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NONLINEAR SEQUENTIAL RIEMANN-LIOUVILLE AND CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL AND INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we discuss the existence and uniqueness of solutions for a new class of sequential fractional differential equations of Riemann-Liouville and Caputo types with nonlocal integral boundary conditions, by using standard fixed point theorems. We also demonstrate the application of the obtained results with the aid of examples.

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1. Introduction

Fractional differential equations have gained considerable importance due to their widespread applications in various disciplines of social and natural sciences, and engineering. In recent years, there has been a significant development in fractional calculus and fractional differential equations, for instance, see the monographs by Kilbas *et al.* [12], Lakshmikantham *et al.* [14], Miller and Ross [15], Podlubny [16], Samko *et al.* [18], Diethelm [9], Ahmad *et al.* [7] and the papers [1, 4–6, 8, 10, 17, 20, 21].

Recently in [2] the authors studied a class of nonlinear differential equations with multiple fractional derivatives and Caputo type integro-differential boundary conditions

$$\begin{cases} D^{\alpha}[D^{\beta}x(t) - g(t, x(t))] = f(t, x(t)),\ t \in J := [0, T], \\ x(0) = 0,\ (D^{\gamma}x)(T) = \lambda(I^{\delta}x)(T), \end{cases} \tag{1.1}$$

where D^{χ} is Caputo fractional derivative of order $\chi \in \{\alpha, \beta, \gamma\}$, $0 < \alpha, \beta, \gamma < 1$, I^{δ} is the Riemann-Liouville fractional integral of order $\delta > 0$, $f, g : J \times \mathbb{R} \to \mathbb{R}$ are given functions and $\lambda \neq \dfrac{\Gamma(\beta + \delta + 1)}{T^{\gamma+\delta}\Gamma(\beta - \gamma + 1)}$. The existence of solutions for the problem (1.1) is established by applying Leray-Schauder nonlinear alternative [11] and Krasnoselskii's fixed point theorem [13]. The uniqueness result for the problem (1.1) is obtained by means of a celebrated fixed point theorem due to Banach.

In [3] existence criteria are developed for the solutions of Caputo-Hadamard type fractional neutral differential equations supplemented with Dirichlet boundary conditions

$$\begin{cases} D^{\omega}[D^{\kappa}x(t) - h(t, x(t))] = f(t, x(t)),\ t \in [1, T],\ T > 1, \\ x(1) = 0,\ x(T) = 0, \end{cases} \tag{1.2}$$

where D^{ρ} denotes the Caputo-Hadamard fractional derivatives of order $\rho \in (0, 1)$ with $\rho \in \{\omega, \kappa\}$ and $f, h : [1, T] \times \mathbb{R} \to \mathbb{R}$ are appropriate functions.

Very recently in [19], the authors discussed existence and uniqueness of solutions for two sequential Caputo-Hadamard and Hadamard-Caputo fractional differential equations subject to separated boundary conditions as

$$\begin{cases} {}^{C}D^{p}({}^{H}D^{q}x)(t) = f(t, x(t)), \qquad t \in (a, b), \\ \alpha_1 x(a) + \alpha_2({}^{H}D^{q}x)(a) = 0, \quad \beta_1 x(b) + \beta_2({}^{H}D^{q}x)(b) = 0, \end{cases} \tag{1.3}$$

and

$$\begin{cases} {}^{H}D^{q}({}^{C}D^{p}x)(t) = f(t, x(t)), \qquad t \in (a, b), \\ \alpha_1 x(a) + \alpha_2({}^{C}D^{p}x)(a) = 0, \quad \beta_1 x(b) + \beta_2({}^{C}D^{p}x)(b) = 0, \end{cases} \tag{1.4}$$

where ${}^{C}D^{p}$ and ${}^{H}D^{q}$ are the Caputo and Hadamard fractional derivatives of orders p and q, respectively, $0 < p, q \leq 1$, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $a > 0$ and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$.

Motivated by the above papers, we consider in the present paper the following boundary value problem

$$\begin{cases} {}^{RL}D^q[{}^{C}D^r x(t) - g(t, x(t))] = f(t, x(t)), \ 0 < t < T, \\ \\ x(\eta) = \phi(x), \ I^p x(T) = h(x), \end{cases} \tag{1.5}$$

where ${}^{RL}D^q$, ${}^{C}D^r$ are Riemann-Liouville and Caputo fractional derivatives of orders $q, r \in (0, 1)$, respectively, I^p is the Riemann-Liouville fractional integral of order $p > 0$, $f, g : J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\phi, h : C(J, \mathbb{R}) \to \mathbb{R}$ are two given functionals.

The rest of the paper is arranged as follows. In Section 2, we establish basic results that lays the foundation for defining a fixed point problem equivalent to the given problem (1.5). The main results, based on Banach's contraction mapping principle, Krasnoselskii's fixed point theorem and nonlinear alternative of Leray-Schauder type, are obtained in Section 3. Examples illustrating the obtained results are also included.

2. Preliminaries

In this section, we recall some basic concepts of fractional calculus [12, 16] and present known results needed in our forthcoming analysis.

Definition 2.1. *The Riemann-Liouville fractional derivative of order q for a function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$${}^{RL}D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_{0+}^{t} (t-s)^{n-q-1} f(s) ds, \quad q > 0, \ n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q, provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. *The Riemann-Liouville fractional integral of order q for a function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_{0+}^{t} (t-s)^{q-1} f(s) ds, \qquad q > 0,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.3. *The Caputo derivative of fractional order q for a n-times derivative function $f : (0, \infty) \to \mathbb{R}$ is defined as*

$${}^{C}D^q f(t) = \frac{1}{\Gamma(n-q)} \int_{0+}^{t} (t-s)^{n-q-1} \left(\frac{d}{ds}\right)^n f(s) ds, \quad q > 0, \ n = [q] + 1.$$

Lemma 2.1. *If $\alpha + \beta > 0$, then the equation $(I^\alpha I^\beta u)(t) = (I^{\alpha+\beta} u)(t)$, $t \in J$ is satisfied for $u \in L^1(J, \mathbb{R})$.*

Lemma 2.2. *Let $\beta > \alpha$. Then the equation $(D^\alpha I^\beta u)(t) = (I^{\beta-\alpha} u)(t)$, $t \in J$ is satisfied for $u \in C(J, \mathbb{R})$.*

Lemma 2.3. *Let $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $n = \alpha$ if $\alpha \in \mathbb{N}$. Then the following relations hold:*

(i) *for $k \in \{0, 1, 2, \ldots, n-1\}$, $D^\alpha t^k = 0$;*

(ii) *if* $\beta > n$ *then* $D^{\alpha} t^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}$;

(iii) $I^{\alpha} t^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} t^{\beta+\alpha-1}$.

Lemma 2.4. *Let* $q > 0$. *Then for* $y \in C(0,T) \cap L(0,T)$ *it holds*

$$^{RL}I^q \left(^{RL}D^q y \right)(t) = y(t) + c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ *and* $n-1 < q < n$.

Lemma 2.5. *Let* $q > 0$. *Then for* $y \in C(0,T) \cap L(0,T)$ *it holds*

$$^{RL}I^q \left(^{C}D^q y \right)(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ *and* $n = [q] + 1$.

In the following, for simplicity, we use the notation I^q for $^{RL}I^q$.

Lemma 2.6. *Let* $p > 0$, $0 < q, r \leq 1$, *with* $q + r > 1$,

$$\Lambda = \frac{\Gamma(q)}{\Gamma(q+r)} \frac{T^p}{\Gamma(p+1)} \eta^{q+r-1} - \frac{\Gamma(q)}{\Gamma(p+q+r)} T^{p+q+r-1} \neq 0, \tag{2.1}$$

and $\hat{g}, y \in C(J, \mathbb{R})$ *and two functionals* $\phi, h : C(J, \mathbb{R}) \to \mathbb{R}$. *The unique solution of the linear problem*

$$\begin{cases} ^{RL}D^q[^{C}D^r x(t) - \hat{g}(t)] = y(t), \ 0 < t < T, \\ \\ x(\eta) = \phi(x), \ I^p x(T) = h(x), \end{cases} \tag{2.2}$$

is given by

$$\begin{aligned} x(t) \ &= \ I^r \hat{g}(t) + I^{q+r} y(t) \\ &+ \frac{t^{q+r-1}}{\Lambda} \frac{\Gamma(q)}{\Gamma(q+r)} \Big[\left(\phi(x) - I^r \hat{g}(\eta) - I^{q+r} y(\eta) \right) \frac{T^p}{\Gamma(p+1)} \\ &- \left(h(x) - I^{p+r} \hat{g}(T) - I^{p+q+r} y(T) \right) \Big] \\ &+ \frac{1}{\Lambda} \Big[\frac{\Gamma(q)}{\Gamma(q+r)} \eta^{q+r-1} \left(h(x) - I^{p+r} \hat{g}(T) - I^{p+q+r} y(T) \right) \\ &- \left(\phi(x) - I^r \hat{g}(\eta) - I^{q+r} y(\eta) \right) \frac{\Gamma(q)}{\Gamma(p+q+r)} T^{p+q+r-1} \Big]. \end{aligned}$$

Proof. Firstly, we apply the Riemann-Liouville fractional integral of order q to both sides of equation (2.2), and using Lemma 2.4, we have

$$^{C}D^r x(t) = \hat{g}(t) + I^q y(t) + c_1 t^{q-1}, \tag{2.3}$$

where a constant $c_1 \in \mathbb{R}$. After that, using Riemann-Liouville fractional integral of order r to both sides the above equation and applying Lemma 2.5, we get

$$x(t) = I^r \hat{g}(t) + I^{q+r} y(t) + c_1 \frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} + c_2, \tag{2.4}$$

where a constant $c_2 \in \mathbb{R}$. Observe that the equation (2.4) is well defined as $q + r > 1$.

Using nonlocal boundary condition of problem (2.2) to the above equation, we obtain the linear system

$$\begin{aligned} c_1 \frac{\Gamma(q)}{\Gamma(q+1)} \eta^{q+r-1} + c_2 &= \phi(x) - I^r \hat{g}(\eta) - I^{q+r} y(\eta), \\ c_1 \frac{\Gamma(q)}{\Gamma(p+q+1)} T^{p+q+r-1} + c_2 \frac{T^p}{\Gamma(p+1)} &= h(x) - I^{p+r} \hat{g}(T) - I^{p+q+r} y(T). \end{aligned}$$

Note that the two functionals $\phi(x)$ and $h(x)$ are constants. Solving the system of linear equations for constants c_1, c_2, we have

$$\begin{aligned} c_1 &= \frac{1}{\Lambda} \Bigg[\frac{T^p}{\Gamma(p+1)} \left(\phi(x) - I^r \hat{g}(\eta) - I^{q+r} y(\eta) \right) \\ &\quad - \left(h(x) - I^{p+r} \hat{g}(T) - I^{p+q+r} y(T) \right) \Bigg], \\ c_2 &= \frac{1}{\Lambda} \Bigg[\frac{\Gamma(q)}{\Gamma(q+r)} \eta^{q+r-1} \left(h(x) - I^{p+r} \hat{g}(T) - I^{p+q+r} y(T) \right) \\ &\quad - \frac{\Gamma(q)}{\Gamma(p+q+r)} T^{p+q+r-1} \left(\phi(x) - I^r \hat{g}(\eta) - I^{q+r} y(\eta) \right) \Bigg]. \end{aligned}$$

Substituting two constants c_1 and c_2 into equation (2.4), we obtain the required solution. The converse follows by direct computation. The proof is completed. □

3. Main results

Let $J = [0, T]$ and $\mathcal{C} = C(J, \mathbb{R})$ denotes the Banach space of all continuous functions from J to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t \in J} |x(t)|$. By Lemma 2.6, we define an operator $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned} &(\mathcal{A}x)(t) \\ = \; & I^r g(s, x(s))(t) + I^{q+r} f(s, x(s))(t) \\ &+ \frac{t^{q+r-1}}{\Lambda} \frac{\Gamma(q)}{\Gamma(q+r)} \Bigg[\left(\phi(x(t)) - I^r g(s, x(s))(\eta) - I^{q+r} f(s, x(s))(\eta) \right) \frac{T^p}{\Gamma(p+1)} \\ &- \left(h(x(t)) - I^{p+r} g(s, x(s))(T) - I^{p+q+r} f(s, x(s))(T) \right) \Bigg] \\ &+ \frac{1}{\Lambda} \Bigg[\frac{\Gamma(q)}{\Gamma(q+r)} \eta^{q+r-1} \left(h(x(t)) - I^{p+r} g(s, x(s))(T) - I^{p+q+r} f(s, x(s))(T) \right) \\ &- \left(\phi(x(t)) - I^r g(s, x(s))(\eta) - I^{q+r} f(s, x(s))(\eta) \right) \frac{\Gamma(q)}{\Gamma(p+q+r)} T^{p+q+r-1} \Bigg], \end{aligned} \tag{3.1}$$

with $\Lambda \neq 0$. It should be noticed that problem (1.5) can be transformed into a fixed point equation $x = \mathcal{A}x$.

To accomplish of the study, we will use fixed point theorems to prove that the operator A has fixed points. For the sake of convenience, we define four constants by

$$\begin{aligned}
\Phi_1 &= \Bigg[\frac{T^r}{\Gamma(r+1)} + \frac{\Gamma(q)}{\Gamma(q+r)}\frac{T^{q+r-1}}{|\Lambda|}\left(\frac{T^p}{\Gamma(p+1)}\frac{\eta^r}{\Gamma(r+1)} + \frac{T^{p+r}}{\Gamma(p+r+1)}\right) \\
&\quad + \frac{1}{|\Lambda|}\left(\frac{\Gamma(q)}{\Gamma(q+1)}\frac{T^{p+r}}{\Gamma(p+r+1)}\eta^{q+r-1} + \frac{\Gamma(q)}{\Gamma(p+q+1)}\frac{\eta^r}{\Gamma(r+1)}T^{p+q+r-1}\right)\Bigg], \\
\Phi_2 &= \Bigg[\frac{T^{q+r}}{\Gamma(q+r+1)} + \frac{\Gamma(q)}{\Gamma(q+r)}\frac{T^{q+r-1}}{|\Lambda|}\bigg(\frac{T^p}{\Gamma(p+1)}\frac{\eta^{q+r}}{\Gamma(q+r+1)} \\
&\quad + \frac{T^{p+q+r}}{\Gamma(p+q+r+1)}\bigg) + \frac{1}{|\Lambda|}\bigg(\frac{\Gamma(q)}{\Gamma(q+1)}\frac{T^{p+q+r}}{\Gamma(p+q+r+1)}\eta^{q+r-1} + \\
&\quad + \frac{\Gamma(q)}{\Gamma(p+q+1)}\frac{\eta^{q+r}}{\Gamma(q+r+1)}T^{p+q+r-1}\bigg)\Bigg], \\
\Phi_3 &= \left[\frac{\Gamma(q)}{|\Lambda|\Gamma(q+r)}\frac{T^{p+q+r-1}}{\Gamma(p+1)} + \frac{\Gamma(q)}{|\Lambda|\Gamma(p+q+1)}T^{p+q+r-1}\right], \\
\Phi_4 &= \left[\frac{\Gamma(q)}{|\Lambda|\Gamma(q+r)}T^{p+q+r-1} + \frac{\Gamma(q)}{|\Lambda|\Gamma(q+1)}\eta^{q+r-1}\right].
\end{aligned}$$

The first existence and uniqueness result is obtained by using Banach contraction mapping principle.

Theorem 3.1. *Let $g, f : J \times \mathbb{R} \to \mathbb{R}$, be continuous functions and $\phi, h : C(J, \mathbb{R}) \to \mathbb{R}$ be two functionals satisfying the assumption:*

(H_1) *there exist positive constants L_i, $i = 1, 2, 3, 4$ such that:*
$|g(t, x) - g(t, y)| \le L_1|x - y|$, $|f(t, x) - f(t, y)| \le L_2|x - y|$, $t \in J, x, y \in \mathbb{R}$,
$|\phi(u) - \phi(v)| \le L_3|u - v|$ *and* $|h(u) - h(v)| \le L_4|u - v|$, $u, v \in C(J, \mathbb{R})$.

If the inequality

$$\Omega_1 := L_1\Phi_1 + L_2\Phi_2 + L_3\Phi_3 + L_4\Phi_4 < 1, \tag{3.2}$$

holds, then the boundary value problem (1.5) has a unique solution on J.

Proof. By using the Banach's contraction mapping principle, we shall show that $\mathcal{A}$ of a fixed point problem, $x = \mathcal{A}x$, has a unique fixed point which is the unique solution of problem (1.5).

To prove the embedding property, we set

$$\sup_{t\in[0,T]} |g(t, 0)| = M_1 < \infty, \sup_{t\in[0,T]} |f(t, 0)| = M_2 < \infty, |\phi(0)| = M_3, |h(0)| = M_4,$$

and choose

$$r \ge \frac{M_1\Phi_1 + M_2\Phi_2 + M_3\Phi_3 + M_4\Phi_4}{1 - \Omega_1}.$$

Now, we show that $\mathcal{A}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. For any $x \in B_r$, and taking into account assumption (H_1), we obtain

$$\begin{aligned}
& |\mathcal{A}x(t)| \\
\leq \; & \sup_{t\in[0,T]} \Bigg\{ I^r g(s,x(s))(t) + I^{q+r} f(s,x(s))(t) \\
& + \frac{t^{q+r-1}}{\Lambda} \frac{\Gamma(q)}{\Gamma(q+r)} \Bigg[\Bigg(\phi(x(t)) - I^r g(s,x(s))(\eta) - I^{q+r} f(s,x(s))(\eta) \Bigg) \frac{T^p}{\Gamma(p+1)} \\
& - \Bigg(h(x(t)) - I^{p+r} g(s,x(s))(T) - I^{p+q+r} f(s,x(s))(T) \Bigg) \Bigg] \\
& + \frac{1}{\Lambda} \Bigg[\frac{\Gamma(q)}{\Gamma(q+r)} \eta^{q+r-1} \Bigg(h(x(t)) - I^{p+r} g(s,x(s))(T) - I^{p+q+r} f(s,x(s))(T) \Bigg) \\
& - \Bigg(\phi(x(t)) - I^r g(s,x(s))(\eta) - I^{q+r} f(s,x(s))(\eta) \Bigg) \frac{\Gamma(q)}{\Gamma(p+q+r)} T^{p+q+r-1} \Bigg] \Bigg\} \\
\leq \; & I^r (|g(s,x(s)) - g(s,0)| + |g(s,0)|)(T) \\
& + I^{q+r} (|f(s,x(s)) - f(s,0)| + |f(s,0)|)(T) \\
& + \frac{T^{q+r-1}}{|\Lambda|} \frac{\Gamma(q)}{\Gamma(q+r)} \Bigg[\frac{T^p}{\Gamma(p+1)} \Bigg(\left(|\phi(x) - \phi(0)| + |\phi(0)|\right)(T) \\
& + I^r \big(|g(s,x(s)) - g(s,0)| + |g(s,0)|\big)(\eta) \\
& + I^{q+r} \big(|f(s,x(s)) - f(s,0)| + |f(s,0)|\big)(\eta) \Bigg) + \Bigg(\left(|h(x) - h(0)| + |h(0)|\right)(T) \\
& + I^{p+r} \big(|g(s,x(s)) - g(s,0)| + |g(s,0)|\big)(T) \\
& + I^{p+q+r} \big(|f(s,x(s)) - f(s,0)| + |f(s,0)|\big)(T) \Bigg) \Bigg] \\
& + \frac{1}{|\Lambda|} \Bigg[\frac{\Gamma(q)}{\Gamma(q+1)} \eta^{q+r-1} \Bigg(\left(|h(x) - h(0)| + |h(0)|\right)(T) \\
& + I^{p+r} \big(|g(s,x(s)) - g(s,0)| + |g(s,0)|\big)(T) \\
& + I^{p+q+r} \big(|f(s,x(s)) - f(s,0)| + |f(s,0)|\big)(T) \Bigg) \\
& + \frac{\Gamma(q)}{\Gamma(p+q+1)} T^{p+q+r-1} \Bigg(\left(|\phi(x) - \phi(0)| + |\phi(0)|\right)(T) \\
& + I^r \big(|g(s,x(s)) - g(s,0)| + |g(s,0)|\big)(\eta) \\
& + I^{q+r} \big(|f(s,x(s)) - f(s,0)| + |f(s,0)|\big)(\eta) \Bigg) \Bigg] \\
\leq \; & (L_1 r + M_1) \frac{T^r}{\Gamma(r+1)} + (L_2 r + M_2) \frac{T^{q+r}}{\Gamma(q+r+1)} \\
& + \frac{T^{q+r-1}}{|\Lambda|} \frac{\Gamma(q)}{\Gamma(q+r)} \Bigg[\frac{T^p}{\Gamma(p+1)} \Bigg((L_3 r + M_3) + (L_1 r + M_1) \frac{\eta^r}{\Gamma(r+1)} \\
& + (L_2 r + M_2) \frac{\eta^{q+r}}{\Gamma(q+r+1)} \Bigg) + \Bigg((L_4 r + M_4) + (L_1 r + M_1) \frac{T^{p+r}}{\Gamma(p+r+1)} \\
& + (L_2 r + M_2) \frac{T^{p+q+r}}{\Gamma(p+q+r+1)} \Bigg) \Bigg]
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{|\Lambda|}\Bigg[\frac{\Gamma(q)}{\Gamma(q+1)}\eta^{q+r-1}\Bigg((L_4r+M_4)+(L_1r+M_1)\,\frac{T^{p+r}}{\Gamma(p+r+1)} \\
& +(L_2r+M_2)\,\frac{T^{p+q+r}}{\Gamma(p+q+r+1)}\Bigg)+\frac{\Gamma(q)}{\Gamma(p+q+r)}\Bigg((L_3r+M_3) \\
& +(L_1r+M_1)\,\frac{\eta^{r}}{\Gamma(r+1)}+(L_2r+M_2)\,\frac{\eta^{q+r}}{\Gamma(q+r+1)}\Bigg)\Bigg] \\
\leq\ & \Bigg[\frac{T^r}{\Gamma(r+1)}+\frac{\Gamma(q)}{\Gamma(q+r)}\frac{T^{q+r-1}}{|\Lambda|}\left(\frac{T^p}{\Gamma(p+1)}\frac{\eta^r}{\Gamma(r+1)}+\frac{T^{p+r}}{\Gamma(p+r+1)}\right) \\
& +\frac{1}{|\Lambda|}\left(\frac{\Gamma(q)}{\Gamma(q+1)}\frac{T^{p+r}}{\Gamma(p+r+1)}\eta^{q+r-1}+\frac{\Gamma(q)}{\Gamma(p+q+1)}\frac{\eta^r}{\Gamma(r+1)}T^{p+q+r-1}\right)\Bigg] \\
& \times(L_1r+M_1)+\Bigg[\frac{T^{q+r}}{\Gamma(q+r+1)}+\frac{\Gamma(q)}{\Gamma(q+r)}\frac{T^{q+r-1}}{|\Lambda|}\Bigg(\frac{T^p}{\Gamma(p+1)}\frac{\eta^{q+r}}{\Gamma(q+r+1)} \\
& +\frac{T^{p+q+r}}{\Gamma(p+q+r+1)}\Bigg)+\frac{1}{|\Lambda|}\Bigg(\frac{\Gamma(q)}{\Gamma(q+1)}\frac{T^{p+q+r}}{\Gamma(p+q+r+1)}\eta^{q+r-1}+ \\
& +\frac{\Gamma(q)}{\Gamma(p+q+1)}\frac{\eta^{q+r}}{\Gamma(q+r+1)}T^{p+q+r-1}\Bigg)\Bigg](L_2r+M_2) \\
& +\left[\frac{\Gamma(q)}{|\Lambda|\Gamma(q+r)}\frac{T^{p+q+r-1}}{\Gamma(p+1)}+\frac{\Gamma(q)}{|\Lambda|\Gamma(p+q+1)}T^{p+q+r-1}\right](L_3r+M_3) \\
& +\left[\frac{\Gamma(q)}{|\Lambda|\Gamma(q+r)}T^{p+q+r-1}+\frac{\Gamma(q)}{|\Lambda|\Gamma(q+1)}\eta^{q+r-1}\right](L_4r+M_4) \\
=\ & \Phi_1(L_1r+M_1)+\Phi_2(L_2r+M_2)+\Phi_3(L_3r+M_3)+\Phi_4(L_4r+M_4) \\
=\ & \Omega_1 r+(M_1\Phi_1+M_2\Phi_2+M_3\Phi_3+M_4\Phi_4) \\
\leq\ & r.
\end{aligned}
$$

This mean that $\|\mathcal{A}x\|\leq r$ which yields $\mathcal{A}B_r\subset B_r$. For all $t\in[0,T]$ and for each $x,y\in\mathcal{C}$, we have

$$
\begin{aligned}
& |\mathcal{A}x(t)-\mathcal{A}y(t)| \\
\leq\ & I^r(|g(s,x(s))-g(s,y(s))|)(T)+I^{q+r}(|f(s,x(s))-f(s,y(s))|)(T) \\
& +\frac{T^{q+r-1}}{|\Lambda|}\frac{\Gamma(q)}{\Gamma(q+r)}\Bigg[\frac{T^p}{\Gamma(p+1)}\Bigg(\left(|\phi(x)-\phi(y)|\right)(T) \\
& +I^r\left(|g(s,x(s))-g(s,y(s))|\right)(\eta)+I^{q+r}\left(|f(s,x(s))-f(s,y(s))|\right)(\eta)\Bigg) \\
& +\Bigg(\left(|h(x)-h(y)|\right)(T)+I^{p+r}\left(|g(s,x(s))-g(s,y(s))|\right)(T) \\
& +I^{p+q+r}\left(|f(s,x(s))-f(s,y(s))|\right)(T)\Bigg)\Bigg] \\
& +\frac{1}{|\Lambda|}\Bigg[\frac{\Gamma(q)}{\Gamma(q+1)}\eta^{q+r-1}\Bigg(\left(|h(x)-h(y)|\right)(T) \\
& +I^{p+r}\left(|g(s,x(s))-g(s,y(s))|\right)(T)+I^{p+q+r}\left(|f(s,x(s))-f(s,y(s))|\right)(T)\Bigg) \\
& +\frac{\Gamma(q)}{\Gamma(p+q+1)}T^{p+q+r-1}\Bigg(\left(|\phi(x)-\phi(y)|\right)(T)+I^r\left(|g(s,x(s))-g(s,x(s))|\right)(\eta)
\end{aligned}
$$

$$
\begin{aligned}
& +I^{q+r}\left(|f(s,x(s))-f(s,y(s))|\right)(\eta)\Big)\Big] \\
\leq\ & (L_1|x-y|)\frac{T^r}{\Gamma(r+1)}+(L_2|x-y|)\frac{T^{q+r}}{\Gamma(q+r+1)} \\
& +\frac{T^{q+r-1}}{|\Lambda|}\frac{\Gamma(q)}{\Gamma(q+r)}\Bigg[\frac{T^p}{\Gamma(p+1)}\Bigg((L_3|x-y|) \\
& +(L_1|x-y|)\frac{\eta^r}{\Gamma(r+1)}+(L_2|x-y|)\frac{\eta^{q+r}}{\Gamma(q+r+1)}\Bigg) \\
& +\Bigg((L_4|x-y|)+(L_1|x-y|)\frac{T^{p+r}}{\Gamma(p+r+1)}+(L_2|x-y|)\frac{T^{p+q+r}}{\Gamma(p+q+r+1)}\Bigg)\Bigg] \\
& +\frac{1}{|\Lambda|}\Bigg[\frac{\Gamma(q)}{\Gamma(q+1)}\eta^{q+r-1}\Bigg((L_4|x-y|)+(L_1|x-y|)\frac{T^{p+r}}{\Gamma(p+r+1)} \\
& +(L_2|x-y|)\frac{T^{p+q+r}}{\Gamma(p+q+r+1)}\Bigg)+\frac{\Gamma(q)}{\Gamma(p+q+r)}\Bigg((L_3|x-y|) \\
& +(L_1|x-y|)\frac{\eta^r}{\Gamma(r+1)}+(L_2|x-y|)\frac{\eta^{q+r}}{\Gamma(q+r+1)}\Bigg)\Bigg] \\
\leq\ & \Bigg[\frac{T^r}{\Gamma(r+1)}+\frac{\Gamma(q)}{\Gamma(q+r)}\frac{T^{q+r-1}}{|\Lambda|}\left(\frac{T^p}{\Gamma(p+1)}\frac{\eta^r}{\Gamma(r+1)}+\frac{T^{p+r}}{\Gamma(p+r+1)}\right) \\
& +\frac{1}{|\Lambda|}\left(\frac{\Gamma(q)}{\Gamma(q+1)}\frac{T^{p+r}}{\Gamma(p+r+1)}\eta^{q+r-1}+\frac{\Gamma(q)}{\Gamma(p+q+1)}\frac{\eta^r}{\Gamma(r+1)}T^{p+q+r-1}\right)\Bigg] \\
& \times(L_1|x-y|)+\Bigg[\frac{T^{q+r}}{\Gamma(q+r+1)}+\frac{\Gamma(q)}{\Gamma(q+r)}\frac{T^{q+r-1}}{|\Lambda|}\Bigg(\frac{T^p}{\Gamma(p+1)}\frac{\eta^{q+r}}{\Gamma(q+r+1)} \\
& +\frac{T^{p+q+r}}{\Gamma(p+q+r+1)}\Bigg)+\frac{1}{|\Lambda|}\Bigg(\frac{\Gamma(q)}{\Gamma(q+1)}\frac{T^{p+q+r}}{\Gamma(p+q+r+1)}\eta^{q+r-1} \\
& +\frac{\Gamma(q)}{\Gamma(p+q+1)}\frac{\eta^{q+r}}{\Gamma(q+r+1)}T^{p+q+r-1}\Bigg)\Bigg]L_2|x-y| \\
& +\left[\frac{\Gamma(q)}{|\Lambda|\Gamma(q+r)}\frac{T^{p+q+r-1}}{\Gamma(p+1)}+\frac{\Gamma(q)}{|\Lambda|\Gamma(p+q+1)}T^{p+q+r-1}\right]L_3|x-y| \\
& +\left[\frac{\Gamma(q)}{|\Lambda|\Gamma(q+r)}T^{p+q+r-1}+\frac{\Gamma(q)}{|\Lambda|\Gamma(q+1)}\eta^{q+r-1}\right]L_4|x-y| \\
=\ & \Phi_1L_1|x-y|)+\Phi_2L_2|x-y|)+\Phi_3L_3|x-y|)+\Phi_4L_4|x-y|) \\
=\ & \Omega_1|x-y|.
\end{aligned}
$$

The above result implies that $\|\mathcal{A}x-\mathcal{A}y\|\leq\Omega_1\|x-y\|$. As $\Omega_1<1$, therefore $\mathcal{A}$ is a contraction operator. Hence, by the Banach contraction mapping principle, we obtain that $\mathcal{A}$ has a unique fixed point which is the unique solution of the problem (1.5). The proof is completed. □

Example 3.1. *Consider the following nonlinear sequential Riemann-Liouville and Caputo fractional differential equation with nonlocal integral boundary conditions*

$$
{}^{RL}D^{\frac{4}{5}}\left({}^{C}D^{\frac{1}{2}}x(t)-\frac{e^t}{(t^2+40)+20}\frac{|x(t)|}{|x(t)|+1}\right)
$$

$$= \frac{\cos^2(2\pi t)}{(t+10)^2+50} \cdot \left(\frac{x^2(t)+2|x(t)|}{|x(t)|+1}\right) + e^t, \quad 0 < t < 3, \tag{3.3}$$

$$x\left(\frac{1}{2}\right) = \frac{x^2(2)+2|x(2)|}{60(|x(2)|+1)} + 30, \quad I^{\frac{2}{3}}x(3) = \frac{|x(1)|}{25(|x(1)|+1)}.$$

Setting constants $q = 4/5$, $r = 1/2$, $p = 2/3$, $\eta = 1/2$, $T = 3$, then we can compute constants as $\Phi_1 = 12.42305820$, $\Phi_2 = 14.24066077$, $\Phi_3 = 6.845515569$, $\Phi_4 = 4.835810257$. Setting functions

$$g(t,x) = \frac{e^t}{(t^2+40)+20}\frac{|x|}{|x|+1}, \quad f(t,x) = \frac{\cos^2(2\pi t)}{(t+10)^2+50} \cdot \left(\frac{x^2+2|x|}{|x|+1}\right) + e^t$$

$$\phi(x) = \frac{x^2+2|x|}{60(|x|+1)} + 30. \quad h(x) = \frac{|x|}{25(|x|+1)},$$

so we get $|g(t,x) - g(t,y)| \le (1/60)|x-y|$, $|f(t,x) - f(t,y)| \le (1/75)|x-y|$, $|\phi(x) - \phi(y)| \le (1/30)|x-y|$ and $|h(x) - h(y)| \le (1/25)|x-y|$. Therefore the condition (H_1) is satisfied with $L_1 = 1/60$, $L_2 = 1/75$, $L_3 = 1/30$ and $L_4 = 1/25$. We can show that

$$\Omega_1 = 0.8185425995 < 1.$$

Hence, by Theorem 3.1, the boundary value problem (3.3) has a unique solution on $[0,3]$.

The second existence result will be proved by using the following Krasnoselskii's fixed point theorem.

Lemma 3.1. *(Krasnoselskii's fixed point theorem) [13]. Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that (a) $Ax+By \in M$ whenever $x, y \in M$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az+Bz$.*

Theorem 3.2. *Assume that $g, f : J\times\mathbb{R} \to \mathbb{R}$, are continuous functions and two functionals $\phi, h : C(J\times\mathbb{R}) \to \mathbb{R}$ satisfying the assumption (H_1). In addition we suppose that:*

- (H_2) *$|g(t,x)| \le \delta_1(t)$, $|f(t,x)| \le \delta_2(t)$, $\forall (t,x) \in J\times\mathbb{R}$ and $\delta_1, \delta_2 \in C(J,\mathbb{R}^+)$,*
  $|\phi(u)| \le \delta_3$, $|h(u)| \le \delta_4$, $\forall u \in C(J\times\mathbb{R})$ and $\delta_3, \delta_4 \in \mathbb{R}^+$.

If the inequality

$$\Omega_2 := L_1\Big(\Phi_1 - \frac{T^q}{\Gamma(r+1)}\Big) + L_2\Big(\Phi_2 - \frac{T^{q+r}}{\Gamma(q+r+1)}\Big) + L_3\Phi_3 + L_4\Phi_4 < 1, \tag{3.4}$$

then the boundary value problem (1.5) has at least one solution on J.

Proof. To applied Lemma 3.1, we let $\sup_{t\in J}|\delta_1(t)| = \|\delta_1\|$, $\sup_{t\in J}|\delta_2(t)| = \|\delta_2\|$, and a positive constant $\bar{r}$ as

$$\bar{r} \ge \|\delta_1\|\Phi_1 + \|\delta_2\|\Phi_2 + \delta_3\Phi_3 + \delta_4\Phi_4.$$

Define a ball $B_{\overline{r}}$ by $B_{\overline{r}} = \{x \in \mathcal{C} : \|x\| \le \overline{r}\}$ which is closed, bounded, convex and nonempty subset of a Banach space $\mathcal{C}$. In addition, we define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\overline{r}}$ as

$$(\mathcal{P}x)(t) \quad = \quad I^r g(s,x(s))(t) + I^{q+r} f(s,x(s))(t), \; t \in [0,T],$$

$$\begin{aligned}
& (\mathcal{Q}x)(t) \\
= \; & \frac{t^{q+r-1}}{\Lambda} \frac{\Gamma(q)}{\Gamma(q+r)} \Bigg[\big(\phi(x(t)) - I^r g(s,x(s))(\eta) - I^{q+r} f(s,x(s))(\eta)\big) \frac{T^p}{\Gamma(p+1)} \\
& - \big(h(x(t)) - I^{p+r} g(s,x(s))(T) - I^{p+q+r} f(s,x(s))(T)\big) \Bigg] \\
& + \frac{1}{\Lambda} \Bigg[\frac{\Gamma(q)}{\Gamma(q+r)} \eta^{q+r-1} \big(h(x(t)) - I^{p+r} g(s,x(s))(T) - I^{p+q+r} f(s,x(s))(T)\big) \\
& - \big(\phi(x(t)) - I^r g(s,x(s))(\eta) - I^{q+r} f(s,x(s))(\eta)\big) \frac{\Gamma(q)}{\Gamma(p+q+r)} T^{p+q+r-1} \Bigg], \\
& t \in [0,T].
\end{aligned}$$

Obvious that $\mathcal{A}x = \mathcal{P}x + \mathcal{Q}x$. To prove that $\mathcal{P}$ and $\mathcal{Q}$ satisfy (a) of Lemma 3.1, for $x, y \in B_{\overline{r}}$, we have

$$\begin{aligned}
& \|\mathcal{P}x + \mathcal{Q}y\| \\
\le \; & \|\delta_1\| \Bigg[\frac{T^r}{\Gamma(r+1)} + \frac{\Gamma(q)}{\Gamma(q+r)} \frac{T^{q+r-1}}{|\Lambda|} \left(\frac{T^p}{\Gamma(p+1)} \frac{\eta^r}{\Gamma(r+1)} + \frac{T^{p+r}}{\Gamma(p+r+1)} \right) \\
& + \frac{1}{|\Lambda|} \left(\frac{\Gamma(q)}{\Gamma(q+1)} \frac{T^{p+r}}{\Gamma(p+r+1)} \eta^{q+r-1} + \frac{\Gamma(q)}{\Gamma(p+q+1)} \frac{\eta^r}{\Gamma(r+1)} T^{p+q+r-1} \right) \Bigg] \\
& + \|\delta_2\| \Bigg[\frac{T^{q+r}}{\Gamma(q+r+1)} + \frac{\Gamma(q)}{\Gamma(q+r)} \frac{T^{q+r-1}}{|\Lambda|} \left(\frac{T^p}{\Gamma(p+1)} \frac{\eta^{q+r}}{\Gamma(q+r+1)} \right. \\
& \left. + \frac{T^{p+q+r}}{\Gamma(p+q+r+1)} \right) + \frac{1}{|\Lambda|} \left(\frac{\Gamma(q)}{\Gamma(q+1)} \frac{T^{p+q+r}}{\Gamma(p+q+r+1)} \eta^{q+r-1} + \right. \\
& \left. + \frac{\Gamma(q)}{\Gamma(p+q+1)} \frac{\eta^{q+r}}{\Gamma(q+r+1)} T^{p+q+r-1} \right) \Bigg] \\
& + \delta_3 \left[\frac{\Gamma(q)}{|\Lambda|\Gamma(q+r)} \frac{T^{p+q+r-1}}{\Gamma(p+1)} + \frac{\Gamma(q)}{|\Lambda|\Gamma(p+q+1)} T^{p+q+r-1} \right] \\
& + \delta_4 \left[\frac{\Gamma(q)}{|\Lambda|\Gamma(q+r)} T^{p+q+r-1} + \frac{\Gamma(q)}{|\Lambda|\Gamma(q+1)} \eta^{q+r-1} \right] \\
= \; & \|\delta_1\|\Phi_1 + \|\delta_2\|\Phi_2 + \delta_3\Phi_3 + \delta_4\Phi_4 \\
\le \; & \overline{r}.
\end{aligned}$$

This shows that $\mathcal{P}x + \mathcal{Q}y \in B_{\overline{r}}$.

The operator $\mathcal{Q}$ satisfies the condition (c) of Lemma 3.1 from assumption (H_1) together with (3.4). The final step is to show that the operator $\mathcal{P}$ is satisfied condition (b) of Lemma 3.1. Since the functions f, g are continuous, we get that the operator $\mathcal{P}$ is continuous. Now we will show that the operator $\mathcal{P}$ is compact.

For any $x \in B_{\overline{r}}$, we obtain

$$\|\mathcal{P}x\| \le \|\delta_1\| \frac{T^r}{\Gamma(q+1)} + \|\delta_2\| \frac{T^{q+r}}{\Gamma(q+r+1)}.$$

Therefore, the set $\mathcal{P}(B_{\overline{r}})$ is uniformly bounded. Let us let $\sup_{(t,x)\in J\times B_{\overline{r}}} |g(t,x)| = \overline{g} < \infty$ and $\sup_{(t,x)\in J\times B_{\overline{r}}} |f(t,x)| = \overline{f} < \infty$. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
& |(\mathcal{P}x)(t_2) - (\mathcal{P}x)(t_1)| \\
\le\ & \frac{\overline{g}}{\Gamma(r)} \left| \int_0^{t_1} \left[(t_2-s)^{r-1} - (t_1-s)^{r-1}\right] ds \right| + \frac{\overline{g}}{\Gamma(r)} \left| \int_{t_1}^{t_2} (t_2-s)^{r-1} ds \right| \\
& + \frac{\overline{f}}{\Gamma(q+r)} \left| \int_0^{t_1} \left[(t_2-s)^{q+r-1} - (t_1-s)^{q+r-1}\right] ds \right| \\
& + \frac{\overline{f}}{\Gamma(q+r)} \left| \int_{t_1}^{t_2} (t_2-s)^{q+r-1} ds \right| \\
\le\ & \frac{\overline{g}}{\Gamma(r+1)} \left[|t_2^r - t_2^r| + 2(t_2-t_1)^r\right] + \frac{\overline{f}}{\Gamma(q+r+1)} \left[\left|t_2^{q+r} - t_1^{q+r}\right| + 2(t_2-t_1)^r\right],
\end{aligned}$$

which is independent of x and tends to zero as $t_1 \to t_2$. Thus, the set $\mathcal{P}(B_{\overline{r}})$ is equicontinuous. Hence, by the Arzelá-Ascoli theorem, the set $\mathcal{P}(B_{\overline{r}})$ is relatively compact. Therefore, the operator $\mathcal{P}$ is compact which is satisfied condition (b) of Lemma 3.1. Thus all the assumptions of Lemma 3.1 are satisfied. So the boundary value problem (1.5) has at least one solution on J. The proof is completed. □

Remark 3.1. *In the above theorem we can interchange the roles of the operators $\mathcal{P}$ and $\mathcal{Q}$ to obtain a second result replacing (3.4) by the following condition:*

$$\Omega_3 := L_1 \frac{T^r}{\Gamma(r+1)} + L_2 \frac{T^{q+r}}{\Gamma(q+r+1)} < 1. \tag{3.5}$$

Remark 3.2. *Since $\Omega_2 < \Omega_1$ and $\Omega_3 < \Omega_1$, the condition* (3.2) *can be relaxed by* (3.4) *and* (3.5)*. However, the conclusion of both theorems has different mentions between uniqueness and multiplicity of solutions.*

Example 3.2. *Consider the following nonlinear sequential Riemann-Liouville and Caputo fractional differential equation with nonlocal integral boundary conditions*

$$\begin{aligned}
& {}^{RL}D^{\frac{1}{2}} \left({}^{C}D^{\frac{2}{3}} x(t) - \frac{e^{2t}}{(t^2+100)^2 + 19300} \cdot \frac{|x(t)|}{|x(t)|+1} \right) \\
& = \frac{\cos^2(2\pi t)}{t^2 + 28000} \cdot \left(\frac{|x(t)|}{|x(t)|+1} \right) + \cos(\pi t), \quad 0 < t < 4,
\end{aligned} \tag{3.6}$$

$$x(2) = \frac{|x(3)|}{9990(|x(3)|+1)}, \quad I^{\frac{2}{3}} x(4) = \frac{|x(2)|}{9840(|x(2)|+1)} + 35.$$

Setting constants $q = 1/2$, $r = 2/3$, $p = 2/3$, $\eta = 2$, $T = 4$, then we can fine that $\Phi_1 = 6717.422119$, $\Phi_2 = 6652.469591$, $\Phi_3 = 3119.677669$, $\Phi_4 = 2175.349828$. Next we set the following functions

$$g(t,x) = \frac{e^{2t}}{(t^2+100)^2 + 19300} \cdot \frac{|x|}{|x|+1},$$

$$f(t,x) = \frac{\cos^2(2\pi t)}{t^2+28000}\cdot\left(\frac{|x|}{|x|+1}\right) + \cos(\pi t)$$
$$\phi(x) = \frac{|x|}{9990(|x|+1)}, \qquad h(x) = \frac{|x|}{9840(|x|+1)} + 35.$$

Since $|g(t,x)-g(t,y)| \le (1/29300)|x-y|$, $|f(t,x)-f(t,y)| \le (1/28000)|x-y|$, $|\phi(x)-\phi(t,y)| \le (1/9990)|x-y|$ and $|h(x)-h(y)| \le (1/9840)|x-y|$, the condition (H_1) fulfilled. It is obvious that

$$|g(t,x)| \le \frac{e^{2t}}{29300}, \quad |f(t,x)| \le 1+\cos(\pi t), \quad |\phi(x)| \le 1, \quad h(x) \le 36.$$

Then the condition (H_2) is satisfied. In addition we have

$$\Omega_2 = 0.999918 < 1.$$

Hence, by Theorem 3.2, the boundary value problem (3.6) has at least one solution on $[0,4]$.

Remark 3.3. *The problem (3.6) can not be applied by Theorem 3.1 since $\Omega_1 = 1.000204 > 1$.*

Now, our third existence result is based on Leray-Schauder's Nonlinear Alternative.

Lemma 3.2. *(Nonlinear alternative for single-valued maps) [11]. Let E be a Banach space, C be a closed, convex subset of E, U be an open subset of C and $0 \in U$. Suppose that $A : \overline{U} \to C$ is a continuous, compact (that is, $A(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *A has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda A(u)$.*

Theorem 3.3. *Assume that $g, f : J\times\mathbb{R} \to \mathbb{R}$ are continuous functions and two functionals $\phi, h : C(J\times\mathbb{R}) \to \mathbb{R}$. In addition we suppose that:*

(H_3) *there exist continuous nondecreasing functions $\psi_1, \psi_2 : [0,\infty) \to (0,\infty)$ and functions $p_1, p_2 \in C(J, \mathbb{R}^+)$ such that*

$$|g(t,x)| \le p_1(t)\psi_1(\|x\|), \ |f(t,x)| \le p_2(t)\psi_2(\|x\|) \ \textit{for each} \ (t,x) \in J\times\mathbb{R};$$

(H_4) *there exists a constant $N > 0$ such that*

$$\frac{N}{\Phi_1\|p_1\|\psi_1(N) + \Phi_2\|p_2\|\psi_2(N) + \Phi_3|\phi(N)| + \Phi_4|h(N)|} > 1.$$

Then the boundary value problem (1.5) has at least one solution on J.

Proof. Let us define a positive number R and let a ball $B_R = \{x \in \mathcal{C} : \|x\| \le R\}$ be a closed, convex subset of $\mathcal{C}$. Next, we will prove that the operator $\mathcal{A}$, defined by (3.1), maps bounded sets (balls) into bounded sets in $\mathcal{C}$. For any $t \in J$ and $x \in B_R$, we have

$$|\mathcal{A}x(t)|$$

$$
\begin{aligned}
&\le I^r|g(s,x(s))|(t)+I^{q+r}|f(s,x(s))|(t)\\
&\quad+\frac{t^{q+r-1}}{|\Lambda|}\frac{\Gamma(q)}{\Gamma(q+r)}\bigg[\Big(|\phi(x(t))|+I^r|g(s,x(s))|(\eta)+I^{q+r}|f(s,x(s))|(\eta)\Big)\frac{T^p}{\Gamma(p+1)}\\
&\quad+\Big(|h(x(t))|+I^{p+r}|g(s,x(s))|(T)+I^{p+q+r}|f(s,x(s))|(T)\Big)\bigg]\\
&\quad+\frac{1}{|\Lambda|}\bigg[\frac{\Gamma(q)}{\Gamma(q+r)}\eta^{q+r-1}\Big(|h(x(t))|+I^{p+r}|g(s,x(s))|(T)+I^{p+q+r}|f(s,x(s))|(T)\Big)\\
&\quad+\Big(|\phi(x(t))|+I^r|g(s,x(s))|(\eta)+I^{q+r}|f(s,x(s))|(\eta)\Big)\frac{\Gamma(q)}{\Gamma(p+q+r)}T^{p+q+r-1}\bigg]\\
&\le \|p_1\|\psi_1(\|x\|)\frac{T^r}{\Gamma(r+1)}+\|p_2\|\psi_2(\|x\|)\frac{T^{q+r}}{\Gamma(q+r+1)}\\
&\quad+\frac{T^{q+r-1}}{|\Lambda|}\frac{\Gamma(q)}{\Gamma(q+r)}\bigg[\frac{T^p}{\Gamma(p+1)}\bigg(|\phi(\|x\|)|+(\|p_1\|\psi_1(\|x\|))\frac{\eta^r}{\Gamma(r+1)}\\
&\quad+(\|p_2\|\psi_2(\|x\|))\frac{\eta^{q+r}}{\Gamma(q+r+1)}\bigg)+\bigg(|h(\|x\|)|+(\|p_1\|\psi_1(\|x\|))\frac{T^{p+r}}{\Gamma(p+r+1)}\\
&\quad+(\|p_2\|\psi_2(\|x\|))\frac{T^{p+q+r}}{\Gamma(p+q+r+1)}\bigg)\bigg]\\
&\quad+\frac{1}{|\Lambda|}\bigg[\frac{\Gamma(q)}{\Gamma(q+1)}\eta^{q+r-1}\bigg(|h(\|x\|)|+(\|p_1\|\psi_1(\|x\|))\frac{T^{p+r}}{\Gamma(p+r+1)}\\
&\quad+(\|p_2\|\psi_2(\|x\|))\frac{T^{p+q+r}}{\Gamma(p+q+r+1)}\bigg)+\frac{\Gamma(q)}{\Gamma(p+q+r)}\bigg(|\phi(\|x\|)|\\
&\quad+(\|p_1\|\psi_1(\|x\|))\frac{\eta^r}{\Gamma(r+1)}+(\|p_2\|\psi_2(\|x\|))\frac{\eta^{q+r}}{\Gamma(q+r+1)}\bigg)\bigg]\\
&\le \bigg[\frac{T^r}{\Gamma(r+1)}+\frac{\Gamma(q)}{\Gamma(q+r)}\frac{T^{q+r-1}}{||\Lambda||}\bigg(\frac{T^p}{\Gamma(p+1)}\frac{\eta^r}{\Gamma(r+1)}+\frac{T^{p+r}}{\Gamma(p+r+1)}\bigg)\\
&\quad+\frac{1}{|\Lambda|}\bigg(\frac{\Gamma(q)}{\Gamma(q+1)}\frac{T^{p+r}}{\Gamma(p+r+1)}\eta^{q+r-1}+\frac{\Gamma(q)}{\Gamma(p+q+1)}\frac{\eta^r}{\Gamma(r+1)}T^{p+q+r-1}\bigg)\bigg]\\
&\quad\times(\|p_1\|\psi_1(\|x\|))+\bigg[\frac{T^{q+r}}{\Gamma(q+r+1)}+\frac{\Gamma(q)}{\Gamma(q+r)}\frac{T^{q+r-1}}{||\Lambda||}\bigg(\frac{T^p}{\Gamma(p+1)}\frac{\eta^{q+r}}{\Gamma(q+r+1)}\\
&\quad+\frac{T^{p+q+r}}{\Gamma(p+q+r+1)}\bigg)+\frac{1}{|\Lambda|}\bigg(\frac{\Gamma(q)}{\Gamma(q+1)}\frac{T^{p+q+r}}{\Gamma(p+q+r+1)}\eta^{q+r-1}+\\
&\quad+\frac{\Gamma(q)}{\Gamma(p+q+1)}\frac{\eta^{q+r}}{\Gamma(q+r+1)}T^{p+q+r-1}\bigg)\bigg]\|p_2\|\psi_2(\|x\|)\\
&\quad+\bigg[\frac{\Gamma(q)}{|\Lambda|\Gamma(q+r)}\frac{T^{p+q+r-1}}{\Gamma(p+1)}+\frac{\Gamma(q)}{|\Lambda|\Gamma(p+q+1)}T^{p+q+r-1}\bigg]|\phi(\|x\|)|\\
&\quad+\bigg[\frac{\Gamma(q)}{|\Lambda|\Gamma(q+r)}T^{p+q+r-1}+\frac{\Gamma(q)}{|\Lambda|\Gamma(q+1)}\eta^{q+r-1}\bigg]|h(\|x\|)|\\
&= \Phi_1\|p_1\|\psi_1(\|x\|)+\Phi_2\|p_2\|\psi_2(\|x\|)+\Phi_3|\phi(\|x\|)|+\Phi_4|h(\|x\|)|\\
&\le \Phi_1\|p_1\|\psi_1(R)+\Phi_2\|p_2\|\psi_2(R)+\Phi_3|\phi(R)|+\Phi_4|h(R)|.
\end{aligned}
$$

Therefore, from the above result, we conclude that

$$\|\mathcal{A}x\|\le\Phi_1\|p_1\|\psi_1(R)+\Phi_2\|p_2\|\psi_2(R)+\Phi_3|\phi(R)|+\Phi_4|h(R)|.$$

Then the set $\mathcal{A}(B_R)$ is uniformly bounded. Next, we show that the operator $\mathcal{A}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $\nu_1, \nu_2 \in J$ with $\nu_1 < \nu_2$ and for any $x \in B_R$, then we have

$$
\begin{aligned}
& |(\mathcal{A}x)(\nu_2) - (\mathcal{A}x)(\nu_1)| \\
\leq\ & I^r |g(s,x(s))(\nu_2) - g(s,x(s))(\nu_1)| + I^{q+r}(|f(s,x(s))(\nu_2) - f(s,x(s))(\nu_1)|) \\
& + \frac{\left|\nu_2^{q+r-1} - \nu_1^{q+r-1}\right|}{|\Lambda|} \frac{\Gamma(q)}{\Gamma(q+r)} \Bigg[(|\phi(x(\nu_2)) - \phi(x(\nu_1))) \frac{T^p}{\Gamma(p+1)} \\
& + (|h(x(\nu_2)) - h(x(\nu_1))|) \Bigg] + \frac{1}{|\Lambda|} \left[\frac{\Gamma(q)}{\Gamma(q+r)} \eta^{q+r-1} (|h(x(\nu_2)) - h(x(\nu_1))|) \right. \\
& \left. + (|\phi(x(\nu_2)) - \phi(x(\nu_1))|) \frac{\Gamma(q)}{\Gamma(p+q+r)} T^{p+q+r-1} \right] \\
\leq\ & \frac{\|p_1\|\psi_1(R)}{\Gamma(r+1)} [|t_2^r - t_2^r| + 2(t_2 - t_1)^r] + \frac{\|p_2\|\psi_2(R)}{\Gamma(q+r+1)} \left[\left|t_2^{q+r} - t_1^{q+r}\right| + 2(t_2 - t_1)^r\right] \\
& + \frac{\left|\nu_2^{q+r-1} - \nu_1^{q+r-1}\right|}{|\Lambda|} \frac{\Gamma(q)}{\Gamma(q+r)} \Bigg[(|\phi(x(\nu_2)) - \phi(x(\nu_1))|) \frac{T^p}{\Gamma(p+1)} \\
& + (|h(x(\nu_2)) - h(x(\nu_1))|) \Bigg] + \frac{1}{|\Lambda|} \left[\frac{\Gamma(q)}{\Gamma(q+r)} \eta^{q+r-1} (|h(x(\nu_2)) - h(x(\nu_1))|) \right. \\
& \left. + (|\phi(x(\nu_2)) - \phi(x(\nu_1))|) \frac{\Gamma(q)}{\Gamma(p+q+r)} T^{p+q+r-1} \right].
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_R$ as $\nu_1 \to \nu_2$, which implies that the set $\mathcal{A}(B_R)$ is equicontinuous. Therefore it follows by the Arzelá-Ascoli theorem that the set $\mathcal{A}(B_R)$ is relative compact. Then the operator $\mathcal{A}$ is compact.

Let $x(t)$ be a solution of problem (1.5). Then, for $t \in J$ and $x \in B_R$, we have

$$\|x\| \leq \Phi_1\|p_1\|\psi_1(\|x\|) + \Phi_2\|p_2\|\psi_2(\|x\|) + \Phi_3|\phi(\|x\|)| + \Phi_4|h(\|x\|)|.$$

Consequently, we have

$$\frac{\|x\|}{\Phi_1\|p_1\|\psi_1(\|x\|) + \Phi_2\|p_2\|\psi_2(\|x\|) + \Phi_3|\phi(\|x\|)| + \Phi_4|h(\|x\|)|} \leq 1.$$

Let us define a subset of B_R as

$$U = \{x \in \mathcal{C} : \|x\| < N\}, \tag{3.7}$$

where N is satisfied the condition (H_4). Note that the operator $\mathcal{A} : \overline{U} \to \mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \theta \mathcal{A}x$ for some $\theta \in (0,1)$. Then, by nonlinear alternative of Leray-Schauder type, Lemma 3.2, we get that the operator $\mathcal{A}$ has a fixed point in $\overline{U}$, which is a solution of the boundary value problem (1.5). This completes the proof. □

Example 3.3. *Consider the following nonlinear sequential Riemann-Liouville and Caputo fractional differential equation with nonlocal integral boundary conditions*

$$\begin{aligned} &{}^{RL}D^{\frac{4}{5}}\left({}^{C}D^{\frac{2}{5}}x(t) - \frac{2e^{-t}\cos^2 t}{1000}\left(\frac{|x|^5}{x^4+1}+1\right)\right) \\ &= \frac{2\sin^4 t}{1000}\left(\frac{x^8}{|x|^7+1}+1\right), \ 0<t<5, \end{aligned} \tag{3.8}$$

$$x(2) = \frac{x(4)}{500}, \qquad I^{\frac{3}{5}}x(5) = \frac{x(3)}{200}.$$

Setting constants $q = 4/5$, $r = 2/5$, $p = 3/5$, $\eta = 2$, $T = 5$, then we get $\Phi_1 = 72.200440$, $\Phi_2 = 129.62057$, $\Phi_3 = 34.389063$ and $\Phi_4 = 2.841029$. Let the following functions

$$\begin{aligned} &g(t,x) = \frac{2e^{-t}\cos^2 t}{1000}\left(\frac{|x|^5}{x^4+1}+1\right), \quad f(t,x) = \frac{2\sin^4 t}{1000}\left(\frac{x^8}{|x|^7+1}+1\right), \\ &\phi(x) = \frac{x}{500}, \qquad h(x) = \frac{x}{200}. \end{aligned}$$

It follows that

$$|g(t,x)| \leq 2\cos^2 t\left(\frac{|x|+1}{1000}\right) \quad \text{and} \quad |f(t,x)| \leq 2\sin^4 t\left(\frac{|x|+1}{1000}\right).$$

Hence, we choose $p_1(t) = 2\cos^2 t$, $\psi_1(|x|) = (|x|+1)/(1000)$, $p_2(t) = 2\sin^4 t$, $\psi_2(|x|) = (|x|+1)/(1000)$. Then there exists a constant $N > 0.97645553$ satisfying inequality

$$\frac{N}{(72.200440)(2)\left(\frac{N+1}{1000}\right) + (129.62057)(2)\left(\frac{N+1}{1000}\right) + (34.389063)\left|\frac{N}{500}\right| + (22.841029)\left|\frac{N}{200}\right|} > 1.$$

Thus, by Theorem 3.3, the boundary value problem (3.8) has at least one solution on $[0,5]$.

The following result can be obtained by substituting $p_1(t), p_2(t) \equiv 1$ and linear functions $\psi_1(|x|) = M_1|x| + K_1$ and $\psi_2(|x|) = M_2|x| + K_2$ in Theorem 3.3.

Corollary 3.1. *Assume that the continuous functions $g, f : J\times\mathbb{R} \to \mathbb{R}$ and two functionals $\phi, h : C(J\times\mathbb{R}) \to \mathbb{R}$ are satisfied*

$$\begin{aligned} &|g(t,x)| \leq M_1|x| + K_1, \quad |f(t,x)| \leq M_2|x| + K_2 \quad \textit{for each } (t,x) \in J\times\mathbb{R}, \\ &|\phi(x)| \leq M_3|x| + K_4, \quad |h(x)| \leq M_4|x| + K_4 \quad \textit{for each } x \in C(J,\mathbb{R}), \end{aligned}$$

where $M_1, M_2, M_3, M_4 > 0$ and $K_1, K_2, K_3, K_4 \geq 0$. If $M_1\Phi_1 + M_2\Phi_2 + M_3\Phi_3 + M_4\Phi_4 < 1$, then boundary value problem (1.5) has at least one solution on $[0,T]$.

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