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GENERALIZED CONVEX FUNCTION AND ASSOCIATED PETROVIĆ'S INEQUALITY

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Abstract. In this paper, Petrović's inequality is generalized for $h-$convex functions, when h is supermultiplicative function. It is noted that the case for $h-$convex functions does not lead the particular cases for $P-$function, Godunova-Levin functions, $s-$Godunova-Levin functions and $s-$convex functions due to the conditions imposed on h. To cover the case, when h is submultiplicative, Petrović's inequality is generalized for $h-$concave functions.

1. Introduction

Let $[c,d]$ be an interval containing $(0,1)$ and $h : [c,d] \to \mathbb{R}$ be a non-negative function. A function $f : [a,b] \to \mathbb{R}$ is said to be an $h-$convex, if f is non-negative and for all $x, y \in [a,b]$, $\alpha \in (0,1)$, one has

$$f(\alpha x + (1-\alpha)y) \le h(\alpha)f(x) + h(1-\alpha)f(y). \tag{1.1}$$

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If above inequality is reversed, then f is said to be h-concave.

The $h-$convex function was introduced by S. Varošanec in [1]. The important thing about these function is that it generalized many other generalization of convex function like $s-$convex functions, Godunova-Levin functions, $s-$Godunova-Levin functions and $P-$functions given in [1–3].

Remark 1.1. *Particular value of h in inequality (1.1) gives us the following results:*

i. $h(\alpha) = \alpha$ *gives the convex functions.*
ii. $h(\alpha) = 1$ *gives the $P-$functions.*
iii. $h(\alpha) = \alpha^s$ *and* $\alpha \in (0, 1)$ *gives the $s-$convex functions of second sense.*
iv. $h(\alpha) = \frac{1}{\alpha}$ *and* $\alpha \in (0, 1)$ *gives the Godunova-Levin functions.*
v. $h(\alpha) = \frac{1}{\alpha^s}$ *and* $\alpha \in (0, 1)$ *gives the $s-$Godunova-Levin functions of second sense.*

In case of $h-$concavity, following results are valid:

vi. $h(\alpha) = 1$ *gives the reverse $P-$functions.*
vii. $h(\alpha) = \alpha^s$ *and* $\alpha \in (0, 1)$ *gives the $s-$concave functions of second sense.*
viii. $h(\alpha) = \frac{1}{\alpha}$ *gives the reverse Godunova-Levin functions.*
ix. $h(\alpha) = \frac{1}{\alpha^s}$ *gives the reverse $s-$Godunova-Levin functions of second sense.*

In [6] (also see [7, p. 154]), M. Petrović proved the following result, which is known as Petrović's inequality in the literature.

Theorem 1.1. *Suppose that $(x_1, ..., x_n)$ and $(p_1, ..., p_n)$ be non-negative n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_i$ for $i = 1, ..., n$ and $\sum_{k=1}^{n} p_k x_k \in [0, a]$. If f is a convex function on $[0, a]$, then the inequality*

$$\sum_{k=1}^{n} p_k f(x_k) \leq f\left(\sum_{k=1}^{n} p_k x_k\right) + \left(\sum_{k=1}^{n} p_k - 1\right) f(0) \tag{1.2}$$

is valid.

In recent years, $h-$Convex functions are considered in literature by many researchers and mathematicians, for example, see [1,3,5,8,9] and references there in. Many authors worked on Petrović's inequality by giving results related to it, for example see [6, 10–12] and it has been generalized for $m-$convex functions by M. Bakula et.al. in [13]. In [14], Petrović's inequality was generalized on coordinates by using the definition of convex functions on coordinates.

In this paper, Petrović's inequality is generalized for $h-$convex functions, in the case, when h is supermultiplicative function. In case, when h is submultiplicative, Petrović's inequality is generalized for $h-$concave functions. Also the results has been generalized on coordinates in the plane.

2. Generalized Petrović's inequality for h-convex function

A function $h : [c, d] \to \mathbb{R}$ is said to be a submultiplicative function if

$$h(xy) \leq h(x)h(y), \tag{2.1}$$

for all $x, y \in [c, d]$. If the above inequality is reversed, then h is said to be supermultiplicative function. If equality holds in the above inequality, then h is said to be multiplicative function.

Here we state important lemma, which is very helpful in proving Petrović's inequality for $h-$convex functions. This lemma is generalization of result given in [7, Page 152].

Lemma 2.1. *Suppose that $(x_1, ..., x_n)$ and $(p_1, ..., p_n)$ be two non-negative n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_i > c$ for $i = 1, ..., n,\ c \in [0, a]$ and $\sum_{k=1}^{n} p_k x_k \in [0, a]$. Also let $h : [0, a] \to \mathbb{R}$ be a positive function and $f : [0, a] \to \mathbb{R}$ be a function. If $\frac{f(x)}{h(x-c)}$ is increasing for $x > c$ on $[0, a]$, then*

$$\sum_{k=1}^{n} p_k f(x_k) \leq \frac{\sum\limits_{k=1}^{n} p_k h(x_k - c)}{h\left(\sum\limits_{k=1}^{n} p_k x_k - c\right)} f\left(\sum_{k=1}^{n} p_k x_k\right). \tag{2.2}$$

Proof. Since $\sum_{k=1}^{n} p_k x_k \geq x_j > c$ for all $j = 1, ..., n$ and $\frac{f(x)}{h(x-c)}$ is increasing on $[0, a]$,

$$\frac{f\left(\sum\limits_{k=1}^{n} p_k x_k\right)}{h\left(\sum\limits_{k=1}^{n} p_k x_k - c\right)} \geq \frac{f(x_j)}{h(x_j - c)},$$

that is,

$$h\left(\sum_{k=1}^{n} p_k x_k - c\right) f(x_j) \leq h(x_j - c) f\left(\sum_{k=1}^{n} p_k x_k\right).$$

Multiplying above inequality by p_j and taking sum for $j = 1, ..., n$, one has

$$h\left(\sum_{k=1}^{n} p_k x_k - c\right) \sum_{j=1}^{n} p_j f(x_j) \leq \sum_{j=1}^{n} p_j h(x_j - c) f\left(\sum_{k=1}^{n} p_k x_k\right).$$

This is equivalent to the required result. □

The following theorem consists of the result for generalized Petrović's inequality for $h-$convex functions.

Theorem 2.1. *Let $(x_1, ..., x_n)$ be non-negative n-tuples and $(p_1, ..., p_n)$ be positive n-tuples such that*

$$\sum_{k=1}^{n} p_k x_k \in [0, a] \text{ and } \sum_{k=1}^{n} p_k x_k \geq x_j \geq c \text{ for } j = 1, ..., n \text{ and } c \in [0, a]. \tag{2.3}$$

Also let $h : [0, a] \to \mathbb{R}^+$ be a supermultiplicative function such that

$$h(\alpha) + h(1 - \alpha) \leq 1, \text{ for all } \alpha \in (0, 1). \tag{2.4}$$

If $f : [0, a] \to \mathbb{R}$ be an $h-$convex function on $[0, a]$, then

$$\sum_{j=1}^{n} p_j f(x_j) \leq \frac{\sum\limits_{j=1}^{n} p_j h(x_j - c)}{h\left(\sum\limits_{k=1}^{n} p_k x_k - c\right)} f\left(\sum_{k=1}^{n} p_k x_k\right) + \left(\sum_{j=1}^{n} p_j - \frac{\sum\limits_{j=1}^{n} p_j h(x_j - c)}{h\left(\sum\limits_{k=1}^{n} p_k x_k - c\right)}\right) f(c). \tag{2.5}$$

Proof. Suppose f is $h-$convex and

$$P_h(x) = \frac{f(x) - f(c)}{h(x - c)}.$$

We take $y > x > c$ and $x = \alpha y + (1 - \alpha)c$, then

$$P_h(x) = \frac{f(\alpha y + (1 - \alpha)c) - f(c)}{h(\alpha y + (1 - \alpha)c - c)} \leq \frac{h(\alpha)f(y) + [h(1 - \alpha) - 1]\, f(c)}{h(\alpha(y - c))}.$$

Using the fact that h is supermultiplicative, one has

$$P_h(x) \leq \frac{h(\alpha)f(y) + [h(1 - \alpha) - 1]\, f(c)}{h(\alpha)h(y - c))}$$

Since $h(1 - \alpha) - 1 \leq -h(\alpha)$, this implies

$$P_h(x) \leq \frac{f(y)}{h(y - c)} - \frac{f(c)}{h(y - c)} = P_h(y).$$

As we have proved if f is $h-$convex, then $\frac{f(x)-f(c)}{h(x-c)}$ is increasing for $x > c$ so substituting $f(x)$ by $f(x) - f(c)$ in Lemma 2.1, one has

$$\sum_{j=1}^{n} p_j \left(f(x_j) - f(c)\right) \leq \frac{\sum\limits_{j=1}^{n} p_j h(x_j - c)}{h\left(\sum\limits_{k=1}^{n} p_k x_k - c\right)} \left[f\left(\sum_{k=1}^{n} p_k x_k\right) - f(c)\right].$$

The above inequality leads to the required result. □

The following theorem is a simple consequence of the above theorem just by taking $c = 0$. It can be considered as Petrović's inequality for $h-$convex functions.

Theorem 2.2. *Let the conditions given in Theorem 2.1 are valid. If $f : [0, a] \to \mathbb{R}$ be an $h-$convex function on $[0, a]$, then*

$$\sum_{j=1}^{n} p_j f(x_j) \leq \frac{\sum\limits_{j=1}^{n} p_j h(x_j)}{h\left(\sum\limits_{k=1}^{n} p_k x_k\right)} f\left(\sum_{k=1}^{n} p_k x_k\right) + \left(\sum_{j=1}^{n} p_j - \frac{\sum\limits_{j=1}^{n} p_j h(x_j)}{h\left(\sum\limits_{k=1}^{n} p_k x_k\right)}\right) f(0). \tag{2.6}$$

From Theorem 2.1, one can get a generalization of Petrović's inequality.

Theorem 2.3. *Suppose that $(x_1, ..., x_n)$ and $(p_1, ..., p_n)$ be two non-negative n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_i > c$ for $i = 1, ..., n$, $c \in [0, a]$ and $\sum_{k=1}^{n} p_k x_k \in [0, a]$. If $f : [0, a] \to \mathbb{R}$ be a convex function on $[0, a]$, then*

$$\sum_{j=1}^{n} p_j f(x_j) \leq \frac{\sum\limits_{j=1}^{n} p_j(x_j - c)}{\left(\sum\limits_{k=1}^{n} p_k x_k - c\right)} f\left(\sum_{k=1}^{n} p_k x_k\right) + \left(\sum_{j=1}^{n} p_j - \frac{\sum\limits_{j=1}^{n} p_j(x_j - c)}{\left(\sum\limits_{k=1}^{n} p_k x_k - c\right)}\right) f(c). \tag{2.7}$$

Proof. Let us consider $h(x) = x$, then clearly h is supermultiplicative and condition (2.4) is valid. Taking this value of h in Theorem 2.1 leads us to required result. □

Remark 2.1. *Taking $h(x) = x$ in Theorem 2.2 or $c = 0$ in Theorem 2.3 leads to Theorem 1.1.*

3. Generalized Petrović's inequality for h-concave function

In the previous section, one can see that the condition on function h given in (2.4) restrict us to give Petrović's type inequalities for particular cases of $h-$convex functions given in Remark 1.1. If we consider reverse inequality in (2.4), then it covers those particular cases but instead of $h-$convex function, we have $h-$concave function.

Lemma 3.1. *Suppose that $(x_1, ..., x_n)$ and $(p_1, ..., p_n)$ be two non-negative n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_i$ for $i = 1, ..., n$ and $\sum_{k=1}^{n} p_k x_k \in [0, a]$. Also let $h : [0, a] \to \mathbb{R}$ be a positive function and $f : [0, a] \to \mathbb{R}$ be a function. If $\frac{f(x)}{h(x-c)}$ is decreasing for $x > c$ on $[0, a]$, then reverse of inequality (2.2) is valid.*

Proof. The proof is similar to the Lemma 2.1. □

In the following theorem, reverse of (2.5) has been concluded. The notable thing is the requirements of submultiplicity and reverse of (2.4) for function h along with h-concavity of the function f.

Theorem 3.1. *Let $(x_1, ..., x_n)$ be non-negative n-tuples and $(p_1, ..., p_n)$ be positive n-tuples and the conditions given in (2.3) are valid. Also let $h : [0, a] \to \mathbb{R}^+$ be a submultiplicative function such that*

$$h(\alpha) + h(1 - \alpha) \geq 1, \ \textit{for all } \alpha \in (0, 1). \tag{3.1}$$

If $f : [0, a] \to \mathbb{R}$ be an $h-$concave function on $[0, a]$, then reverse of (2.5) is valied, that is,

$$\begin{aligned} \sum_{j=1}^{n} p_j f(x_j) \geq & \frac{\sum\limits_{j=1}^{n} p_j h(x_j - c)}{h\left(\sum\limits_{k=1}^{n} p_k x_k - c\right)} f\left(\sum_{k=1}^{n} p_k x_k\right) \\ & + \left(\sum_{j=1}^{n} p_j - \frac{\sum\limits_{j=1}^{n} p_j h(x_j - c)}{h\left(\sum\limits_{k=1}^{n} p_k x_k - c\right)}\right) f(c). \end{aligned} \tag{3.2}$$

Proof. First we will prove that $\frac{f(x)-f(c)}{h(x-c)}$ is decreasing for $x > c$ when f is $h-$concave function. For this purpose consider

$$P_h(x) = \frac{f(x) - f(c)}{h(x-c)}.$$

We take $y > x > c$ and $x = \alpha y + (1-\alpha)c$, then

$$P_h(x) = \frac{f(\alpha y + (1-\alpha)c) - f(c)}{h(\alpha y + (1-\alpha)c - c)} \geq \frac{h(\alpha)f(y) + [h(1-\alpha) - 1]\, f(c)}{h(\alpha(y-c))}.$$

Using the fact that h is submultiplicative, so we have

$$P_h(x) \geq \frac{h(\alpha)f(y) + [h(1-\alpha) - 1]\, f(c)}{h(\alpha)h(y-c))}$$

Since $h(1-\alpha) - 1 \geq -h(\alpha)$, this implies

$$P_h(x) \geq \frac{f(y)}{h(y-c)} - \frac{f(c)}{h(y-c)} = P_h(y).$$

This proves that $\frac{f(x)-f(c)}{h(x-c)}$ is decreasing in $[0,a]$ for $x > c$. Now substituting $f(x)$ by $f(x) - f(c)$ in Lemma 3.1 give us the required result.

□

In the following theorem, we give Petrović's inequality for $h-$concave functions. It is simple consequence of the previous theorem by just taking $c = 0$.

Theorem 3.2. *Let the conditions given in Theorem 3.1 are valid. If $f : [0,a] \to \mathbb{R}$ be an $h-$concave function on $[0,a]$, then the reverse of (2.6) is valid.*

In the following theorem, we give the generalized Petrović's inequality for concave functions.

Theorem 3.3. *Suppose that $(x_1, ..., x_n)$ and $(p_1, ..., p_n)$ be two non-negative n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_i > c$ for $i = 1, ..., n$, $c \in [0,a]$ and $\sum_{k=1}^{n} p_k x_k \in [0,a]$. If $f : [0,a] \to \mathbb{R}$ be concave function on $[0,a]$, then the reverse of (2.7) is valid.*

Proof. If we take $h(x) = x$ in (3.2), we get the required result. □

Remark 3.1. *By Taking $h(x) = x$ in Theorem 3.2 or $c = 0$ in Theorem 3.3, one can get the reverse of inequality (1.2) in the case when f is concave function.*

In the following theorem, we give the Petrović's type inequality for reverse $P-$functions.

Theorem 3.4. *Suppose that $(x_1, ..., x_n)$ and $(p_1, ..., p_n)$ be two non-negative n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_i$ for $i = 1, ..., n$ and $\sum_{k=1}^{n} p_k x_k \in [0,a]$. If $f : [0,a] \to \mathbb{R}$ be a reverse $P-$function on $[0,a]$, then*

$$\sum_{j=1}^{n} p_j f(x_j) \geq \sum_{j=1}^{n} p_j f\left(\sum_{k=1}^{n} p_k x_k\right) \tag{3.3}$$

Proof. If we take $h(x) = 1$, then it fulfils the condition of Theorem 3.1 and follows the required result. □

In the following theorem, we give the generalized Petrović's type inequality for reverse Godunova-Levin functions.

Theorem 3.5. *Suppose that $(x_1, ..., x_n)$ and $(p_1, ..., p_n)$ be two non-negative n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_i > c$ for $i = 1, ..., n$, $c \in [0, a]$ and $\sum_{k=1}^{n} p_k x_k \in [0, a]$. If $f : [0, a] \to \mathbb{R}$ be a reverse Godunova-Levin function on $[0, a]$, then*

$$\begin{aligned} \sum_{j=1}^{n} p_j f(x_j) \geq & \left(\sum_{k=1}^{n} p_k x_k - c\right) \sum_{j=1}^{n} \frac{p_j}{x_j - c} f\left(\sum_{k=1}^{n} p_k x_k\right) \\ & + \left(\sum_{j=1}^{n} p_j - \left(\sum_{k=1}^{n} p_k x_k - c\right) \sum_{j=1}^{n} \frac{p_j}{x_j - c}\right) f(c). \end{aligned} \tag{3.4}$$

Proof. Consider $h(x) = \frac{1}{x}$, then

$$h(\alpha) + h(1 - \alpha) = \frac{1}{\alpha} + \frac{1}{1 - \alpha} > 1 \text{ for all } \alpha \in (0, 1).$$

Using above value of h in Theorem 3.1 gives the required result. □

The following theorem is a simple consequence of the previous theorem. It is worth stating as Petrović's type inequality for reverse Godunova-Levin functions.

Theorem 3.6. *Let the conditions given in Theorem 3.1 are valid. If $f : [0, a] \to \mathbb{R}$ be a reverse Godunova-Levin function on $[0, a]$, then*

$$\begin{aligned} \sum_{j=1}^{n} p_j f(x_j) \geq & \left(\sum_{k=1}^{n} p_k x_k\right) \sum_{j=1}^{n} \frac{p_j}{x_j} f\left(\sum_{k=1}^{n} p_k x_k\right) \\ & + \left(\sum_{j=1}^{n} p_j - \left(\sum_{k=1}^{n} p_k x_k\right) \sum_{j=1}^{n} \frac{p_j}{x_j}\right) f(0). \end{aligned} \tag{3.5}$$

Proof. Putting $c = 0$ in Theorem 3.6 leads to required result. □

Before giving two important theorems, let us consider

$$H(h) = h(\alpha) + h(1 - \alpha) - 1, \alpha \in (0, 1),$$

then for different values of h, that is, for α^s and $\frac{1}{\alpha^s}$, we take

$$g_1(\alpha) := H(\alpha^s) = \alpha^s + (1 - \alpha)^s - 1$$

and

$$g_2(\alpha) := H\left(\frac{1}{\alpha^s}\right) = \frac{1}{\alpha^s} + \frac{1}{(1 - \alpha)^s} - 1, \text{ where } s \in (0, 1).$$

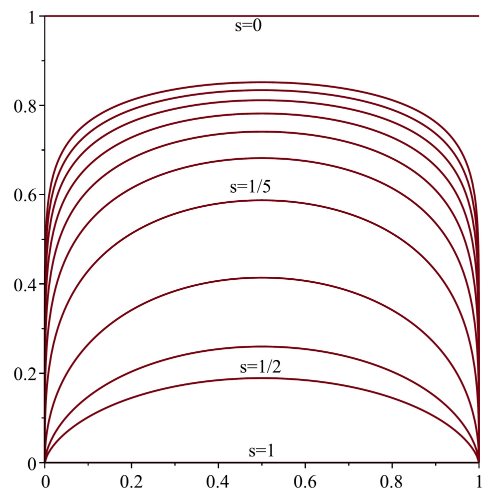

FIGURE 1. Graph of g_1 at different value of s. One can see that g_1 is positive for $\alpha \in (0, 1)$ and at different value of s. The line at bottom is at $s = 1$, the next curve is for $s = \frac{1}{2}$ and so on.

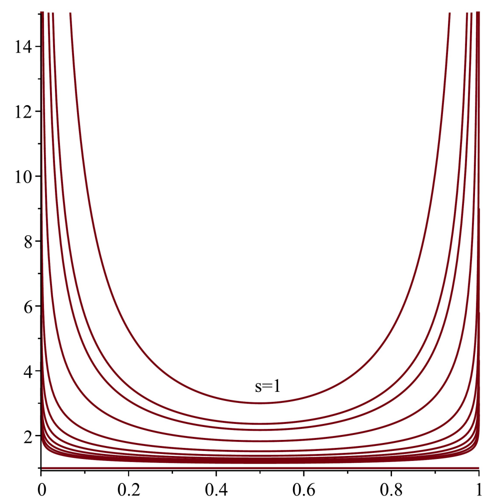

FIGURE 2. Graph of g_2 at different value of s. One can see that g_2 is positive for $\alpha \in (0, 1)$ and at different value of s. The curve at top is at $s = 1$, the below one is for $s = \frac{1}{2}$ and so on.

From Figures 1 and 2, one can see that g_1 and g_2 are positive, therefore $h(\alpha) = \alpha^s$ and $h(\alpha) = \frac{1}{\alpha^s}$ for $\alpha, s \in (0, 1)$ satisfied the conditions of Theorem 3.1, but these functions does not satisfy the conditions of Theorem 2.1. Hence the above two particular values of h in Theorem 3.1 leads us the following two theorems.

Theorem 3.7. *Suppose that* $(x_1, ..., x_n)$ *and* $(p_1, ..., p_n)$ *be two non-negative n-tuples such that* $\sum_{k=1}^{n} p_k x_k \geq x_i > c$ *for* $i = 1, ..., n,$ $c \in [0, a]$ *and* $\sum_{k=1}^{n} p_k x_k \in [0, a]$*. If* $f : [0, a] \to \mathbb{R}$ *be a reverse s-Godunova-Levin* *function on* $[0,a]$. *Then*

$$\sum_{j=1}^{n} p_j f(x_j) \geq \left(\sum_{k=1}^{n} p_k x_k - c\right)^{s} \sum_{j=1}^{n} \frac{p_j}{(x_j - c)^s} f\left(\sum_{k=1}^{n} p_k x_k\right) + \left(\sum_{j=1}^{n} p_j - \left(\sum_{k=1}^{n} p_k x_k - c\right)^{s} \sum_{j=1}^{n} \frac{p_j}{(x_j - c)^s}\right) f(c). \tag{3.6}$$

Theorem 3.8. *Suppose that* $(x_1, ..., x_n)$ *and* $(p_1, ..., p_n)$ *be two non-negative n-tuples such that* $\sum_{k=1}^{n} p_k x_k \geq x_i > c$ *for* $i = 1, ..., n$, $c \in [0,a]$ *and* $\sum_{k=1}^{n} p_k x_k \in [0,a]$. *If* $f : [0,a] \to \mathbb{R}$ *be a s-concave function on* $[0,a]$, *then*

$$\sum_{j=1}^{n} p_j f(x_j) \geq \left(\sum_{k=1}^{n} p_k x_k - c\right)^{s} \sum_{j=1}^{n} \frac{p_j}{(x_j - c)^s} f\left(\sum_{k=1}^{n} p_k x_k\right) + \left(\sum_{j=1}^{n} p_j - \left(\sum_{k=1}^{n} p_k x_k - c\right)^{s} \sum_{j=1}^{n} \frac{p_j}{(x_j - c)^s}\right) f(c). \tag{3.7}$$

If we take $c = 0$, then we have following Petrović's type inequalities.

Theorem 3.9. *Let the conditions given in Theorem 1.1 are valid. If* $f : [0,a] \to \mathbb{R}$ *be a reverse s-Godunova-Levin function on* $[0,a]$, *then*

$$\sum_{j=1}^{n} p_j f(x_j) \geq \left(\sum_{k=1}^{n} p_k x_k\right)^{s} \sum_{j=1}^{n} \frac{p_j}{(x_j)^s} f\left(\sum_{k=1}^{n} p_k x_k\right) + \left(\sum_{j=1}^{n} p_j - \left(\sum_{k=1}^{n} p_k x_k\right)^{s} \sum_{j=1}^{n} \frac{p_j}{(x_j)^s}\right) f(c). \tag{3.8}$$

Theorem 3.10. *Let the conditions given in Theorem 1.1 are valid. If* $f : [0,a] \to \mathbb{R}$ *be a s-concave function on* $[0,a]$, *then*

$$\sum_{j=1}^{n} p_j f(x_j) \geq \left(\sum_{k=1}^{n} p_k x_k\right)^{s} \sum_{j=1}^{n} \frac{p_j}{(x_j)^s} f\left(\sum_{k=1}^{n} p_k x_k\right) + \left(\sum_{j=1}^{n} p_j - \left(\sum_{k=1}^{n} p_k x_k\right)^{s} \sum_{j=1}^{n} \frac{p_j}{(x_j)^s}\right) f(0). \tag{3.9}$$

Proof. Put $c = 0$ in Theorem 3.8, one has the required result. □

4. Concluding remarks

This paper generalized the Petrović's inequality for $h-$convex ($h-$concave) functions. It has been noted that under certain conditions on h, Theorem 2.1 provide the generalization of Petrović's inequality for $h-$convex functions, but this generally does not leads to Godunova-Levin functions, $P-$functions, $s-$Godunova-Levin functions and $s-$convex functions. Theorem 3.1 give the Petrović's inequality for $h-$concave under certain condition on h. Interesting, it give reverse of those particular cases for which Theorem 2.1 fails.

It is an open problem to find such generalization of Petrović's inequality for $h-$convex functions with some suitable conditions on h, which lead to all particular cases of $h-$convex functions specially mentioned in Remark 1.1.

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References

[1] S. Varošanec, On $h-$Convexity, J. Math. Anal. Appl. 326(1) (2007), 303–311.

[2] S. S. Dragomir, J. E. Pečarić and L. E. Persson , Some Inequalities of Hadamard inequality, Soochow J. Math. 21(3) (1995), 335–341.

[3] A. Házy, Bernstein-Doetsch Type Results for h-Convex Functions, Math. Inequal. Appl. 14(3) (2011), 499–508.

[4] S. S Dragomir, On Hadamards Inequality for Convex Functions on the Co-ordinates in a Rectangle from the Plane, Taiwanese J. Math. 5(4) (2001), 775–788.

[5] M. Alomari, M. A. Latif, On Hadmard-Type Inequalities for h-Convex Functions on the Co-ordinates, Int. J. Math. Anal. 3(33) (2009), 1645–1656.

[6] M. Petrović's, Sur Une Fonctionnelle, Publ. Math. Univ. Belgrade 1 (1932), 146–149.

[7] J. E. Pečarić, F. Proschan and Y. L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press, New York, 1991.

[8] M. Bombardelli, S. Varoanec, Properties of h-Convex Functions Related to the HermiteHadamardFejr Inequalities, Comput. Math. Appl. 58(9) (2009), 1869–1877.

[9] A. Olbryś, On Separation by h-Convex Function, Tatra Mt. Math. Publ. 62(1) (2015), 105–111.

[10] S. Butt, J. Pečarić and A. U. Rehman, Exponential Convexity of Petrović and Related Functional, J. Inequal. Appl. 2011(1) (2011), 16 pp.

[11] J. Pečarić and V. Čuljak, Inequality of Petrović and Giaccardi for Convex Function of Higher Order, Southeast Asian Bull. Math. 26(1) (2003), 57–61.

[12] J. E. Pečarić, On the Petrović Inequality for Convex Functions, Glasnik Matematicki 18(38) (1983), 77–85.

[13] M. K. Bakula, J. Pečarić and M. Ribičić, Companion Inequalities to Jensen's Inequality for m-Convex and (α, m)-Convex Functions, J. Inequal. Pure Appl. Math. 7(5) (2006), 32 pp.

[14] A. U. Rehman, M. Mudessir, H. T. Fazal and G. Farid, Petrović's Inequality on Coordinates and Related Results, Cogent Math. 3(1) (2016), 11 pp.

[15] J. Pečarić and J. Peric, Improvements of the Giaccardi and the Petrovic Inequality and Related Stolarsky Type Means, An. Univ. Craiova, Ser. Mat. Inf. 39(1) (2012), 65–75.