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GROWTH PROPERTIES OF WRONSKIANS IN THE LIGHT OF RELATIVE ORDER

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Abstract. In this paper we study the comparative growth properties of composition of entire and meromorphic functions on the basis of relative order (relative lower order) of Wronskians generated by entire and meromorphic functions.

1. Introduction, Definitions and Notations.

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The function $M(r,f)$ on $|z|=r$ known as maximum modulus function corresponding to f is defined as follows:

$$M(r,f)=\max_{|z|=r}|f(z)| \ .$$

When f is meromorphic, $M(r,f)$ can not be defined. In this situation one may define another function $T(r,f)$ known as Nevanlinna's Characteristic function of f, playing the same role as $M(r,f)$ in the following manner:

$$T(r,f)=N(r,f)+m(r,f) \ .$$

When f is an entire function, $T(r,f)$ reduces to $m(r,f)$.

We call the function $N(r,a;f)$ $\left(\bar{N}(r,a;f)\right)$ as counting function of a-points (distinct a-points) of f. In many occasions $N(r,\infty;f)$ and $\bar{N}(r,\infty;f)$ are denoted by $N(r,f)$ and $\bar{N}(r,f)$ respectively.We put

$$N(r,a;f)=\int_0^r \frac{n(t,a;f)-n(0,a;f)}{t}dt+\bar{n}(0,a;f)\log r \ ,$$

where we denote by $n(t,a;f)$ $\left(\bar{n}(t,a;f)\right)$ the number of a-points (distinct a-points) of f in $|z|\le t$ and an ∞ -point is a pole of f .

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On the other hand $m\left(r, \frac{1}{f-a}\right)$ is denoted by $m(r, a; f)$ and we mean $m(r, \infty; f)$ by $m(r, f)$, which is called the proximity function of f. We also put

$$m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta, \quad \text{where}$$
$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0 \ .$$

If the entire function g is non-constant then $T_g(r)$ is strictly increasing and continuous and its inverse $T_g^{-1} : (T_g(0), \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} T_g^{-1}(s) = \infty$.

Lahiri and Banerjee [4] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 1.1. [4] *Let f be meromorphic and g be entire. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as*

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}. \end{aligned}$$

Analogously, one can define the relative lower order of a meromorphic function f with respect to an entire function g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

If we consider $g(z) = \exp z$, Definition 1.1 coincides $\{cf.[4]\}$ with the classical definition of order and lower order of meromorphic function which are as follows:

Definition 1.2. *The order ρ_f and lower order λ_f of a meromorphic function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$, $k = 1, 2, 3, \ldots$ *and* $\log^{[0]} x = x$.

The following definitions are also well known:

Definition 1.3. *A meromorphic function $a \equiv a(z)$ is called small with respect to f if $T(r, a) = S(r, f)$ where $S(r, f) = o\{T(r, f)\}$ i.e., $\frac{S(r,f)}{T(r,f)} \to 0$ as $r \to \infty$.*

Definition 1.4. *Let $a_1, a_2, \ldots. a_k$ be linearly independent meromorphic functions and small with respect to f . We denote by $L(f) = W(a_1, a_2, \ldots. a_k; f)$, the Wronskian determinant of $a_1, a_2, \ldots, a_k$, f i.e.,*

$$L(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1' & a_2' & . & . & . & a_k' & f' \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$

Definition 1.5. *If $a \in \mathbb{C} \cup \{\infty\}$,the quantity*

$$\delta(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T_f(r)} = \liminf_{r\to\infty} \frac{m(r, a; f)}{T_f(r)}$$

is called the Nevanlinna's deficiency of the value "a".

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf [3],.*p*.43). If in particular, $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

In this connection the following two definitions are also relevant :

Definition 1.6. [1] *A non-constant entire function f is said have the property (A) if for any $\delta > 1$ and for all large r, $[M_f(r)]^2 \leq M_f(r^\delta)$ holds. For exapmles of functions with or without the Property (A), one may see* [1].

Definition 1.7. *Two entire functions g and h are said to be asymptotically equivalent if there exists l $(0 < l < \infty)$ such that*

$$\frac{M_g(r)}{M_h(r)} \to l \text{ as } r \to \infty$$

and in this case we write $g \sim h$. Clearly if $g \sim h$ then $h \sim g$.

In this paper we establish some newly developed results based on the growth properties of relative order and relative lower order of wronskians generated by entire and meromorphic functions. We do not explain the standard notations and definitions in the theory of entire and meromorphic functions because those are available in [3] and [5].

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [1] *Let g be an entire function and $\alpha > 1, 0 < \beta < \alpha$. Then*

$$M_g(\alpha r) > \beta M_g(r) \text{ for all sufficiently large } r.$$

Lemma 2.2. [1] *Let f be an entire function which satisfies the Property (A). Then for any positive integer n and for all large r,*

$$[M_f(r)]^n \leq M_f(r^\delta)$$

holds where $\delta > 1$.

Lemma 2.3. *Let g be entire. Then for all sufficiently large values of r,*

$$T_g(r) \leq \log M_g(r) \leq 3T_g(2r) \ .$$

Lemma 3 *follows from Theorem* 1.6 $\{$*cf. p.18,* [3]$\}$ *on putting $R = 2r$.*

Lemma 2.4. [2] *If f be a transcendental meromorphic function with the maximum deficiency sum and g be a transcendental entire function of regular growth having non zero finite order and $\sum_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$, then the relative order and relative lower order of $L(f)$ with respect to $L(g)$ are same as those of f with respect to g i.e.,*

$$\rho_{L[g]}(L[f]) = \rho_g(f) \text{ and } \lambda_{L[g]}(L[f]) = \lambda_g(f) \ .$$

Lemma 2.5. *Let g and h be any two transcendental entire functions of regular growth having non zero finite order with* $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ *and* $\sum\limits_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ *respectively. Then for any transcendental meromorphic function f with the maximum deficiency sum,*

$$\rho_{L[g]}(L[f])=\rho_{L[h]}(L[f])$$

and

$$\lambda_{L[g]}(L[f])=\lambda_{L[h]}(L[f]) \ ,$$

if g and h have the Property (A) and $g\sim h$.

Proof. Let $\varepsilon>0$ be arbitrary.
Now we get from Definition 1.7 and Lemma 2.1 for all sufficiently large values of r that

$$M_g(r)<(l+\varepsilon)M_h(r)\leq M_h(\alpha r) \ , \tag{1}$$

where $\alpha>1$ is such that $l+\varepsilon<\alpha$.
Now from Lemma 2.3 and in view of Definition 1.1, we obtain for all sufficiently large values of r that

$$\begin{aligned} T_f(r) &\leq T_g\left[(r)^{(\rho_g(f)+\varepsilon)}\right] \\ i.e.,\ T_f(r) &\leq \log M_g\left[(r)^{(\rho_g(f)+\varepsilon)}\right] \ . \end{aligned}$$

Therefore in view of (1), for any $\delta>1$ it follows from above by using Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned} T_f(r) &\leq \frac{1}{3}\log\left[M_h\left[(\alpha r)^{(\rho_g(f)+\varepsilon)}\right]\right]^3 \\ i.e.,\ T_f(r) &\leq \frac{1}{3}\log M_h\left[(\alpha r)^{\delta(\rho_g(f)+\varepsilon)}\right] \\ i.e.,\ T_f(r) &\leq T_h\left[(2\alpha r)^{\delta(\rho_g(f)+\varepsilon)}\right] \\ i.e.,\ \frac{\log T_h^{-1}T_f(r)}{\log r} &\leq \delta(\rho_g(f)+\varepsilon)\frac{\log(2\alpha r)}{\log r} \ . \end{aligned}$$

Letting $\delta\to 1+$ we get from above that

$$\rho_h(f)\leq\rho_g(f) \ . \tag{2}$$

Since $h\sim g$, we also obtain that

$$\rho_g(f)\leq\rho_h(f) \ . \tag{3}$$

Now in view of Lemma 2.4 we obtain from (2) and (3) that

$$\rho_{L[g]}(L[f])=\rho_{L[h]}(L[f]) \ .$$

Similarly we have

$$\lambda_{L[g]}(L[f])=\lambda_{L[h]}(L[f]) \ .$$

Thus the lemma follows. □

3. **Theorems.**

In this section we present the main results of the paper.

Theorem 3.1. *Suppose f be a transcendental meromorphic function having the maximum deficiency sum. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ and g be any entire function such that $0<\lambda_h(f\circ g)\leq\rho_h(f\circ g)<\infty$ and $0<\lambda_h(f)\leq\rho_h(f)<\infty$ .Then*

$$\frac{\lambda_h(f\circ g)}{\rho_h(f)}\leq\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{L[h]}^{-1}T_{L[f]}(r)}\leq\frac{\lambda_h(f\circ g)}{\lambda_h(f)}$$

$$\leq\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{L[h]}^{-1}T_{L[f]}(r)}\leq\frac{\rho_h(f\circ g)}{\lambda_h(f)}\ .$$

Proof. From the definition of $\rho_h(f)$ and $\lambda_h(f\circ g)$ and Lemma 2.4 we have for arbitrary positive ε and for all sufficiently arge values of r that

$$\log T_h^{-1}T_{f\circ g}(r)\geqslant(\lambda_h(f\circ g)-\varepsilon)\log r \tag{4}$$

and

$$\log T_{L[h]}^{-1}T_{L[f]}(r)\leq\left(\rho_{L[h]}(L[f])+\varepsilon\right)\log r$$

$$i.e.,\ \log T_{L[h]}^{-1}T_{L[f]}(r)\leq(\rho_h(f)+\varepsilon)\log r\ . \tag{5}$$

Now from (4) and (5) it follows for all sufficiently large values of r that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{L[h]}^{-1}T_{L[f]}(r)}\geqslant\frac{(\lambda_h(f\circ g)-\varepsilon)\log r}{(\rho_h(f)+\varepsilon)\log r}\ .$$

As $\varepsilon\,(>0)$ is arbitrary , we obtain that

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{L[h]}^{-1}T_{L[f]}(r)}\geqslant\frac{\lambda_h(f\circ g)}{\rho_h(f)}\ . \tag{6}$$

Again for a sequence of values of r tending to infinity ,

$$\log T_h^{-1}T_{f\circ g}(r)\leq(\lambda_h(f\circ g)+\varepsilon)\log r \tag{7}$$

and for all sufficiently large values of r ,

$$\log T_{L[h]}^{-1}T_{L[f]}(r)\geqslant\left(\lambda_{L[h]}(L[f])-\varepsilon\right)\log r$$

$$i.e.,\ \log T_{L[h]}^{-1}T_{L[f]}(r)\geqslant(\lambda_h(f)-\varepsilon)\log r\ . \tag{8}$$

Combining (7) and (8) we get for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{L[h]}^{-1}T_{L[f]}(r)}\leq\frac{(\lambda_h(f\circ g)+\varepsilon)\log r}{(\lambda_h(f)-\varepsilon)\log r}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{L[h]}^{-1}T_{L[f]}(r)}\leq\frac{\lambda_h(f\circ g)}{\lambda_h(f)}\ . \tag{9}$$

Also for a sequence of values of r tending to infinity,

$$\log T_{L[h]}^{-1} T_{L[f]}(r) \leq \left(\lambda_{L[h]}(L[f]) + \varepsilon\right) \log r$$

$$i.e., \ \log T_{L[h]}^{-1} T_{L[f]}(r) \leq (\lambda_h(f) + \varepsilon) \log r \ . \tag{10}$$

Now from (4) and (10) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[f]}(r)} \geq \frac{(\lambda_h(f\circ g) - \varepsilon) \log r}{(\lambda_h(f) + \varepsilon) \log r} \ .$$

As $\varepsilon\,(> 0)$ is arbitrary, we get from above that

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[f]}(r)} \geq \frac{\lambda_h(f\circ g)}{\lambda_h(f)} \ . \tag{11}$$

Also for all sufficiently large values of r ,

$$\log T_h^{-1} T_{f\circ g}(r) \leq (\rho_h(f\circ g) + \varepsilon) \log r \ . \tag{12}$$

Now it follows from (8) and (12) for all sufficiently large values of r that

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[f]}(r)} \leq \frac{(\rho_h(f\circ g) + \varepsilon) \log r}{(\lambda_h(f) - \varepsilon) \log r} \ .$$

Since $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[f]}(r)} \leq \frac{\rho_h(f\circ g)}{\lambda_h(f)} \ . \tag{13}$$

Thus the theorem follows from $(6), (9), (11)$ and (13). □

The following theorem can be proved in the line of Theorem 3.1 and so its proof is omitted.

Theorem 3.2. *Let g be a transcendental entire function with $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a\neq\infty} \delta(a;h) + \delta(\infty;h) = 2$ and f be any meromorphic function such that $0 < \lambda_h(f\circ g) \leq \rho_h(f\circ g) < \infty$ and $0 < \lambda_h(g) \leq \rho_h(g) < \infty$. Then*

$$\frac{\lambda_h(f\circ g)}{\rho_h(g)} \leq \liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[g]}(r)} \leq \frac{\lambda_h(f\circ g)}{\lambda_h(g)}$$

$$\leq \limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[g]}(r)} \leq \frac{\rho_h(f\circ g)}{\lambda_h(g)} \ .$$

Theorem 3.3. *Suppose f be a transcendental meromorphic function with $\sum_{a\neq\infty} \delta(a;f)$ $+\delta(\infty;f) = 2$. Also let g be entire and h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a\neq\infty} \delta(a;h) + \delta(\infty;h) = 2$, $0 < \rho_h(f\circ g) < \infty$ and $0 < \rho_h(f) < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[f]}(r)} \leq \frac{\rho_h(f\circ g)}{\rho_h(f)} \leq \limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[f]}(r)} \ .$$

Proof. From the definition of $\rho_{L[h]}\left(L[f]\right)$ and in view of Lemma 2.4 we get for a sequence of values of r tending to infinity that

$$\log T_{L[h]}^{-1} T_{L[f]}\left(r\right) \geqslant \left(\rho_{L[h]}\left(L[f]\right)-\varepsilon\right)\log r$$

$$\text{(14)} \qquad \text{i.e., } \log T_{L[h]}^{-1} T_{L[f]}\left(r\right) \geqslant \left(\rho_{h}\left(f\right)-\varepsilon\right)\log r\ .$$

Now from (12) and (14) it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_{h}^{-1} T_{f\circ g}\left(r\right)}{\log T_{L[h]}^{-1} T_{L[f]}\left(r\right)} \leq \frac{\left(\rho_{h}\left(f\circ g\right)+\varepsilon\right)\log r}{\left(\rho_{h}\left(f\right)-\varepsilon\right)\log r}\ .$$

As $\varepsilon\left(>0\right)$ is arbitrary, we obtain that

$$\text{(15)} \qquad \liminf_{r\rightarrow\infty}\frac{\log T_{h}^{-1} T_{f\circ g}\left(r\right)}{\log T_{L[h]}^{-1} T_{L[f]}\left(r\right)} \leq \frac{\rho_{h}\left(f\circ g\right)}{\rho_{h}\left(f\right)}\ .$$

Again for a sequence of values of r tending to infinity ,

$$\text{(16)} \qquad \log T_{h}^{-1} T_{f\circ g}\left(r\right) \geqslant \left(\rho_{h}\left(f\circ g\right)-\varepsilon\right)\log r\ .$$

So combining (5) and (16) we get for a sequence of values of r tending to infinity that

$$\frac{\log T_{h}^{-1} T_{f\circ g}\left(r\right)}{\log T_{L[h]}^{-1} T_{L[f]}\left(r\right)} \geqslant \frac{\left(\rho_{h}\left(f\circ g\right)-\varepsilon\right)\log r}{\left(\rho_{h}\left(f\right)+\varepsilon\right)\log r}\ .$$

Since $\varepsilon\left(>0\right)$ is arbitrary, it follows that

$$\text{(17)} \qquad \limsup_{r\rightarrow\infty}\frac{\log T_{h}^{-1} T_{f\circ g}\left(r\right)}{\log T_{L[h]}^{-1} T_{L[f]}\left(r\right)} \geqslant \frac{\rho_{h}\left(f\circ g\right)}{\rho_{h}\left(f\right)}\ .$$

Thus the theorem follows from (15) and (17). □

The following theorem can be carried out in the line of Theorem 3.3 and therefore we omit its proof.

Theorem 3.4. *Let f be meromorphic and g,h be both transcendental entire functions with the maximum deficiency sums and $0<\rho_{h}\left(f\circ g\right)<\infty$, $0<\rho_{h}\left(g\right)<\infty$. In addition, let h of regular growth having non zero finite order. Then*

$$\liminf_{r\rightarrow\infty}\frac{\log T_{h}^{-1} T_{f\circ g}\left(r\right)}{\log T_{L[h]}^{-1} T_{L[g]}\left(r\right)} \leq \frac{\rho_{h}\left(f\circ g\right)}{\rho_{h}\left(g\right)} \leq \limsup_{r\rightarrow\infty}\frac{\log T_{h}^{-1} T_{f\circ g}\left(r\right)}{\log T_{L[h]}^{-1} T_{L[g]}\left(r\right)}\ .$$

The following theorem is a natural consequence of Theorem 3.1 and Theorem 3.3 :

Theorem 3.5. *Suppose f be a transcendental meromorphic function having the maximum deficiency sum. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\neq\infty}\delta\left(a;h\right)+\delta\left(\infty;h\right)=2$ and g be any entire function such that $0<\lambda_{h}\left(f\circ g\right)\leq\rho_{h}\left(f\circ g\right)<\infty$ and $0<\lambda_{h}\left(f\right)\leq\rho_{h}\left(f\right)<\infty$. Then*

$$\liminf_{r\rightarrow\infty}\frac{\log T_{h}^{-1} T_{f\circ g}\left(r\right)}{\log T_{L[h]}^{-1} T_{L[f]}\left(r\right)} \leq \min\left\{\frac{\lambda_{h}\left(f\circ g\right)}{\lambda_{h}\left(f\right)},\frac{\rho_{h}\left(f\circ g\right)}{\rho_{h}\left(f\right)}\right\}$$

$$\leq \max\left\{\frac{\lambda_{h}\left(f\circ g\right)}{\lambda_{h}\left(f\right)},\frac{\rho_{h}\left(f\circ g\right)}{\rho_{h}\left(f\right)}\right\} \leq \limsup_{r\rightarrow\infty}\frac{\log T_{h}^{-1} T_{f\circ g}\left(r\right)}{\log T_{L[h]}^{-1} T_{L[f]}\left(r\right)}\ .$$

The proof is omitted.

Analogously one may state the following theorem without its proof.

Theorem 3.6. *Let f be meromorphic and g, h be both transcendental entire functions with the maximum deficiency sums and $0 < \lambda_h (f \circ g) \leq \rho_h (f \circ g) < \infty$, $0 < \lambda_h (g) \leq \rho_h (g) < \infty$. In addition, let h of regular growth having non zero finite order. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[g]}(r)} \leq \min\left\{\frac{\lambda_h(f\circ g)}{\lambda_h(g)}, \frac{\rho_h(f\circ g)}{\rho_h(g)}\right\}$$

$$\leq \max\left\{\frac{\lambda_h(f\circ g)}{\lambda_h(g)}, \frac{\rho_h(f\circ g)}{\rho_h(g)}\right\} \leq \limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[g]}(r)} .$$

Theorem 3.7. *Suppose f be a transcendental meromorphic function having the maximum deficiency sum. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a\neq\infty} \delta(a;h) + \delta(\infty;h) = 2$ and g be any entire function such that $\rho_h(f) < \infty$ and $\lambda_h(f\circ g) = \infty$ .Then*

$$\lim_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{L[h]}^{-1} T_{L[f]}(r)} = \infty .$$

Proof. Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\beta > 0$ such that for a sequence of values of r tending to infinity,

$$\log T_h^{-1} T_{f\circ g}(r) \leq \beta \log T_{L[h]}^{-1} T_{L[f]}(r) . \tag{18}$$

Again from the definition of $\rho_{L[g]}(L[f])$ it follows for all sufficiently large values of r and in view of Lemma 2.4 that

$$\begin{aligned} \log T_{L[h]}^{-1} T_{L[f]}(r) &\leq \left(\rho_{L[h]}(L[f]) + \varepsilon\right) \log r \\ i.e.,\ \log T_{L[h]}^{-1} T_{L[f]}(r) &\leq (\rho_h(f) + \varepsilon) \log r . \end{aligned} \tag{19}$$

Thus from (18) and (19), we have for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f\circ g}(r) &\leq \beta(\rho_h(f) + \varepsilon) \log r \\ i.e.,\ \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log r} &\leq \frac{\beta(\rho_h(f) + \varepsilon) \log r}{\log r} \\ i.e.,\ \liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log r} &= \lambda_h(f\circ g) < \infty. \end{aligned}$$

This is a contradiction.
Hence the theorem follows. □

Remark 3.8. *Theorem 3.7 is also valid with "limit superior" instead of "limit" if $\lambda_h(f\circ g) = \infty$ is replaced by $\rho_h(f\circ g) = \infty$ and the other conditions remain the same.*

Corollary 3.9. *Under the assumptions of Theorem 3.7 and Remark 3.8,*

$$\lim_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_{L[h]}^{-1} T_{L[f]}(r)} = \infty \text{ and } \limsup_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_{L[h]}^{-1} T_{L[f]}(r)} = \infty .$$

respectively hold.

The proof is omitted.

Analogously one may also state the following theorem and corollaries without their proofs as those may be carried out in the line of Remark 3.8, Theorem 3.7 and Corollary 3.9 respectively.

Theorem 3.10. *Let g be a transcendental entire function with $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ and f be any meromorphic function such that $\rho_h(g)<\infty$ and $\rho_h(f\circ g)=\infty$ .Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{L[h]}^{-1}T_{L[g]}(r)}=\infty\ .$$

Corollary 3.11. *Theorem 3.10 is also valid with "limit" instead of "limit superior" if $\rho_h(f\circ g)=\infty$ is replaced by $\lambda_h(f\circ g)=\infty$ and the other conditions remain the same.*

Corollary 3.12. *Under the assumptions of Theorem 3.7 and Corollary 3.11,*

$$\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[g]}(r)}=\infty \text{ and } \lim_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{L[h]}^{-1}T_{L[g]}(r)}=\infty$$

respectively hold.

Theorem 3.13. *Suppose f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)$ $+\delta(\infty;f)=2$. Also let h be a transcendental entire function of regular growth having non zero finite order with the maximum deficiency sum and g be any entire function such that $0<\rho_h(f\circ g)<\infty$ and $0<\rho_h(f)<\infty$ and $g\sim h$. Then*

$$\liminf_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_{L[h]}^{-1}T_{L[f]}(r)}\le 1\le\limsup_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_{L[h]}^{-1}T_{L[f]}(r)}\ .$$

Proof. From the definition of $\rho_g(f)$ we get for all sufficiently large values of r that

$$\log T_g^{-1}T_f(r)\le(\rho_g(f)+\varepsilon)\log r \tag{20}$$

and for a sequence of values of r tending to infinity that

$$\log T_g^{-1}T_f(r)\ge(\rho_g(f)-\varepsilon)\log r\ . \tag{21}$$

Now from (14) and (20) it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_g^{-1}T_f(r)}{\log T_{L[h]}^{-1}T_{L[f]}(r)}\le\frac{(\rho_g(f)+\varepsilon)\log r}{(\rho_h(f)-\varepsilon)\log r}\ .$$

As $\varepsilon\,(>0)$ is arbitrary we obtain that

$$\liminf_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_{L[h]}^{-1}T_{L[f]}(r)}\le\frac{\rho_g(f)}{\rho_h(f)}\ . \tag{22}$$

Now as $g\sim h$, in view of Lemma 2.4 and Lemma 2.5 we obtain from (22) that

$$\liminf_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_{L[h]}^{-1}T_{L[f]}(r)}\le 1\ . \tag{23}$$

Again combining (5) and (21) we get for a sequence of values of r tending to infinity that

$$\frac{\log T_g^{-1} T_f(r)}{\log T_{L[h]}^{-1} T_{L[f]}(r)} \geqslant \frac{(\rho_g(f) - \varepsilon)\log r}{(\rho_h(f) + \varepsilon)\log r} .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_{L[h]}^{-1} T_{L[f]}(r)} \geqslant \frac{\rho_g(f)}{\rho_h(f)} . \tag{24}$$

Now as $g \sim h$, in view of Lemma 2.4 and Lemma 2.5 we obtain from (24) that

$$\limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_{L[h]}^{-1} T_{L[f]}(r)} \geqslant 1 . \tag{25}$$

Thus the theorem follows from (23) and (25). □

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