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FIXED POINT THEOREM OF ĆIRIĆ-PATA TYPE

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Abstract. In this article, we proved a fixed point theorem of Ćirić-Pata type in metric space. This result extends several results in the existing literature. Moreover, an example is given in the support of our result. In particular, the main result provides a complete solution to an open problem raised by Kadelburg and Radenović (J. Egypt. Math. Soc. 24 (2016) 77-82).

1. Introduction

Throughout this paper, (X, d) will be a complete metric space. Fix an arbitrary point $x_0 \in X$ and denote $\|x\| = d(x, x_0)$, for each $x \in X$. Also, $\psi : [0,1] \to [0,\infty)$ is an increasing function, continuous at zero, with $\psi(0) = 0$. Given a function $f : X \to X$.

In 2011, Pata [1] obtained the following result which is a generalization of the classical Banach contraction principle.

Theorem 1.1. *[1] Let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ be fixed constants. If the inequality*

$$d(fx, fy) \leq (1-\varepsilon)d(x,y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x\| + \|y\|]^{\beta} \tag{1.1}$$

is satisfied for every $\varepsilon \in [0,1]$ and all $x, y \in X$, then f has a unique fixed point $z \in X$.

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Afterward many pata type fixed point theorems have been established by various authors; see ([2], [3], [4], [5], [6], [7], [8], [9]). Particularly, Kadelburg and Radenović [7] proved some fixed point theorems of Pata type and raised the following open question on Pata-version of Ćirić contraction principle (see [10]).

Problem 1.1. *[7] Prove or disprove the following. Let $f : X \to X$ and let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ be fixed constants. If the inequality*

$$\begin{aligned} d(fx, fy) \leq (1-\varepsilon)\max\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\} \\ + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x\| + \|y\|]^{\beta} \end{aligned} \tag{1.2}$$

is satisfied for every $\varepsilon \in [0,1]$ and all $x, y \in X$, then f has a unique fixed point $z \in X$. Furthermore, the sequence $\{f^n x_0\}$ converges to z.

Very recently, Jacobe et al. give the following result.

Theorem 1.2. *[5] Let $f : X \to X$ and let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ be fixed constants. If the inequality*

$$\begin{aligned} d(fx, fy) \leq (1-\varepsilon)\max\left\{d(x,y), \frac{d(x,fx)+d(y,fy)}{2}, \frac{d(x,fy)+d(y,fx)}{2}\right\} \\ + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x\| + \|y\| + \|fx\| + \|fy\|]^{\beta} \end{aligned} \tag{1.3}$$

is satisfied for every $\varepsilon \in [0,1]$ and all $x, y \in X$, then f has a unique fixed point in X.

In this paper, we give a fixed point theorem of Ćirić-Pata type in metric space. This theorem extends the main results in ([1], [5], [7]) and provides a complete solution to the above Problem 1.1. Finally, an example is given to illustrate the superiority of the main results.

2. Main results

Our result of this paper are stated as follows.

Theorem 2.1. *Let $\Lambda \geq 0$, $\alpha \geq 1$ be fixed constants. For $x, y \in X$, we denote*

$$M(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\}.$$

If the inequality

$$d(fx, fy) \leq (1-\varepsilon)M(x,y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x\| + \|y\| + \|fx\| + \|fy\|]^{\alpha} \tag{2.1}$$

is satisfied for every $\varepsilon \in [0,1]$ and all $x, y \in X$, then f has a unique fixed point $z \in X$. Furthermore, the sequence $\{f^n x_0\}$ converges to z.

Proof. Starting from x_0, construct a sequence $\{x_n\}$ such that $x_n = fx_{n-1} = f^n x_0$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then x_{n_0} is a fixed point of f. Thus, we always assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

We prove that $x_n \neq x_m$ for all $m, n \in \mathbb{N}$ and $n \neq m$. Assume that there exist $n_0, m_0 \in \mathbb{N}$ such that $n_0 < m_0$ and $x_{n_0} = x_{m_0}$. Denote $A = \max\{d(x_i, x_j) : n_0 \leq i < j \leq m_0\}$ and $B = \max\{\|x_i\| : n_0 \leq i \leq m_0 + 1\}$. It is obvious that $A = \max\{d(x_i, x_j) : n_0 + 1 \leq i < j \leq m_0\}$ and $A > 0$. For each $i, j \in \mathbb{N}$ such that $n_0 + 1 \leq i < j \leq m_0$, we have

$$\begin{aligned} d(x_i, x_j) &\leq (1-\varepsilon)M(x_{i-1}, y_{j-1}) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|x_{i-1}\| + \|x_{j-1}\| + \|x_i\| + \|x_j\|]^\alpha \\ &\leq (1-\varepsilon)A + \Lambda\varepsilon^\alpha\psi(\varepsilon)(1+4B)^\alpha. \end{aligned}$$

It follows that

$$A \leq (1-\varepsilon)A + \Lambda\varepsilon^\alpha\psi(\varepsilon)(1+4B)^\alpha$$

and

$$A \leq \Lambda\varepsilon^{\alpha-1}\psi(\varepsilon)(1+4B)^\alpha.$$

Letting $\varepsilon \to 0$, we can see $A \leq 0$. This is a contradiction with $A > 0$.

Denote $D_n = \max\{d(x_i, x_j) : 0 \leq i < j \leq n\}$ and $\delta_n = \sup\{d(x_i, x_j) : n \leq i < j\}$. In order to prove $\{x_n\}$ is a Cauchy sequence, we divide into the following three steps.

Step 1. We show that $d(fx, fy) < M(x, y)$ for all $x, y \in X$ and $x \neq y$. Let $\varepsilon = 0$ in (2.1), we have $d(fx, fy) \leq M(x, y)$ for all $x, y \in X$. Assume that there exist $x_0, y_0 \in X$ and $x_0 \neq y_0$ such that $d(fx_0, fy_0) = M(x_0, y_0)$. Using (2.1), we get

$$M(x_0, y_0) = d(fx_0, fy_0) \leq (1-\varepsilon)M(x_0, y_0) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|x_0\| + \|y_0\| + \|fx_0\| + \|fy_0\|]^\alpha.$$

It follows that

$$M(x_0, y_0) \leq \Lambda\varepsilon^{\alpha-1}\psi(\varepsilon)[1 + \|x_0\| + \|y_0\| + \|fx_0\| + \|fy_0\|]^\alpha.$$

Passing to the limit as $\varepsilon \to 0$, we see $M(x_0, y_0) \leq 0$, a contradiction.

Step 2. We prove that $\{D_n\}$ is bounded. By step 1, we see that

$$d(x_i, x_j) = d(fx_{i-1}, fx_{j-1}) < M(x_{i-1}, x_{j-1}) \leq D_n,$$

for all $i, j \in \mathbb{N}$ such that $0 < i < j \leq n$. Thus there exists $\ell_n \in \mathbb{N}$ such that $1 \leq \ell_n \leq n$ and $D_n = d(x_0, x_{\ell_n})$. Using (2.1), we have

$$\begin{aligned} D_n &= d(x_0, x_{\ell_n}) \\ &\leq d(x_0, x_1) + d(x_1, x_{\ell_n}) \\ &\leq d(x_0, x_1) + (1-\varepsilon)M(x_0, x_{\ell_n - 1}) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|x_0\| + \|x_{\ell_n-1}\| + \|x_1\| + \|x_{\ell_n}\|]^\alpha \\ &\leq (1-\varepsilon)D_n + \Lambda\varepsilon^\alpha\psi(\varepsilon)(1+3D_n)^\alpha + d(x_0, x_1). \end{aligned}$$

This implies that

$$\varepsilon D_n \leq \Lambda \varepsilon^\alpha \psi(\varepsilon)(1+3D_n)^\alpha + d(x_0, x_1).$$

Suppose that $\{D_n\}$ is unbounded. Then there exists a subsequence $\{D_{n_k}\}$ with $\{D_n\}$ such that $D_{n_k} \to \infty$ $(k \to \infty)$ and $D_{n_k} \geq 1 + d(x_0, x_1)$. Let $\varepsilon = \varepsilon_k = \frac{1+d(x_0,x_1)}{D_{n_k}}$. Then we get

$$\frac{1+d(x_0,x_1)}{D_{n_k}} \cdot D_{n_k} \leq \Lambda[\frac{1+d(x_0,x_1)}{D_{n_k}}]^\alpha \psi(\varepsilon_k)(1+3D_{n_k})^\alpha + d(x_0, x_1)$$

and

$$1 \leq \Lambda(\frac{1}{D_{n_k}} + 3)^\alpha \psi(\varepsilon_k)[1 + d(x_0, x_1)]^\alpha.$$

Letting $k \to \infty$, we have $\varepsilon_k \to 0$ and

$$\Lambda(\frac{1}{D_{n_k}} + 3)^\alpha \psi(\varepsilon_k)[1 + d(x_0, x_1)]^\alpha \to 0.$$

This is a contradiction. Thus $\{D_n\}$ is bounded and there exists a constant $M > 0$ such that $D_n \leq M$.

Step 3. We show that $\delta_n \to 0$. Observe that

$$d(x_i, x_j) = d(fx_{i-1}, x_{j-1}) < M(x_{i-1}, x_{j-1}) \leq \delta_n$$

for every $i, j \in \mathbb{N}$ with $n + 1 \leq i < j$. Thus we get $\delta_{n+1} \leq \delta_n \leq \cdots \leq \delta_0 \leq M$. It is easy to see that $\{\delta_n\}$ is decreasing and bounded sequence. It follows that $\lim\limits_{n\to\infty} \delta_n = \delta$ for some $\delta \geq 0$. Assume that $\delta > 0$. From (2.1), it holds for each $i, j \in \mathbb{N}$ with $n + 1 \leq i < j$,

$$d(x_i, x_j) \leq (1-\varepsilon)M(x_{i-1}, x_{j-1}) + \Lambda\varepsilon^\alpha\psi(\varepsilon)(1 + 4M)^\alpha.$$

This implies that

$$\delta_{n+1} \leq (1-\varepsilon)\delta_n + \Lambda\varepsilon_\alpha\psi(\varepsilon)(1 + 4M)^\alpha. \tag{2.2}$$

Letting $n \to \infty$ in (2.2), we get

$$\delta \leq (1-\varepsilon)\delta + \Lambda\varepsilon^\alpha\psi(\varepsilon)(1 + 4M)^\alpha$$

and

$$\delta \leq \Lambda\varepsilon^{\alpha-1}\psi(\varepsilon)(1 + 4M)^\alpha.$$

From

$$\Lambda\varepsilon^{\alpha-1}\psi(\varepsilon)(1 + 4M)^\alpha \to 0 \ (\varepsilon \to 0),$$

we see $\delta \leq 0$, a contradiction. For each $p \in \mathbb{N}$, we get $d(x_n, x_{n+p}) \leq \delta_n \to 0$ $(n \to \infty)$. Hence, $\{x_n\}$ is Cauchy sequence. Since X is complete, there exists $z \in X$ such that $x_n \to z$ $(n \to \infty)$.

Now, we show that $fz = z$. Using (2.1), we get

$$\begin{aligned} d(fz, x_{n+1}) \leq (1-\varepsilon)\max\{&d(z, x_n), d(z, fz), d(x_n, x_{n+1}), d(z, x_{n+1}), d(x_n, fz)\} \\ &+ \Lambda\varepsilon^\alpha\psi(\varepsilon)(1 + 4M)^\alpha. \end{aligned}$$

By taking limits on both sides when $n \to \infty$, we obtain

$$d(fz, z) \le (1-\varepsilon)d(fz, z) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)(1+4M)^{\alpha}.$$

Then

$$d(fz, z) \le \Lambda\varepsilon^{\alpha-1}\psi(\varepsilon)(1+4M)^{\alpha} \to 0 \ (\varepsilon \to 0).$$

This implies that $d(fz, z) = 0$ and $fz = z$.

Finally, we prove the uniqueness of z. If $fu = u$, $fv = v$ for any two fixed $u, v \in X$, then we can write (2.1) in the form

$$\begin{aligned} d(u, v) &= d(fu, fv) \\ &\le (1-\varepsilon)\max\{d(u, v), d(u, fu), d(v, fv), d(u, fv), d(v, fu)\} \\ &\quad + \Lambda\epsilon^{\alpha}\psi(\varepsilon)[1 + \|u\| + \|v\| + \|fu\| + \|fv\|]^{\alpha} \\ &\le (1-\varepsilon)d(u, v) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + 2\|u\| + 2\|v\|]^{\alpha}. \end{aligned}$$

Therefore

$$d(u, v) \le \Lambda\varepsilon^{\alpha-1}\psi(\varepsilon)[1 + 2\|u\| + 2\|v\|]^{\alpha} \to 0 \ (\varepsilon \to 0),$$

which implies that $d(u, v) = 0$ and $u = v$. Hence, f has a unique fixed point $z \in X$. □

Remark 2.1. *It is easy to see that the condition* (2.1) *is weaker than the condition* (1.2)*. Hence, Theorem 2.1 provides a solution to Problem 1.1.*

From Theorem 2.1 we get the following Corollaries.

Corollary 2.1. *Let $f : X \to X$ and Let $\Lambda \ge 0$, $\alpha \ge 1$ be fixed constants. If the inequality*

$$\begin{aligned} d(fx, fy) &\le (1-\varepsilon)d(x, y) \\ &\quad + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x\| + \|y\| + \|fx\| + \|fy\|]^{\alpha} \end{aligned} \tag{2.3}$$

is satisfied for every $\varepsilon \in [0, 1]$ and all $x, y \in X$, then f has a unique fixed point $z \in X$. Furthermore, the sequence $\{f^n x_0\}$ converges to z.

Remark 2.2. *It is easy to see that the condition* (2.3) *is weaker than the condition* (1.1)*. Thus Corollary 2.1 is an extension of Theorem 1.1.*

Corollary 2.2. *Let $f : X \to X$ and Let $\Lambda \ge 0$, $\alpha \ge 1$ be fixed constants. If the inequality*

$$\begin{aligned} d(fx, fy) &\le \frac{1-\varepsilon}{2}(d(x, fy) + d(y, fx)) \\ &\quad + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x\| + \|y\| + \|fx\| + \|fy\|]^{\alpha} \end{aligned} \tag{2.4}$$

is satisfied for every $\varepsilon \in [0, 1]$ and all $x, y \in X$, then f has a unique fixed point $z \in X$. Furthermore, the sequence $\{f^n x_0\}$ converges to z.

Remark 2.3. *It is easy to see that the condition 2.4 is weaker than the condition 2.1 in [7]. Thus Corollary 2.2 is an extension of Theorem 2.1 in [7].*

Corollary 2.3. *Let $f : X \to X$ and Let $\Lambda \geq 0$, $\alpha \geq 1$ be fixed constants. If the inequality*

$$\begin{aligned} d(fx, fy) \leq (1-\varepsilon)\max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\} \\ + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\|x\|+\|y\|+\|fx\|+\|fy\|]^{\beta} \end{aligned} \tag{2.5}$$

is satisfied for every $\varepsilon \in [0,1]$ and all $x, y \in X$, then f has a unique fixed point $z \in X$. Furthermore, the sequence $\{f^n x_0\}$ converges to z.

Remark 2.4. *It is easy to see that the condition (2.5) is weaker than the condition (1.3). Thus Corollary 2.3 is an extension of Theorem 1.2.*

The following is an example which can apply to Theorem 2.1 but not Corollary 2.3 or Theorem 1.2.

Example 2.1. *Let $X = \{0,1,2,4,8,9\} \bigcup \{\frac{1}{2^n} : n = 1, 2, \cdots\}$ with the usual metric. It is easily to check that X is a complete metric space. Define $f : X \to X$ by*

$$fx = \begin{cases} 8 & x = 9, \\ \frac{1}{2}x & others\ . \end{cases}$$

Then mapping f satisfies the condition (2.1) with $\Lambda = \frac{8}{9}$, $\beta = 1$ and $\psi(\varepsilon) = \varepsilon^{\frac{1}{8}}$ (for all $\varepsilon \in [0,1]$). Moreover, it is worth mentioning that

$$\frac{8}{9} - 1 + \varepsilon \leq \frac{8}{9}[1 + \frac{9}{8}(\varepsilon - 1)] \leq \frac{8}{9}\varepsilon^{\frac{9}{8}} \leq \frac{8}{9}\varepsilon\varepsilon^{\frac{1}{8}}.$$

Thus we have the following two cases.

(1) If $x = 9$ and $y \neq 9$, then

$$\begin{aligned} d(fx, fy) &= d(8, \frac{1}{2}y) \ = \ 8 - \frac{1}{2}y \\ &\leq \frac{8}{9}(9 - \frac{1}{2}y) \ = \ \frac{8}{9}d(9, fy) \\ &\leq \frac{8}{9}M(9, y) \\ &= (1-\varepsilon)M(9,y) + (\frac{8}{9} - 1 + \varepsilon)M(9,y) \\ &\leq (1-\varepsilon)M(9,y) + (\frac{8}{9} - 1 + \varepsilon)[1 + \|9\| + \|y\| + \|f9\| + \|fy\|] \\ &\leq (1-\varepsilon)M(9,y) + \frac{8}{9}\varepsilon\varepsilon^{\frac{1}{8}}[1 + \|9\| + \|y\| + \|f9\| + \|fy\|] \end{aligned}$$

(2) If $x \neq 9$ and $y \neq 9$, then

$$\begin{aligned} d(fx, fy) &= \frac{1}{2}(x-y) \leq \frac{8}{9}(x-y) \\ &= \frac{8}{9}d(x,y) \leq \frac{8}{9}M(x,y) \\ &= (1-\varepsilon)M(x,y) + (\frac{8}{9} - 1 + \varepsilon)M(x,y) \\ &\leq (1-\varepsilon)M(x,y) + (\frac{8}{9} - 1 + \varepsilon)[1 + \|x\| + \|y\| + \|fx\| + \|fy\|] \\ &\leq (1-\varepsilon)M(x,y) + \frac{8}{9}\varepsilon\varepsilon^{\frac{1}{8}}[1 + \|x\| + \|y\| + \|fx\| + \|fy\|] \end{aligned}$$

Hence, f satisfies all conditions of Theorem 2.1. This leads to f has a unique fixed point. Indeed, 0 is the fixed point for the mapping f.

Now, let $\varepsilon = 0$, $x = 9$ and $y = 4$, we have

$$\begin{aligned} d(f9, f4) = 6 > \frac{11}{2} &= \max\{5, 1, 2, \frac{11}{2}\} \\ &= \max\{d(9,4), d(9, f9), d(4, f4), \frac{d(9, f4) + d(4, f9)}{2}\} \end{aligned}$$

It is easy to see that f does not satisfy the condition (2.5) of Corollary 2.3. Also, f does not satisfy the condition (1.3) of Theorem 1.2.

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