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FRACTIONAL INTEGRAL INEQUALITIES OF GRÜSS TYPE VIA GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract. We use generalized fractional integral operator containing the generalized Mittag-Leffler function to establish some new integral inequalities of Grüss type. A cluster of fractional integral inequalities have been identified by setting particular values to parameters involved in the Mittag-Leffler special function. Presented results contain several fractional integral inequalities which reflects their importance.

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1. Introduction

In 1935, Grüss [5] proved the following inequality

$$\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f_1(t)f_2(t)dt-\left(\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f_1(t)dt\right)\left(\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f_2(t)dt\right) \le \frac{(M-m)(N-n)}{4}, \tag{1.1}$$

where f and g are two integrable functions on [a,b] and satisfying the following conditions

$$m \le f_1(x) \le M, \quad n \le f_2(x) \le N \quad m, M, n, N \in \mathbb{R}, \quad x \in [a,b].$$

In the literature inequality (1.1) is well known as the Grüss inequality. Inequality (1.1) remains in the focus of researchers especially working in the field of mathematical analysis. A lot of authors are working on (1.1) and have produced important results for different kinds of functions. In recent years, many important and fascinating Grüss type inequalities have been established (see for example [7, 9, 15]). Our interest in this paper is to give some generalized fractional integral inequalities of Grüss type by use of generalized fractional integral operators due to the Mittag-Leffler function.

In the following we define an extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$ as fallows:

Definition 1.1. *[3] Let $\mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \ge 0$, $\delta > 0$ and $0 < k \le \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$ is defined by*

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n+\alpha)} \frac{t^n}{(l)_{n\delta}}, \tag{1.2}$$

Here $(c)_{nk}$ denotes the generalized Pochhammer symbol

$$(c)_{nk} = \frac{\Gamma(c+nk)}{\Gamma(c)},$$

B_p *is an extension of the beta function*

$$\mathrm{B}_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}} dt \quad (\Re(x), \Re(y), \Re(p) > 0).$$

The corresponding generalized fractional integral operator $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f$ is defined as fallows:

Definition 1.2. *[3] Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \ge 0$, $\delta > 0$ and $0 < k \le \delta + \Re(\mu)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the generalized fractional integral operator $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f$ is defined by:*

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f\right)(x;p) = \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p) f(t)dt. \tag{1.3}$$

From generalized fractional integral operator we have

$$\begin{aligned}
&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}1\right)(x;p)\\
&=\int_a^x (x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(w(x-t)^{\mu};p)dt\\
&=\int_a^x (x-t)^{\alpha-1}\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\gamma+nk,c-\gamma)}{\mathrm{B}(\gamma,c-\gamma)}\frac{(c)_{nk}}{\Gamma(\mu n+\alpha)}\frac{w^n(x-t)^{\mu n}}{(l)_{n\delta}}dt\\
&=\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\gamma+nk,c-\gamma)}{\mathrm{B}(\gamma,c-\gamma)}\frac{(c)_{nk}}{\Gamma(\mu n+\alpha)}\frac{w^n}{(l)_{n\delta}}\int_a^x (x-t)^{\mu n+\alpha-1}dt\\
&=(x-a)^{\alpha}\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\gamma+nk,c-\gamma)}{\mathrm{B}(\gamma,c-\gamma)}\frac{(c)_{nk}}{\Gamma(\mu n+\alpha)}\frac{w^n}{(l)_{n\delta}}(x-a)^{\mu n}\frac{1}{\mu n+\alpha}.
\end{aligned}$$

Hence

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}1\right)(x;p)=(x-a)^{\alpha}\,E^{\gamma,\delta,k,c}_{\mu,\alpha+1,l}(w(x-a)^{\mu};p).$$

We use the following notation in our results

$$C_{\alpha}(x;p)=\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}1\right)(x;p). \tag{1.4}$$

Integral operators are very useful in solving integral as well as differential equations. Several types of integral operators have been studied by the mathematicians (see for example [1, 2, 4, 6, 8, 11–14]).

In this paper at first some generalized fractional integral inequalities and their particular cases are established. Then a generalized fractional Korkine's identity is proved. At the end Grüss fractional integral inequality via generalized fractional integral operator have been obtained. The presented inequality contained several versions of Grüss inequality in fractional calculus.

2. Main results

First we prove the following fractional inequality.

Theorem 2.1. *Let $f,\psi_1,\psi_2\in L_1[a,b]$ such that*

$$\psi_1(x)\le f(x)\le\psi_2(x)\qquad\forall x\in[a,b]. \tag{2.1}$$

Then for extended generalized fractional integral operator (1.3) we have the following inequality:

$$\begin{aligned}
&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\psi_2\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}f\right)(x;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}\psi_1\right)(x;p)\\
&\ge\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\psi_2\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}\psi_1\right)(x;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}f\right)(x;p)
\end{aligned} \tag{2.2}$$

Proof. From (2.1) we have

$$(\psi_2(u)-f(u))(f(v)-\psi_1(v))\ge 0\qquad\forall u,v\in[a,b]. \tag{2.3}$$

This gives the following inequality:

$$\psi_2(u)f(v)+\psi_1(v)f(u)\geq\psi_1(v)\psi_2(u)+f(u)f(v). \tag{2.4}$$

Multiplying (2.4) by $(x-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu};p)$ on both sides and integrating with respect to u over $[a,x]$, the following inequality is obtained:

$$\begin{aligned}&\int_a^x(x-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu};p)\psi_2(u)f(v)du\\&+\int_a^x(x-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu};p)\psi_1(v)f(u)du\\&\geq\int_a^x(x-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu};p)\psi_1(v)\psi_2(u)du\\&+\int_a^x(x-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu};p)f(u)f(v)du.\end{aligned} \tag{2.5}$$

Using the Definition 1.2 we get

$$\begin{aligned}&f(v)\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}\psi_2\right)(x;p)+\psi_1(v)\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}f\right)(x;p)\\&\geq\psi_1(v)\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}\psi_2(u)\right)(x;p)+f(v)\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}f\right)(x;p).\end{aligned} \tag{2.6}$$

Now multiplying (2.6) by $(x-v)^{\beta-1}E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(x-v)^{\mu};p)$ on both sides and integrating with respect to v over $[a,x]$, the following inequality is obtained:

$$\begin{aligned}&\left(\epsilon_{\mu,\beta,l,\omega,a}^{\gamma,\delta,k,c}\psi_2\right)(x;p)\int_a^x(x-v)^{\beta-1}E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(x-v)^{\mu};p)f(v)dv\\&+\left(\epsilon_{\mu,\beta,l,\omega,a}^{\gamma,\delta,k,c}f\right)(x;p)\int_a^x(x-v)^{\beta-1}E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(x-v)^{\mu};p)\psi_1(v)dv\\&\geq\left(\epsilon_{\mu,\beta,l,\omega,a}^{\gamma,\delta,k,c}\psi_2\right)(x;p)\int_a^x(x-v)^{\beta-1}E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(x-v)^{\mu};p)\psi_1(v)dv\\&+\left(\epsilon_{\mu,\beta,l,\omega,a}^{\gamma,\delta,k,c}f\right)(x;p)\int_a^x(x-v)^{\beta-1}E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(x-v)^{\mu};p)f(v)dv.\end{aligned} \tag{2.7}$$

By using the Definition 1.2 and then after simple calculation we get the required inequality (2.2). □

A particular case is given as follows.

Corollary 2.1. *Let $f\in L_1[a,b]$ and m_1,m_2 be two real numbers such that*

$$m_1\leq f(x)\leq m_2\qquad\forall x\in[a,b].$$

Then we have

$$\begin{aligned}&m_2C_{\alpha}(x;p)\left(\epsilon_{\mu,\beta,l,\omega,a}^{\gamma,\delta,k,c}f\right)(x;p)+m_1\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}f\right)(x;p)C_{\beta}(x;p)\\&\geq m_1m_2C_{\alpha}(x;p)C_{\beta}(x;p)+\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}f\right)(x;p)\left(\epsilon_{\mu,\beta,l,\omega,a}^{\gamma,\delta,k,c}f\right)(x;p).\end{aligned} \tag{2.8}$$

Proof. Proof follows on the same lines as the proof of Theorem 2.1 just use $\psi_1(x) = m_1$ and $\psi_2(x) = m_2$ as constant functions. □

Some more inequalities are given in the next result.

Theorem 2.2. *Let $f, \psi_1, \psi_2 \in L_1[a,b]$ such that (2.1) holds. Also let $g \in L_1[a,b]$ and there exist ϕ_1 and ϕ_2 such that*

$$\phi_1(x) \leq g(x) \leq \phi_2(x) \quad \forall x \in [a,b]. \tag{2.9}$$

Then for extended generalized fractional integral (1.3) we have the following inequalities:

$$\begin{aligned}
&(i) \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}\phi_1\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}\psi_2\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p) \\
&\geq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}\phi_1\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\psi_2\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}g\right)(x;p), \\
&(ii) \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}\psi_1\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}\phi_2\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f\right)(x;p) \\
&\geq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}\psi_1\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\phi_2\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}f\right)(x;p), \\
&(iii) \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}\psi_2\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\phi_2\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}f\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p) \\
&\geq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}\psi_2\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\phi_2\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}f\right)(x;p), \\
&(iv) \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}\psi_1\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\phi_1\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}f\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p) \\
&\geq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}\psi_1\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\phi_1\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a}f\right)(x;p),
\end{aligned} \tag{2.10}$$

Proof. (i) From (2.1) and (2.9) we have

$$(\psi_2(u) - f(u))(g(v) - \phi_1(v)) \geq 0, \tag{2.11}$$

that gives

$$\psi_2(u)g(v) + \phi_1(v)f(u) \geq \psi_1(v)\psi_2(u) + f(u)g(v). \tag{2.12}$$

Multiplying (2.12) by $(x-u)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-u)^{\mu};p)(x-v)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(x-v)^{\mu};p)$ on both sides and integrating with respect to u and v over $[a,x]$ then by using Definition 1.2 we get (i).
To prove $(ii)-(iv)$, we use the following inequalities instead of (2.11) respectively
(ii) $(\phi_2(u) - g(u))(f(v) - \psi_1(v)) \geq 0$,
(iii) $(\psi_2(u) - f(u))(g(v) - \phi_2(v)) \leq 0$,
(iv) $(\psi_1(u) - f(u))(g(v) - \phi_1(v)) \leq 0$,
then on the same lines as done to obtain (i) one can get inequalities $(ii)-(iv)$. □

Special cases are stated as follows.

Corollary 2.2. *Let $f, g \in L_1[a,b]$. Also let m_1, m_2, n_1 and n_2 be real constants such that*

$$m_1 \le f(x) \le m_2, \qquad n_1 \le g(x) \le n_2, \quad \forall x \in [a,b].$$

Then we have

$$\begin{aligned}
&(i) \quad n_1 C_\beta(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f\right)(x;p) + m_2 C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a} g\right)(x;p)\\
&\ge n_1 m_2 C_\beta(x;p) C_\alpha(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a} g\right)(x;p),\\
&(ii) \quad m_1 C_\beta(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} g\right)(x;p) + n_2 C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a} f\right)(x;p)\\
&\ge m_1 n_2 C_\beta(x;p) C_\alpha(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a} f\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} g\right)(x;p),\\
&(iii) \quad m_2 C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a} g\right)(x;p) + n_2 C_\beta(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f\right)(x;p)\\
&\ge m_2 n_2 C_\beta(x;p) C_\alpha(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a} g\right)(x;p),\\
&(iv) \quad m_1 C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a} g\right)(x;p) + n_1 C_\beta(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f\right)(x;p)\\
&\ge m_1 n_1 C_\beta(x;p) C_\alpha(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,a} g\right)(x;p).
\end{aligned}$$

Proof. Proof follows on the same lines as the proof of Theorem 2.1 just use $\psi_1(x) = m_1, \psi_2(x) = m_2, \phi_1(x) = n_1$ and $\phi_2(x) = n_2$ as constant functions. □

Next we give the Korkine's identity which is used in the next result.

Theorem 2.3. *Let $f, \psi_1, \psi_2 \in L_1[a,b]$ such that (2.1) holds. Then for extended generalized fractional integral (1.3) we have the following equality:*

$$\begin{aligned}
&C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f^2\right)(x;p) - \left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f\right)(x;p)\right]^2 \qquad (2.13)\\
&= \left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} \psi_2\right)(x;p) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f\right)(x;p)\right]\\
&\times \left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f\right)(x;p) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} \psi_1\right)(x;p)\right]\\
&- C_\alpha(x;p)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} \psi_2\right)(x;p) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f\right)(x;p)\right]\\
&\times \left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f\right)(x;p) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} \psi_1\right)(x;p)\right]\\
&+ C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} \psi_1 f\right)(x;p) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} \psi_1\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f\right)(x;p)\\
&+ C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} \psi_2 f\right)(x;p) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} \psi_2\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} f\right)(x;p)\\
&- C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} \psi_1 \psi_2\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} \psi_1\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a} \psi_2\right)(x;p).
\end{aligned}$$

Proof. For any $u, v \in [a, b]$ we have

$$(\psi_2(v) - f(v))(f(u) - \psi_1(u)) + (\psi_2(u) - f(u))(f(v) - \psi_1(v)) \tag{2.14}$$
$$- (\psi_2(u) - f(u))(f(u) - \psi_1(u)) - (\psi_2(v) - f(v))(f(v) - \psi_1(v))$$
$$= f^2(u) + f^2(v) - 2f(u)f(v) + \psi_2(v)f(u) + \psi_1(u)f(v)$$
$$- \psi_1(u)\psi_2(v) + \psi_2(u)f(v) + \psi_1(v)f(u) - \psi_1(v)\psi_2(v)$$
$$- \psi_2(u)f(u) + \psi_1(u)\psi_2(u) - \psi_1(u)f(u) - \psi_2(v)f(v)$$
$$+ \psi_1(v)\psi_2(v) - \psi_1(v)f(v).$$

Now multiplying (2.14) by $(x-u)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-u)^{\mu};p)(x-v)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-v)^{\mu};p)$ on both sides and integrating with respect to u and v over $[a, x]$ then by using Definition 1.2 we get the required identity (2.13). □

The last result is the generalized fractional Grüss inequality.

Theorem 2.4. *Let f and g be a two functions such that $f, g \in L_1[a, b]$. Also let ψ_1, ψ_2, ϕ_1 and ϕ_2 be four integrable functions satisfying (2.1) and (2.9). Then for extended generalized fractional integral (1.3) we have the following inequality:*

$$\left|C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}fg\right)(x;p) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p)\right| \tag{2.15}$$
$$\leq \sqrt{G(f,\psi_1,\psi_2)G(g,\phi_1,\phi_2)},$$

where

$$G(u,v,w)$$
$$= \left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}w\right)(x;p) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}u\right)(x;p)\right]$$
$$\times \left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}u\right)(x;p) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}v\right)(x;p)\right]$$
$$+ C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}vu\right)(x;p) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}v\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}u\right)(x;p)$$
$$+ C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}wu\right)(x;p) - \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}w\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}u\right)(x;p)$$
$$- C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}vw\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}v\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}w\right)(x;p).$$

Proof. Since f and g are two integrable functions we have

$$[f(u) - f(v)]\,[g(u) - g(v)] \tag{2.16}$$
$$= f(u)g(u) + f(v)g(v) - f(u)g(v) - f(v)g(u).$$

Multiplying (2.16) by $\frac{1}{2}(x-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu};p)(x-v)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-v)^{\mu};p)$ and integrating with respect to u and v over $[a,x]$, the following inequality is obtained:

$$\Big(\frac{1}{2}\int_a^x\int_a^x (x-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu};p)(x-v)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-v)^{\mu};p) \tag{2.17}$$
$$\times\left[f(u)-f(v)\right]\left[g(u)-g(v)\right]dudv)$$
$$=\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}fg\right)(x;p)-\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}f\right)(x;p)\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}g\right)(x;p).$$

Now by using Cauchy-Schwarz inequality we have

$$\Big(\frac{1}{2}\int_a^x\int_a^x (x-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu};p)(x-v)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-v)^{\mu};p) \tag{2.18}$$
$$\times\left[f(u)-f(v)\right]\left[g(u)-g(v)\right]dudv)^2$$
$$\leq\frac{1}{2}\int_a^x\int_a^x (x-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu};p)(x-v)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-v)^{\mu};p)$$
$$\times\left[f(u)-f(v)\right]^2dudv$$
$$\times\frac{1}{2}\int_a^x\int_a^x (x-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu};p)(x-v)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-v)^{\mu};p)$$
$$\times\left[g(u)-g(v)\right]^2dudv.$$

From (2.18) one can have

$$\frac{1}{2}\int_a^x\int_a^x (x-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu};p)(x-v)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-v)^{\mu};p) \tag{2.19}$$
$$\times\left[f(u)-f(v)\right]^2dudv$$
$$=C_\alpha(x;p)\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}f^2\right)(x;p)-\left[\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}f\right)(x;p)\right]^2.$$

Similarly,

$$\frac{1}{2}\int_a^x\int_a^x (x-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu};p)(x-v)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-v)^{\mu};p) \tag{2.20}$$
$$\times\left[g(u)-g(v)\right]^2dudv$$
$$=C_\alpha(x;p)\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}g^2\right)(x;p)-\left[\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}g\right)(x;p)\right]^2.$$

Using (2.19) and (2.20) in (2.18) we have

$$\Big(\frac{1}{2}\int_a^x\int_a^x (x-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-u)^{\mu};p)(x-v)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-v)^{\mu};p) \tag{2.21}$$
$$\times\left[f(u)-f(v)\right]\left[g(u)-g(v)\right]dudv)^2$$
$$\leq C_\alpha(x;p)\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}f^2\right)(x;p)-\left[\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}f\right)(x;p)\right]^2$$
$$\times C_\alpha(x;p)\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}g^2\right)(x;p)-\left[\left(\epsilon_{\mu,\alpha,l,\omega,a}^{\gamma,\delta,k,c}g\right)(x;p)\right]^2.$$

Now combining (2.17) and (2.21) we have

$$\left(\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}fg\right)(x;p)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p)\right)^2 \tag{2.22}$$
$$\leq C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f^2\right)(x;p)-\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f\right)(x;p)\right]^2$$
$$\times C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g^2\right)(x;p)-\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p)\right]^2.$$

Since

$$(\psi_2(x)-f(x))(f(x)-\psi_1(x))\geq 0$$

and

$$(\phi_2(x)-g(x))(g(x)-\phi_1(x))\geq 0,$$

therefore

$$C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}(\psi_2-f)(f-\psi_1)\right)(x;p)\geq 0,$$

and

$$C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}(\phi_2-g)(g-\phi_1)\right)(x;p)\geq 0.$$

By Theorem 2.3 we have

$$C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f^2\right)(x;p)-\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f\right)(x;p)\right]^2 \tag{2.23}$$
$$\leq\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\psi_2\right)(x;p)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f\right)(x;p)\right]$$
$$\times\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f\right)(x;p)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\psi_1\right)(x;p)\right]$$
$$+C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\psi_1 f\right)(x;p)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\psi_1\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f\right)(x;p)$$
$$+C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\psi_2 f\right)(x;p)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\psi_2\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}f\right)(x;p)$$
$$-C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\psi_1\psi_2\right)(x;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\psi_1\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\psi_2\right)(x;p).$$
$$=G(f,\psi_1,\psi_2).$$

Similarly

$$\begin{aligned}
&C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g^2\right)(x;p)-\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p)\right]^2 \qquad (2.24)\\
&\leq\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\phi_2\right)(x;p)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p)\right]\\
&\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\phi_1\right)(x;p)\right]\\
&+C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\phi_1 f\right)(x;p)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\phi_1\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p)\\
&+C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\phi_2 g\right)(x;p)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\phi_2\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}g\right)(x;p)\\
&-C_\alpha(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\phi_1\phi_2\right)(x;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\phi_1\right)(x;p)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a}\phi_2\right)(x;p).\\
&=G(g,\phi_1,\phi_2).
\end{aligned}$$

Combining (2.23),(2.24) with (2.22), we get the required inequality (2.15). □

Concluding remarks

Since the extended generalized fractional integral operator contains itself several known fractional integral operators for particular values of involved parameters. For example selecting $p = 0$, fractional integral inequalities for fractional integral operators defined by Salim and Faraj in [12], selecting $l = \delta = 1$, fractional integral inequalities for fractional integral operators defined by Rahman et al. in [11], selecting $p = 0$ and $l = \delta = 1$, fractional integral inequalities for fractional integral operators defined by Shukla and Prajapati in [13] and see also [14], selecting $p = 0$ and $l = \delta = k = 1$, fractional integral inequalities for fractional integral operators defined by Prabhakar in [10], selecting $p = \omega = 0$ fractional integral inequalities for Riemann-Liouville fractional integrals. Therefore the presented results contain all such results for these particular fractional integral operators as special cases.

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