n} \sum_{g \in S_n \setminus L_n} \left| \frac{f(g)}{h(g)} - 1 \right| \\ &\leq M \frac{1}{\lambda_n} \left| \left\{ g \in S_n : \left| \frac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \right| + \frac{\varepsilon}{2}. \end{aligned}$$

Then,

$$\begin{aligned} &\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{g \in S_n} \left| \frac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \\ &\qquad \subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \left| \left\{ g \in S_n : \left| \frac{f(g)}{h(g)} - 1 \right| \geq \frac{\varepsilon}{2} \right\} \right| \geq \frac{\varepsilon}{2M} \right\} \in \mathcal{I}. \end{aligned}$$

Therefore, $f \sim^{V_{\lambda_{\mathcal{I}}}(G)} h$. □

Theorem 3.7. *If $f \sim^{C_1(\mathcal{I})(G)} h$, then, $f \sim^{S(G)} h$.*

Proof. Let $f \sim^{C_1(\mathcal{I})(G)} h$, and $\varepsilon > 0$ given. Then,

$$\begin{aligned} \tfrac{1}{|S_n|} \sum_{g \in S_n} \left| \tfrac{f(g)}{h(g)} - 1 \right| &\geq \tfrac{1}{|S_n|} \sum_{g \in S_n \& \left| \frac{f(g)}{h(g)} - 1 \right| \geq \varepsilon} \left| \tfrac{f(g)}{h(g)} - 1 \right| \\ &\geq \varepsilon . \left| \left\{ k \leq |S_n| : \left| \tfrac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \right| \end{aligned}$$

and so

$$\frac{1}{\varepsilon . |S_n|} \sum_{g \in S_n} \left| \frac{f(g)}{h(g)} - 1 \right| \geq \frac{1}{|S_n|} \left| \left\{ k \leq |S_n| : \left| \frac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \right|.$$

So for a given $\delta > 0$,

$$\begin{aligned} &\left\{ n \in \mathbb{N} : \tfrac{1}{|S_n|} \left| \left\{ k \leq |S_n| : \left| \tfrac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \\ &\qquad \subseteq \left\{ n \in \mathbb{N} : \tfrac{1}{|S_n|} \sum_{g \in S_n} \left| \tfrac{f(g)}{h(g)} - 1 \right| \geq \varepsilon \delta \right\} \in \mathcal{I}. \end{aligned}$$

Therefore $f \sim^{S(G)} h$. □

Theorem 3.8. *Let $f,\, h \in m\,(G)$. If $f \sim^{S(G)} h$ then $f \sim^{C_1(\mathcal{I})(G)} h$.*

Proof. Suppose that $f,\, h \in m\,(G)$ and $f \sim^{S_\lambda(G)} h$. Then, there is a M such that $\left|\frac{f(g)}{h(g)} - 1\right| \leq M$ for all k. Given $\varepsilon > 0$, we have

$$\begin{aligned}
\tfrac{1}{|S_n|} \sum_{g\in S_n} \left|\tfrac{f(g)}{h(g)} - 1\right| &= \tfrac{1}{|S_n|} \sum_{g\in S_n \& \left|\frac{f(g)}{h(g)}-1\right| \geq \varepsilon} \left|\tfrac{f(g)}{h(g)} - 1\right| \\
&+ \tfrac{1}{|S_n|} \sum_{g\in S_n \& \left|\frac{f(g)}{h(g)}-1\right| < \varepsilon} \left|\tfrac{f(g)}{h(g)} - 1\right| \\
&\leq \tfrac{1}{|S_n|}.M \left|\left\{k \leq |S_n| : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| \\
&\quad + \tfrac{1}{|S_n|}.\varepsilon \left|\left\{k \leq |S_n| : \left|\tfrac{f(g)}{h(g)} - 1\right| < \varepsilon\right\}\right| \\
&\leq \frac{M}{|S_n|} \left|\left\{k \leq |S_n| : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| + \varepsilon.
\end{aligned}$$

Then, for any $\delta > 0$,

$$\left\{n \in \mathbb{N} : \tfrac{1}{|S_n|} \sum_{g\in S_n} \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \delta\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \tfrac{1}{|S_n|} \left|\left\{k \leq |S_n| : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| \geq \frac{\delta}{M}\right\} \in \mathcal{I}.$$

Therefore $f \sim^{C_1(\mathcal{I})(G)} h$. □

Theorem 3.9. *Let $f, h \in w(G)$ two nonnegative functions. Then,*

(a) If $f \sim^{N_{\mathcal{I}}^{\theta}(G)} h$, then $f \sim^{S_{\mathcal{I}}^{\theta}(G)} h$,

(b) If $f, h \in m\,(G)$ and $f \sim^{S_{\mathcal{I}}^{\theta}(G)} h$, then $f \sim^{N_{\mathcal{I}}^{\theta}(G)} h$.

Proof. (a). Let $\varepsilon > 0$ and $f \sim^{N_{\mathcal{I}}^{\theta}(G)} h$. Then, we can write

$$\sum_{g\in S_n} \left|\frac{f\,(g)}{h\,(g)} - 1\right| \geq \varepsilon. \left|\left\{g \in S_n : \left|\frac{f\,(g)}{h\,(g)} - 1\right| \geq \varepsilon\right\}\right|$$

and so

$$\frac{1}{\varepsilon h_r} \sum_{g\in S_n} \left|\frac{f\,(g)}{h\,(g)} - 1\right| \geq \frac{1}{h_r} \left|\left\{g \in S_n : \left|\frac{f\,(g)}{h\,(g)} - 1\right| \geq \varepsilon\right\}\right|.$$

Then, for any $\delta > 0$

$$\left\{r \in \mathbb{N} : \tfrac{1}{h_r} \left|\left\{g \in S_n : \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon\right\}\right| \geq \delta\right\}$$

$$\subseteq \left\{r \in \mathbb{N} : \tfrac{1}{h_r} \sum_{g\in S_n} \left|\tfrac{f(g)}{h(g)} - 1\right| \geq \varepsilon.\delta\right\} \in \mathcal{I}.$$

Hence, we have $f \sim^{S_{\mathcal{I}}^{\theta}(G)} h$.

(b) Suppose that $f, h \in m(G)$ and $f \sim^{S_{\mathcal{I}}^{\theta}(G)} h$. Since $f, h \in m(G)$, they are bounded, assume that $\left|\frac{f(g)}{h(g)} - 1\right| < M$ for all g.

$$\begin{aligned}\frac{1}{h_r}\sum_{g\in S_n}\left|\frac{f(g)}{h(g)}-1\right| &= \frac{1}{h_r}\sum_{g\in S_n}\left|\frac{f(g)}{h(g)}-1\right| + \frac{1}{h_r}\sum_{g\in S_n\setminus L_n}\left|\frac{f(g)}{h(g)}-1\right| \\ &\leq \frac{M}{h_r}\left|\left\{g\in S_n : \left|\frac{f(g)}{h(g)}-1\right| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2}.\end{aligned}$$

And define the sets

$$D_1 = \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{g\in S_n}\left|\frac{f(g)}{h(g)}-1\right| \geq \varepsilon\right\}$$

and

$$D_2 = \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{g\in S_n : \left|\frac{f(g)}{h(g)}-1\right| \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{\varepsilon}{2M}\right\}.$$

If $r \notin D_2$, then $\frac{1}{h_r}\left|\left\{g\in S_n : \left|\frac{f(g)}{h(g)}-1\right| \geq \frac{\varepsilon}{2}\right\}\right| < \frac{\varepsilon}{2M}$.

Also we can get

$$\begin{aligned}\frac{1}{h_r}\sum_{g\in S_n}\left|\frac{f(g)}{h(g)}-1\right| &\leq \frac{M}{h_r}\left|\left\{g\in S_n : \left|\frac{f(g)}{h(g)}-1\right| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus, $r \notin D_1$. Consequently, we have

$$\begin{aligned}&\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{g\in S_n}\left|\frac{f(g)}{h(g)}-1\right| \geq \varepsilon\right\} \\ &\subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{g\in S_n : \left|\frac{f(g)}{h(g)}-1\right| \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{\varepsilon}{2M}\right\} \in \mathcal{I}.\end{aligned}$$

Therefore, $f \sim^{N_{\mathcal{I}}^{\theta}(G)} h$. This completes the proof. □

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