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A NOTE ON GENERALIZED INDEXED NORLUND SUMMABILITY FACTOR OF AN INFINITE SERIES

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ABSTRACT. In the present article, we have established a result on generalized indexed absolute Norlund summability factor by generalizing results of Mishra and Srivastava on indexed absolute Cesaro summabilty factors and Padhy et.al. on the absolute indexed Norlund summability.

1. Introduction

In 1930, J.M.Whittaker [18] was the 1st to establish a result on the absolute summability of Fourier series and in 1932, M. Fekete [6] established a result on generalized indexed summability. Later on the researchers like Daniel [4] in 1964, Das [5] in 1966, Siya Ram [15] in 1969, Mazhar [11] in 1971, Mishra and Srivastava [13] in 1984, Sulaiman [16] in 2011 etc. have established results on indexed summability factors of an infinite series.

Let $\sum a_n$ be a given infinite series with sequence of partial sums $\{s_n\}$. Let ${t_n}^{\alpha}$ be the nth (C, α) mean (with order $\alpha > -1$) of the sequence $\{s_n\}$ and is given by

$$t_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} s_k,\ n \in N, where\ A_n^{\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)},$$

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then the series $\sum a_n$is said to be summable $|C,\alpha|_k, k \geq 1$, [7] if

$$\sum_{n=1}^{\infty}(n)^{k-1}|t_n^{\alpha} - t_{n-1}^{\alpha}|^k < \infty.$$

Let t_n be the nth $(C,1)$- mean of the sequence $\{s_n\}$ and is given by

$$t_n = \frac{1}{n+1}\sum_{k=0}^{n}s_k,$$

then the series $\sum a_n$is said to be summable $|C,1|_k, k \geq 1$, [3] if

$$\sum_{n=1}^{\infty}(n)^{k-1}|t_n - t_{n-1}|^k < \infty. \tag{1.1}$$

Suppose $\{q_n\}$ be a sequence of real numbers with $q_n > 0$, such that

$$Q_n = \sum_{\nu=0}^{n}q_\nu \to \infty,\ as\ n \to \infty\,(Q_{-i} = q_{-i} = 0, i \geq 1) \tag{1.2}$$

The sequence to sequence transformation

$$T_n = \frac{1}{Q_n}\sum_{\nu=0}^{n}q_{n-\nu}s_\nu \tag{1.3}$$

defines the sequence $\{T_n\}$ of the (N,q_n)- means of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{q_n\}$.

The series $\sum a_n$ is said to be summable $|N,q_n|$ if the sequence $\{T_n\}$ is of bonded variation i.e; $\sum|T_n - T_{n-1}|$ is convergent.

The series $\sum a_n$ is said to be summable $|N,q_n|_k, k \geq 1$,if (see [8])

$$\sum_{n=1}^{\infty}\left(\frac{Q_n}{q_n}\right)^{k-1}|T_n - T_{n-1}|^k < \infty \tag{1.4}$$

Clearly,$|N,q_n|_k$-summabiity is same as $|C,1|$-summabiity,when $q_n = 1$, for all values of n. Further any sequence $\{\alpha_n\}$ of positive numbers the series $\sum a_n$ is said to be summable $|N,q_n,\alpha_n|_k, k \geq 1$ if

$$\sum_{n=1}^{\infty}(\alpha_n)^{k-1}|T_n - T_{n-1}|^k < \infty \tag{1.5}$$

and is said to be summable $|N,q_n,\alpha_n;\delta|_k, k \geq 1, \delta \geq 0$ if

$$\sum_{n=1}^{\infty}(\alpha_n)^{\delta k+k-1}|T_n - T_{n-1}|^k < \infty \tag{1.6}$$

For any sequence $\{\mu_n\}$,$\sum_{n=1}^{\infty}a_n\mu_n$ is an infinite series.

We define

$$\Delta\mu_n = \mu_n - \mu_{n-1}, |\Delta\mu_n| = |\mu_n - \mu_{n-1}|$$

Also, for any sequence $\{\mu_n\}$, by $\mu_n = O(n)$, we mean that the sequence $\{\frac{\mu_n}{n}\}$ is bounded.

2. Known Theorems

Concerning with $|C, 1|$ and $|N, q_n|$ summability Kishore [10] has proved the following theorem:

Theorem 2.1. *Let $q_0 > 0, q_n \geq 0$ and (q_n) be a non-decreasing sequence. If $\sum a_n$ is summable $|C, 1|$ then the series $\sum a_n Q_n (n+1)^{-1}$ is summable $|N, q_n|$.*

Later on Ram [15] has proved the following theorem related to absolute Norlund factors of infinite series.

Theorem 2.2. *Let (q_n) be a non-increasing sequence with $q_0 > 0, q_n \geq 0$. If*

$$\sum_{k=1}^{n} \frac{1}{k} |s_k| = O(Y_n) \text{ as } n \to \infty;$$

where (Y_n) is a positive non-decreasing sequence and (μ_n) is a sequence such that

$$\sum_{n=1}^{\infty} n |\Delta^2 \mu_n| Y_n < \infty;$$

$$|\mu_n| Y_n = O(1) \text{ as } n \to \infty,$$

then the series $\sum a_n Q_n (n+1)^{-1}$ is summable $|N, q_n|$.

Also verma [17] has proved the following summability factor theorem:

Theorem 2.3. *Let (q_n) be a non-increasing sequence with $q_0 > 0, q_n \geq 0$. If $\sum a_n$ is summable $|C, 1|_k$ then the series $\sum a_n Q_n (n+1)^{-1}$ is summable $|N, q_n|_k, k \geq 1$.*

In 1984, Mishra and Srivatava [13] proved the following theorem for $|C, 1|_k$ summability.

Theorem 2.4. *Let (Y_n) be a positive non-decreasing sequence and let there be sequnces $\{\beta_n\}$ and $\{\mu_n\}$ such that*

$$|\Delta \mu_n| \leq \beta_n; \tag{2.1}$$

$$\beta_n \to 0 \text{ as } n \to \infty; \tag{2.2}$$

$$|\mu_n| Y_n = O(1) \text{ as } n \to \infty; \tag{2.3}$$

$$\sum_{n=1}^{\infty} n |\Delta \beta_n| Y_n < \infty; \tag{2.4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} |s_n|^k = O(Y_m) \text{ as } m \to \infty, \tag{2.5}$$

then the series $\sum_{n=1}^{\infty} a_n \mu_n$ is summable $|C, 1|_k, k \geq 1$.

Very recently, Padhy et al. [14] have proved a theorem on $|N, q_n|_k$-summability by extending theorem 2.4, in the following form:

Theorem 2.5. *Let for a positive non-decreasing sequence* (Y_n)*,there be sequences* $\{\beta_n\}$ *and* $\{\mu_n\}$ *satisfying the conditions* 2.1 *to* 2.5 *and* $\{q_n\}$ *be a sequence with* $\{q_n\} \in R^+$ *such that*

$$Q_n = O(nq_n); \tag{2.6}$$

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n} |s_n|^k = O(Y_m) \text{ as } m \to \infty; \tag{2.7}$$

$$\frac{Q_{n-r-1}}{Q_n} = O\left(\frac{q_{n-r-1}}{Q_n} \frac{Q_r}{q_r}\right); \tag{2.8}$$

$$\sum_{n=r+1}^{m+1} \left(\frac{Q_n}{q_n}\right)^{k-1} \frac{q_{n-r}}{Q_n} = O\left(\frac{q_r}{Q_r}\right), \tag{2.9}$$

then the series $\sum_{n=1}^{\infty} a_n \mu_n$ *is summable* $|N, q_n|_k, k \geq 1$*,where* $0 \leq r \leq n$*.*

It should be noted that if we take $q_n = 1 \forall n$ then condition 2.7 will be reduced to 2.5.
In what follows, we have generalized known theorems 2.4 and 2.5 to $|N, q_n, \alpha_n; \delta|_k$ - summability in the form of the following theorem after studying [1] and [2] :

3. Main Theorem

Theorem 3.1. *Let* (Y_n) *be a positive non-decreasing sequence and there be sequences* $\{\beta_n\}$ *and* $\{\mu_n\}$ *such that the conditions* 2.1 *to* 2.5 *are satisfied.Further let* $\{q_n\}$ *be a sequence of real numbers with* $q_n > 0$*, such that*

$$Q_n = O(nq_n); \tag{3.1}$$

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n} |s_n|^k = O(Y_m) \text{ as } m \to \infty; \tag{3.2}$$

$$\frac{Q_{n-r-1}}{Q_n} = O\left(\frac{q_{n-r-1}}{Q_n} \frac{Q_r}{q_r}\right); \tag{3.3}$$

$$\sum_{n=r+1}^{m+1} (\alpha_n)^{\delta k + k - 1} \frac{q_{n-r}}{Q_n} = O\left(\frac{q_r}{Q_r}\right), \tag{3.4}$$

then the series $\sum_{n=1}^{\infty} a_n \mu_n$ *is summable* $|N, q_n, \alpha_n; \delta|_k, k \geq 1, \delta \geq 0$*.*

We require the below mentioned lemma to prove our main theorem:

4. Lemma [5]

Let (Y_n) be a positive non decreasing sequence and there be sequences $\{\beta_n\}$ and $\{\mu_n\}$ such that the conditions 2.1 to 2.5 are satisfied.Then

$$\beta_n Y_n = O(1) \ as \ n \to \infty, \tag{4.1}$$

$$\sum_{n=1}^{\infty} \beta_n Y_n < \infty. \tag{4.2}$$

5. Proof of the Main theorem

Suppose (τ_n) refers to the (N, q_n)- mean of the series $\sum_{n=1}^{\infty} a_n \mu_n$. Then by definition, we have

$$\begin{aligned}
\tau_n &= \frac{1}{Q_n} \sum_{r=0}^{n} q_{n-r} \sum_{s=0}^{r} a_s \mu_s \\
&= \frac{1}{Q_n} \sum_{s=0}^{n} a_s \mu_s \sum_{r=s}^{n} q_{n-r} \\
&= \frac{1}{Q_n} \sum_{s=0}^{n} a_s \mu_s Q_{n-s} \\
&= \frac{1}{Q_n} \sum_{r=0}^{n} a_r \mu_r Q_{n-r}
\end{aligned}$$

Thus

$$\begin{aligned}
&\tau_n - \tau_{n-1} = \frac{1}{Q_n} \sum_{r=1}^{n} Q_{n-r} a_r \mu_r - \frac{1}{Q_{n-1}} \sum_{r=1}^{n-1} Q_{n-r-1} a_r \mu_r \\
&= \sum_{r=1}^{n} \left(\frac{Q_{n-r}}{Q_n} - \frac{Q_{n-r-1}}{Q_{n-1}} \right) a_r \mu_r \\
&= \frac{1}{Q_n Q_{n-1}} \sum_{r=1}^{n} (Q_{n-r} Q_{n-1} - Q_{n-r-1} Q_n) a_r \mu_r \\
&= \frac{1}{Q_n Q_{n-1}} \left[\sum_{r=1}^{n-1} \Delta \{ (Q_{n-r} Q_{n-1} - Q_{n-r-1} Q_n) \, \mu_r \} \right] \sum_{\nu=1}^{n} a_\nu, \ with \ p_0 = 0 \\
&= \frac{1}{Q_n Q_{n-1}} \left[\sum_{r=1}^{n-1} (q_{n-r} Q_{n-1} - q_{n-r-1} Q_n) \, \mu_r s_r + \sum_{r=1}^{n-1} (Q_{n-r-1} Q_{n-1} - Q_{n-r-2} Q_n) \, \Delta \mu_r s_r \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{(By Abel's transformation)} \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \ (say)
\end{aligned}$$

Now, to show $\sum_{n=1}^{\infty} a_n \mu_n$ is summable $|N, q_n, \alpha_n; \delta|_k, k \geq 1, \delta \geq 0$, by 1.6,we need to show that

$$\sum_{n=1}^{\infty} (\alpha_n)^{\delta k + k - 1} |\tau_n - \tau_{n-1}|^k < \infty.$$

i.e; to show that

$$\sum_{n=1}^{\infty}(\alpha_n)^{\delta k+k-1}|T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4}|^k<\infty.$$

It will be enough to show that

$$\sum_{n=1}^{\infty}(\alpha_n)^{\delta k+k-1}|T_{n,j}|^k<\infty \ for \ j=1,2,3,4.$$

to establish the main theorem by using the inequality given by Minkowski.

Now we have

$$\begin{aligned}
&\sum_{n=2}^{m+1}(\alpha_n)^{\delta k+k-1}|T_{n,1}|^k\\
&\sum_{n=2}^{m+1}(\alpha_n)^{\delta k+k-1}\left|\frac{1}{Q_nQ_{n-1}}\sum_{r=1}^{n-1}q_{n-r}Q_{n-1}\mu_r s_r\right|^k\\
&\leq\sum_{n=2}^{m+1}(\alpha_n)^{\delta k+k-1}\frac{1}{Q_n}\left(\sum_{r=1}^{n-1}q_{n-r}|\mu_r|^k|s_r|^k\right)\left(\frac{1}{Q_n}\sum_{r=1}^{n-1}q_{n-r}\right)^{k-1} \quad (Using\ Holder's\ inequality)\\
&=O(1)\sum_{r=1}^{m}|\mu_r|^k|s_r|^k\sum_{n=r+1}^{m+1}(\alpha_n)^{\delta k+k-1}\left(\frac{q_{n-r}}{Q_n}\right)\\
&=O(1)\sum_{r=1}^{m}|\mu_r|^k|s_r|^k\frac{q_r}{Q_r},\ by\ 3.4\\
&=O(1)\sum_{r=1}^{m}\frac{q_r}{Q_r}|s_r|^k|\mu_r||\mu_r|^{k-1}\\
&=O(1)\sum_{r=1}^{m-1}\Delta|\mu_r|\sum_{w=1}^{r}\frac{q_w}{Q_w}|s_w|^k+O(1)|\mu_m|\sum_{r=1}^{m}\frac{q_r}{Q_r}|s_r|^k\\
&=O(1)\sum_{r=1}^{m-1}|\Delta\mu_r|Y_r+O(1)|\mu_m|Y_m\ ,\ by\ 3.2\\
&=O(1),\ as\ m\to\infty
\end{aligned}$$

(By the lemma and 2.3)

Next,

$$\begin{aligned}
&\sum_{n=2}^{m+1}(\alpha_n)^{\delta k+k-1}|T_{n,2}|^k\\
&=\sum_{n=1}^{m+1}(\alpha_n)^{\delta k+k-1}\left|\frac{1}{Q_nQ_{n-1}}\sum_{r=1}^{n-1}q_{n-r-1}Q_n\mu_r s_r\right|^k\\
&\leq\sum_{n=2}^{m+1}(\alpha_n)^{\delta k+k-1}\frac{1}{Q_{n-1}}\left(\sum_{r=1}^{n-1}q_{n-r-1}|\mu_r|^k|s_r|^k\right)\left(\frac{1}{Q_{n-1}}\sum_{r=1}^{n-1}q_{n-r-1}\right)^{k-1}
\end{aligned} \tag{5.1}$$

$$
\begin{aligned}
&= O(1)\sum_{r=1}^{m}|\mu_r|^k|s_r|^k\sum_{n=r+1}^{m+1}(\alpha_n)^{\delta k+k-1}\left(\frac{q_{n-r-1}}{Q_{n-1}}\right)\\
&= O(1)\sum_{r=1}^{m}|\mu_r|^k|s_r|^k\frac{q_r}{Q_r}\\
&= O(1),\ as\ m\to\infty,\ As\ in\ proof\ of\ the\ 1st\ part.
\end{aligned}
$$

Further,

$$
\begin{aligned}
&\sum_{n=2}^{m+1}(\alpha_n)^{\delta k+k-1}|T_{n,3}|^k\\
&= \sum_{n=1}^{m+1}(\alpha_n)^{\delta k+k-1}|\frac{1}{Q_nQ_{n-1}}\sum_{r=1}^{n-1}Q_{n-r-1}Q_{n-1}\Delta\mu_r s_r|^k\\
&\le \sum_{n=2}^{m+1}(\alpha_n)^{\delta k+k-1}\frac{1}{Q_n}\left(\sum_{r=1}^{n-1}Q_{n-r-1}|\Delta\mu_r||s_r|^k\right)\left(\frac{1}{Q_n}\sum_{r=1}^{n-1}Q_{n-r-1}|\Delta\mu_r|\right)^{k-1}\\
&Since,\ \left(\frac{1}{Q_n}\sum_{r=1}^{n-1}Q_{n-r-1}|\Delta\mu_r|\right)\le\sum_{r=1}^{n-1}|\Delta\mu_n|\le n|\Delta\mu_r|\le n\beta_n
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{n=2}^{m+1}(\alpha_n)^{\delta k+k-1}|T_{n,3}|^k\\
&\le O(1)\sum_{r=1}^{m}(r\beta_r)^{k-1}|\Delta\mu_r||s_r|^k\sum_{n=r+1}^{m+1}(\alpha_n)^{\delta k+k-1}\frac{Q_{n-r-1}}{Q_n}\\
&= O(1)\sum_{r=1}^{m}|\Delta\mu_r||s_r|^k\frac{q_r}{Q_r}\\
&\le O(1)\sum_{r=1}^{m}\beta_r|s_r|^k\frac{q_r}{Q_r}\\
&= O(1)\sum_{r=1}^{m-1}\Delta(\beta_r)\sum_{w=1}^{r}\frac{q_w}{Q_w}|s_w|^k+O(1)(\beta_m)\sum_{r=1}^{m}\frac{q_r}{Q_r}|s_r|^k\\
&= O(1)\sum_{r=1}^{m-1}|\Delta\beta_r|Y_r+O(1)(\beta_m)Y_m\\
&= O(1)\ as\ m\to\infty
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\sum_{n=2}^{m+1}(\alpha_n)^{\delta k+k-1}|T_{n,4}|^k\\
&= \sum_{n=2}^{m+1}(\alpha_n)^{\delta k+k-1}|\frac{1}{Q_nQ_{n-1}}\sum_{r=1}^{n-1}Q_{n-r-2}Q_n\Delta\mu_r s_r|^k
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
&\leq \sum_{n=2}^{m+1}(\alpha_n)^{\delta k+k-1}\frac{1}{Q_{n-1}}\left(\sum_{r=1}^{n-1}Q_{n-r-2}|\Delta\mu_r||s_r|^k\right)\frac{1}{Q_{n-1}}\sum_{r=1}^{n-1}Q_{n-r-2}|\Delta\mu_r|^{k-1}\\
&= O(1)\sum_{r=1}^{m}(r\beta_r)^{k-1}|\Delta\mu_r||s_r|^k\sum_{n=r+1}^{m+1}(\alpha_n)^{\delta k+k-1}\left(\frac{Q_{n-r-1}}{Q_n}\right),\ (as\ above)\\
&= O(1)\sum_{r=1}^{m}|\Delta\mu_r||s_r|^k\frac{q_r}{Q_r}\\
&= O(1)\ as\ m\to\infty.\ (as\ above)
\end{aligned}
$$

This completes the proof of the theorem.

6. Conclusion

If (Y_n) is a positive non-decreasing sequence and there be sequences $\{\beta_n\}$ and $\{\mu_n\}$ such that the conditions 2.1 to 2.5 along with the conditions 4.1 and 4.2 are satisfied then the series $\sum_{n=1}^{\infty} a_n\mu_n$ is summable $|N, q_n, \alpha_n; \delta|_k, k \geq 1, \delta \geq 0$, under the conditions 3.1 to 3.4.Thus, our result generalizes the result of Mishra and Srivastava [13] and Padhy et. al [14].

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