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FIXED POINT RESULTS FOR $\phi-(\gamma,\eta,n,m)-$CONTRACTIONS WITH APPLICATIONS TO NONLINEAR INTEGRAL EQUATIONS

HASANEN A. HAMMAD[1] AND MANUEL DE LA SEN[2,*]

[1] *Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt*
https://orcid.org/0000-0001-8724-9367

[2] *Institute of Research and Development of Processes University of the Basque Country 48940- Leioa (Bizkaia), Spain*

[*] *Corresponding author:* manuel.delasen@ehu.eus

Abstract. The aim of this paper is to introduce a new class of pair of contraction mappings, called $\phi-(\gamma,\eta,n,m)$-contraction pairs, and obtain common fixed point theorems for a pair of mappings in this class, satisfying a weakly compatible condition. As an application, we use mappings of this class to find the existence of solutions for nonlinear integral equations on the space of continuous functions and in some of its subspaces. Moreover, some examples are given here to illustrate the applicability of these results.

1. Introduction and preliminaries

The Banach contraction mapping plays an important role in solving nonlinear problems. Then a lot of publications are devoted to the study and solutions of many practical and theoretical problems by using this condition ([1]- [7]).

Continuity in this line, we establish some common fixed point results for a class of contraction mappings wherein contractive inequality is controlled by a positive function satisfying a stability condition at 0. After that, we use the class of mappings to establish the existence and uniqueness results for solutions of nonlinear

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integral equations. Finally, we present some concentrated examples to illustrate the usability of the obtained results.

Definition 1.1. *Let (X, d) be a metric space. A pair of self-mappings (S, T) is said to be:*
i- compatible [8] iff $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, *whenever $\{x_n\} \subset X$ is such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = r \text{ for some } r \in X.$$

ii- non-compatible [9] if there exists at least one sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = r$ *for some $r \in X$ such that* $\lim_{n\to\infty} d(STx_n, TSx_n)$ *is either non-existent or nonzero.*
iii- satisfies the property (E.A) [10] if there exists a sequence $\{x_n\} \subset X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = r \text{ for some } r \in X.$$

iv-satisfies the common limit in the range of T property (CLR_T) [11] if there exists a sequence $\{x_n\} \subset X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tr \text{ for some } r \in X.$$

v- satisfies non-trivially weakly compatible (WC) condition [12] if they commute at their coincidence points, whenever the set of coincidences is nonempty.

Definition 1.2. *A point $x \in X$ is called:*
1- a coincidence point (CP) of S and T if $Sx = Tx$, the set of coincidence points of S and T will be denoted by $C(S, T)$ and if $x \in C(S, T)$, then $w = Sx = Tx$ is called a point of coincidence (POC) of S and T.
2- common fixed point of S and T if $Sx = Tx = x$.

Remark 1.1. *It may be noted that:*
1- non-trivial weak compatibility is a necessary, hence minimal condition for the existence of common fixed points of contractive type mapping pairs.
2- commutativity at coincidence points of S and T is a coincidence point, whenever x is a coincidence point of S and T.
3- non-trivially weakly compatible mappings may equivalently be called as coincidence preserving mappings.
4- compatible mappings are necessarily coincidence preserving since compatible mappings commute at each coincidence points. However, the converse need not be true.

The following lemma will be used to prove our results:

Lemma 1.1. *[13] Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence in X, then there exist $\varepsilon > 0$ and sequences of integers positive $(m(k))$ and $(n(k))$ with

$$m(k) > n(k) > k$$

such that,

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon,\ \ d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$$

and

(1) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon$,

(2) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon$,

(3) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon$.

2. $\phi - (\gamma, \eta, n, m)$-contraction pairs and their coincidence point

As in [3], we will use functions $\gamma, \eta : \mathbb{R}^+ \to [0, 1)$ satisfying that $\gamma(r) + \eta(r) < 1$ for $r \in \mathbb{R}^+$ and

$$\begin{cases} \limsup\limits_{s\to 0^+} \gamma(s) < 1, \\ \limsup\limits_{s\to r^+} \frac{\eta(s)}{1-\gamma(s)} < 1, \quad \forall r > 0. \end{cases} \tag{2.1}$$

Now, we present our contraction mappings.

Definition 2.1. *Let (X, d) be a metric space and let $S, T : X \to X$ be mappings. The pair (S, T) is called a $\phi - (\gamma, \eta, n, m)$-contraction pair if for all $x, y \in X$*

$$\phi(d(Sx, Sy)) \leq \gamma(d(Tx, Ty))\phi(n(x, y)) + \eta(d(Tx, Ty))\phi(m(x, y)),$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying that

$$\phi(r_n) \to 0 \iff r_n \to 0, \tag{2.2}$$

and

$$\begin{aligned} n(x, y) &= \max\left\{d(Tx, Ty), \frac{d(Sx, Tx).d(Sy, Ty)}{1 + d(Tx, Ty)}\right\}, \\ m(x, y) &= \min\left\{d(Sx, Tx), d(Sy, Ty), d(Sy, Tx), d(Sx, Ty)\right\}. \end{aligned} \tag{2.3}$$

Proposition 2.1. *Let $S, T : X \to X$ be mappings on a metric space X with $S(X) \subset T(X)$. If the pair (S, T) is a $\phi - (\gamma, \eta, n, m)$-contraction pair, then for any $x_\circ \in X$, a sequence $\{y_n\}$ defined by*

$$y_n = Sx_n = Tx_{n+1} \quad n = 0, 1, ..$$

satisfies:

(i) $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$,

(ii) *$\{y_n\} \subset X$ is a Cauchy sequence in $T(X)$.*

Proof. Let $x_\circ \in X$ be an arbitrary point. Since $S(X) \subset T(X)$, there exists $x_1 \in X$ such that $Sx_\circ = Tx_1$. By continuing this process inductively we get a sequence $\{x_n\}$ in X such that

$$y_n = Sx_n = Tx_{n+1}.$$

Now,

$$\begin{aligned}\phi(d(Tx_{n+1}, Tx_{n+2})) &= \phi(d(Sx_n, Sx_{n+1})) \\ &\leq \gamma(d(Tx_n, Tx_{n+1}))\phi(n(x_n, x_{n+1})) \\ &\quad +\eta(d(Tx_n, Tx_{n+1})))\phi(m(x_n, x_{n+1})),\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}n(x_n, x_{n+1}) &= \max\left\{d(Tx_n, Tx_{n+1}), \frac{d(Sx_n, Tx_n).d(Sx_{n+1}, Tx_{n+1})}{1 + d(Tx_n, Tx_{n+1})}\right\} \\ &= \max\left\{d(Tx_n, Tx_{n+1}), \frac{d(Tx_{n+1}, Tx_n).d(Tx_{n+2}, Tx_{n+1})}{1 + d(Tx_n, Tx_{n+1})}\right\} \\ &\leq \max\{d(Tx_n, Tx_{n+1}), d(Tx_{n+2}, Tx_{n+1})\}\end{aligned}$$

and

$$\begin{aligned}m(x_n, x_{n+1}) &= \min\{d(Sx_n, Tx_n), d(Sx_{n+1}, Tx_{n+1}), d(Sx_{n+1}, Tx_n), d(Sx_n, Tx_{n+1})\} \\ &= \min\{d(Tx_{n+1}, Tx_n), d(Tx_{n+2}, Tx_{n+1}), d(Tx_{n+2}, Tx_n), d(Tx_{n+1}, Tx_{n+1})\} = 0.\end{aligned}$$

If $n(x_n, x_{n+1}) = d(Tx_{n+2}, Tx_{n+1})$, then from (2.4) we get

$$\phi(d(Tx_{n+1}, Tx_{n+2})) \leq \gamma(d(Tx_n, Tx_{n+1}))\phi(d(Tx_{n+2}, Tx_{n+1})) < \phi(d(Tx_{n+2}, Tx_{n+1})),$$

which is a contradiction. So $n(x_n, x_{n+1}) = d(Tx_n, Tx_{n+1})$, then from (2.4) and using the properties of the function γ, we have

$$\phi(d(Tx_{n+1}, Tx_{n+2})) \leq \gamma(d(Tx_n, Tx_{n+1}))\phi(d(Tx_n, Tx_{n+1})) < \phi(d(Tx_n, Tx_{n+1})).$$

Thus $\{v_n\} = \{\phi(d(Tx_n, Tx_{n+1}))\}$ is a decreasing sequence of positive numbers bounded below by zero, and so converges to $b \geq 0$. Now if $b > 0$, then by taking $\limsup$ on both sides of the above inequality we have a contradiction. Thus,

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \phi(d(Tx_n, Tx_{n+1})) = 0.$$

Consequently, from the stability condition at zero (2.2) we conclude that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = \lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = \lim_{n\to\infty} d(Tx_{n+1}, Tx_{n+2}) = 0. \tag{2.5}$$

This proves (i).

To prove (ii) we are going to suppose that $\{y_n\} \subset T(X)$ is not a Cauchy sequence. Then there exist an $\varepsilon > 0$ and sequences of integers positive $(m(k))$ and $(n(k))$ with $m(k) > n(k) > k$ such that,

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon,\ d(x_{m(k)-1}, x_{n(k)}) < \varepsilon.$$

From Lemma 1.1 and the continuity of ϕ we have

$$\begin{aligned} \phi(\varepsilon) &= \limsup_{k\to\infty} \phi(d(Tx_{m(k)+1}, Tx_{n(k)+1})) \\ &= \limsup_{k\to\infty} \phi(d(Sx_{m(k)}, Sx_{n(k)})) \\ &\leq \limsup_{k\to\infty} \gamma(d(Tx_{m(k)}, Tx_{n(k)}))\phi(n(x_{m(k)}, x_{n(k)})) \\ &\quad + \limsup_{k\to\infty} \eta(d(Tx_{m(k)}, Tx_{n(k)}))\phi(m(x_{m(k)}, x_{n(k)})), \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} n(x_{m(k)}, x_{n(k)}) &= \max\left\{ \begin{array}{c} d(Tx_{m(k)}, Tx_{n(k)}), \\ \frac{d(Sx_{m(k)}, Tx_{m(k)}).d(Sx_{n(k)}, Tx_{n(k)})}{1+d(Tx_{m(k)}, Tx_{n(k)})} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d(Tx_{m(k)}, Tx_{n(k)}), \\ \frac{d(Tx_{m(k)+1}, Tx_{m(k)}).d(Tx_{n(k)+1}, Tx_{n(k)})}{1+d(Tx_{m(k)}, Tx_{n(k)})} \end{array} \right\} \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} m(x_{m(k)}, x_{n(k)}) &= \min\left\{ \begin{array}{c} d(Sx_{m(k)}, Tx_{m(k)}), d(Sx_{n(k)}, Tx_{n(k)}), \\ d(Sx_{n(k)}, Tx_{m(k)}), d(Sx_{m(k)}, Tx_{n(k)}) \end{array} \right\} \\ &= \min\left\{ \begin{array}{c} d(Tx_{m(k)+1}, Tx_{m(k)}), d(Tx_{n(k)+1}, Tx_{n(k)}), \\ d(Tx_{n(k)+1}, Tx_{m(k)}), d(Tx_{m(k)+1}, Tx_{n(k)}) \end{array} \right\} \end{aligned} \tag{2.8}$$

Letting $k \to \infty$ in (2.7), (2.8) and by (2.5), we can write

$$\begin{aligned} \lim_{k\to\infty} n(x_{m(k)}, x_{n(k)}) &= \max\{\varepsilon, 0\} = \varepsilon, \\ \lim_{k\to\infty} m(x_{m(k)}, x_{n(k)}) &= \min\{0, 0, \varepsilon, \varepsilon\} = 0. \end{aligned}$$

Therefore, (2.6) is now

$$\phi(\varepsilon) \leq \limsup_{k\to\infty} \gamma(d(Tx_{m(k)}, Tx_{n(k)}))\phi(\varepsilon) < \phi(\varepsilon).$$

Which is a contradiction, hence $\{y_n\} \subset X$ is a Cauchy sequence. $\square$

Lemma 2.1. *Let S and T be self-mappings on a metric space (X, d) and the pair (S, T) is a $\phi-(\gamma, \eta, n, m)$-contraction pair. If S and T have a POC in X then it is unique.*

Proof. Let $z \in X$ be a POC of the pair (S,T). Then there exits $x \in X$ such that $Sx = Tx = z$. Suppose that $y \in X$, $Sy = Ty = u$ with $u \neq z$. Then

$$\begin{aligned} \phi(d(z,u)) &= \phi(d(Sx,Sy)) \\ &\leq \gamma(d(Tx,Ty))\phi(n(x,y)) + \eta(d(Tx,Ty))\phi(m(x,y)) \\ &\leq \gamma(d(z,u))\phi(n(x,y)) + \eta(d(z,u))\phi(m(x,y)). \end{aligned} \tag{2.9}$$

Using (2.3) we have

$$\begin{aligned} n(x,y) &= \max\left\{d(Tx,Ty), \frac{d(Sx,Tx).d(Sy,Ty)}{1+d(Tx,Ty)}\right\} \\ &= \max\left\{d(z,u), \frac{d(z,z).d(u,u)}{1+d(z,u)}\right\} = d(z,u), \end{aligned}$$

and

$$\begin{aligned} m(x,y) &= \min\{d(Sx,Tx), d(Sy,Ty), d(Sy,Tx), d(Sx,Ty)\} \\ &= \min\{d(z,z), d(u,u), d(u,z), d(z,u)\} = 0. \end{aligned}$$

Substituting it into (2.9) we get

$$\phi(d(z,u)) \leq \gamma(d(z,u))\phi(d(z,u)) < \phi(d(z,u)),$$

which is a contradiction, therefore $z = u$. □

Theorem 2.1. *Let S and T be self-mappings on a metric space (X,d) such that*

(i) $S(X) \subset T(X)$,

(ii) $T(X) \subset X$ is a complete subspace of X,

(iii) the pair (S,T) is a $\phi-(\gamma,\eta,n,m)$-contraction pair.

Then, the pair (S,T) has a unique POC.

Proof. Let $y_n = Sx_n = Tx_{n+1}$, $n = 0,1,..$ be a Cauchy sequence defined by Proposition 2.1 satisfies $\{y_n\} = \{Tx_{n+1}\} \subset T(X)$.

Since $T(X) \subset X$ is a complete subspace of X, then there exists $v \in T(X)$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_{n+1} = v,$$

thus we can find $u \in X$ such that $Tu = v$.

Now, we shall prove that $Tu = Su$, then

$$\phi(d(v,Su)) = \phi(d(Sx_{n+1},Su)) \leq \gamma(d(Tx_{n+1},Tu))\phi(n(x_{n+1},u)) + \eta(d(Tx_{n+1},Tu))\phi(m(x_{n+1},u)), \tag{2.10}$$

where

$$\begin{aligned} n(x_{n+1}, u) &= \max\left\{d(Tx_{n+1}, Tu), \frac{d(Sx_{n+1}, Tx_{n+1}).d(Su, Tu)}{1 + d(Tx_{n+1}, Tu)}\right\} \\ &= \max\left\{d(v, Tu), \frac{d(v, v).d(Su, Tu)}{1 + d(v, Tu)}\right\} = 0 \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} m(x_{n+1}, u) &= \min\{d(Sx_{n+1}, Tx_{n+1}), d(Su, Tu), d(Su, Tx_{n+1}), d(Sx_{n+1}, Tu)\} \\ &= \min\{d(v, v), d(Su, Tu), d(Su, v), d(v, Tu)\} = 0. \end{aligned} \tag{2.12}$$

Applying (2.11) and (2.12) in (2.10), we have $\phi(d(v, Su)) \leq 0$, this holds only if $d(v, Su) = 0$. So $Tu = Su = v$, therefore v is a POC of S and T.

From Lemma 2.1 we conclude that v is a unique POC. □

3. Common fixed point for $\phi - (\gamma, \eta, n, m)$-contraction pairs

In this section general common fixed point results for a pair of mappings belonging to the $\phi - (\gamma, \eta, n, m)$-contraction class, under a minimal commutativity condition are given.

Theorem 3.1. *Let S and T be self-mappings on a metric space X satisfying the conditions of Theorem 2.1, if the pair (S, T) is non-trivially weakly compatible pair, then there are a unique common fixed point of S and T.*

Proof. Since the pair (S, T) is non-trivially weakly compatible, then they commute at their unique coincidence point. Hence, $SSu = STu = TSu = TTu$, using uniqueness of the POC, we obtain that $v = Su$ is a common fixed point of (S, T).

Uniqueness of the common fixed point can be proved using the same reasoning as above. □

Now, we omit the condition $S(X) \subset T(X)$ from the above theorem and obtain the following result:

Theorem 3.2. *Let S,T : $X \to X$ be mappings on a metric space (X, d) satisfying the property (E.A). Consider the pair (S, T) is non-trivially weakly compatible $\phi - (\gamma, \eta, n, m)$-contraction pair. If $T(X) \subset X$ is closed, then S and T have a unique common fixed point.*

Proof. Since the pair (S, T) satisfies the property (E.A), there exists a sequence $\{x_n\} \subset X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = v,$$

for some $v \in X$. Since $T(X)$ is closed, so $v \in T(X)$ and $v = Tu$ for some $u \in X$. As in the proof of the Theorem 2.1, we can prove that $v = Tu = Su$ and that v is a unique POC of S and T. The existence of the unique common fixed point follows as in the proof of Theorem 3.1. □

Remark 3.1. *Since noncompatible mappings on a metric space (X, d) satisfy the property (E. A). Therefore, conclusion of Theorem 3.2 still valid if we consider S and T noncompatible mappings.*

We can replace conditions (i) and (ii) of Theorem 2.1 by a single condition and obtain the following result. Here $\overline{S(X)}$ denotes the closure of the range of the mapping S.

Theorem 3.3. *Let S and T be self-mappings on a metric space (X, d) such that*

(i) $\overline{S(X)} \subset X$ is a complete subspace of X,

(ii) the pair (S, T) is a $\phi - (\gamma, \eta, n, m)$-contraction pair.

Then the pair (S, T) has a unique POC. Furthermore, if the pair (S, T) is nontrivially weakly compatible, then S and T have a unique common fixed point.

In the next result, we drop the closeness of the range of mapping and replace the property (E. A) by CLR_T property.

Theorem 3.4. *Let (X, d) be a metric space and S,$T : X \to X$ satisfying the CLR_T property. Let us suppose that the pair (S, T) is $\phi - (\gamma, \eta, n, m)$-contraction pair. If the pair (S, T) is non-trivially weakly compatible, then S and T have a unique common fixed point.*

Proof. Since the pair (S, T) satisfies the CLR_T property, there exists a sequence $\{x_n\} \subset X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tv,$$

for some $v \in X$. The rest of the proof runs with similarities to the proof of the previous results. □

Remark 3.2. *Notice that by considering particular functions, as constants, for the functions γ, η as well as by considering $\phi = id$ (the identity mapping), or by choosing a particular form for $m(x, y)$ and $n(x, y)$ in the class of $\phi - (\gamma, \eta, n, m)$-contraction pair, we can obtain known several subclasses of mappings.*

4. Application to a class of nonlinear integral equations

In this section, we will study the existence of solutions for a class of nonlinear integral equations by using the existence of coincidence and common fixed points for mappings belonging to the $\phi - (\gamma, \eta, n, m)$-contraction class.

Let $X = C([0, T], \mathbb{R})$ denote the space of all continuous functions on $[0, T]$, it is a complete metric space equipped with the uniform metric d

$$d(u, v) = \sup_{t\in[0,T]} \{|u(t) - v(t)|\}, \ u, v \in X. \tag{4.1}$$

Now, following the idea in [4], we discuss an application of fixed point techniques to obtain the solution of the nonlinear integral equation:

$$x(t) = f_1(t) - f_2(t) + \delta \int_0^t h_1(t,s)K_1(t,s,x(s))ds + \lambda \int_0^T h_2(t,s)K_2(t,s,x(s))ds, \tag{4.2}$$

where $t \in [0,T]$, δ, λ real numbers, $f_1, f_2 \in C([0,T], \mathbb{R})$ are known, $f_1(t) \geq f_2(t)$ and K_1, K_2, h_1, h_2 are continuous real-valued functions in $[0,T] \times \mathbb{R}$.

To attain our aim, we will use some functional associated with h-concave and quasilinear functions [14]. Let C be a convex cone in the linear space X over $\mathbb{R}$ and let $L \neq 0$ be a real number. A functional $\phi : C \to \mathbb{R}$ is called *L-superadditive* on C if

$$\phi(x+y) \geq L(\phi(x) + \phi(y)), \text{ for any } x, y \in C.$$

Let G be a real non-negative function and ϕ a functional satisfying

$$\phi(tx) \leq G(t)\phi(x),$$

for any $t \geq 0$ and $x \in C$, is called G-positive homogeneous. Notice that necessarily $G(1) = 1$.

The following lemma is very important in the sequel:

Lemma 4.1. *[14] Let $u, v \in C$ and $\phi : C \to \mathbb{R}$ be a non-negative, L-superadditive and G-positive homogeneous functional on C. If $M \geq m > 0$ are such that $u - mv$ and $Mv - u \in C$, then*

$$LG(m)\phi(v) \leq \phi(u) \leq \frac{1}{L} G(M)\phi(v).$$

Now, our theorem concerned with the existence solution of system (4.2) become affordable.

Theorem 4.1. *Suppose that the following conditions are satisfied:*

(i) $\int_0^T \sup_{t\in[0,T]} |h_i(t,s)| \leq 1,\ i = \{1,2\}$,

(ii) for each $s \in [0,T]$ and for all $x, y \in X$, there is $M_i \geq 0$ such that

$$|K_i(t,s,x(s)) - K_i(t,s,y(s))| \leq M_i |x(s) - y(s)| \leq M_i \|x - y\|,\ i = \{1,2\},$$

(iii) Lemma 4.1 holds for ϕ being a non-negative, L-superadditive with $L > 0$ and G-positive homogeneous functional.

Then the integral equation (4.2) has at least one solution in X, provided that

$$|\delta| M_1 + |\lambda| M_2 = 1. \tag{4.3}$$

Proof. We define the following operators, for each $x \in X$

$$Sx(t) = -f_2(t) + \delta \int_0^t h_1(t,s)K_1(t,s,x(s))ds \tag{4.4}$$

and

$$Tx(t) = x(t) - f_1(t) - \lambda \int_0^T h_2(t,s)K_2(t,s,x(s))ds. \tag{4.5}$$

Clearly, S and T are self operators on X.

Now, for all $x, y \in X$ by using (i)-(ii), we have

$$\begin{aligned} |Sx(t) - Sy(t)| &\leq |\delta| \int_0^t |h_1(t,s)| \, |K_1(t,s,x(s)) - K_1(t,s,y(s))| \, ds \\ &\leq |\delta| \int_0^t \sup_{t\in[0,T]} |h_1(t,s)| \, |K_1(t,s,x(s)) - K_1(t,s,y(s))| \, ds \\ &\leq |\delta| \int_0^t \sup_{t\in[0,T]} |h_1(t,s)| \, M_1 \, |x(s) - y(s)| \, ds \\ &\leq |\delta| \, M_1 \, \|x - y\| \int_0^t \sup_{t\in[0,T]} |h_1(t,s)| \leq |\delta| \, M_1 \, \|x - y\| \, . \end{aligned}$$

This implies that

$$\|Sx - Sy\| = |Sx(t) - Sy(t)| \leq |\delta| \, M_1 \, \|x - y\| \, . \tag{4.6}$$

By a similar way we get

$$\begin{aligned} &\left| \lambda \int_0^T h_2(t,s)K_2(t,s,x(s))ds - \lambda \int_0^T h_2(t,s)K_2(t,s,y(s))ds \right| \\ \leq\ & |\lambda| \int_0^T |h_2(t,s)| \, |K_2(t,s,x(s)) - K_2(t,s,y(s))| \, ds \\ \leq\ & |\lambda| \int_0^T \sup_{t\in[0,T]} |h_2(t,s)| \, M_2 \, |x(s) - y(s)| \, ds \\ \leq\ & |\lambda| \, M_2 \, \|x - y\| \, . \end{aligned}$$

Consequently, it follows that

$$\begin{aligned} \|Tx - Ty\| &\geq \|x - y\| - \left| \lambda \int_0^T K_2(t,s,x(s))ds - \lambda \int_0^T K_2(t,s,y(s))ds \right| \\ &\geq (1 - |\lambda| \, M_2) \, \|x - y\| \, , \end{aligned} \tag{4.7}$$

since condition (4.3) implies that $|\lambda| \, M_2 < 1$, So (4.7) yields

$$\|x - y\| \leq \frac{1}{(1 - |\lambda| \, M_2)} \, \|Tx - Ty\| \, . \tag{4.8}$$

Using (4.6), (4.8) and condition (4.3), we obtain that

$$\|Sx - Sy\| \leq \frac{|\delta| \, M_1}{(1 - |\lambda| \, M_2)} \, \|Tx - Ty\| = \|Tx - Ty\| \, .$$

Moreover, there exists $0 \leq m < 1$ depending of x and y such that

$$m(x,y) \, \|Tx - Ty\| \leq \|Sx - Sy\| \leq \|Tx - Ty\| \, . \tag{4.9}$$

Now, by (iii), ϕ is a non-negative, continuous, 2-superadditive and G-positive homogeneous functional on the cone $\mathbb{R}^+$ satisfying (2.2). For $u = \|Sx - Sy\|$, $v = \|Tx - Ty\|$ and the inequality (4.9), the Lemma 4.1 allows us to conclude that,

$$\phi(\|Sx - Sy\|) \leq \frac{1}{2}\phi(\|Tx - Ty\|).$$

Let $\gamma, \eta : \mathbb{R}^+ \to [0, 1)$ satisfying (2.1) with $\gamma(t) \geq \frac{1}{2}$ for any $t \in \mathbb{R}^+$. Hence we obtain

$$\phi(\|Sx - Sy\|) \leq \frac{1}{2}\phi(\|Tx - Ty\|) \leq \gamma(\|Tx - Ty\|)\phi(n(x, y)) + \eta(\|Tx - Ty\|)\phi(m(x, y)).$$

Therefore, (S, T) is a $\phi - (\gamma, \eta, n, m)-$contraction pair.

Since S is a continuous mapping and X is a complete, $S(X)$ is a complete subspace of X, therefore from Theorem 3.3, the pair (S, T) has a unique POC (say $p_\circ$); i.e., $p_\circ = Sx^*(t) = Tx^*(t)$. Thus,

$$-f_2(t) + \delta \int_0^t h_1(t, s)K_1(t, s, x^*(s))ds = x^*(t) - f_1(t) - \lambda \int_0^T h_2(t, s)K_2(t, s, x^*(s))ds,$$

or equivalently,

$$x^*(t) = f_1(t) - f_2(t) + \delta \int_0^t h_1(t, s)K_1(t, s, x^*(s))ds + \lambda \int_0^T h_2(t, s)K_2(t, s, x^*(s))ds.$$

Therefore, $x^* \in X$ is a solution of the nonlinear integral equation (4.2). □

Under the notion of non-trivial weak compatibility of the pair (S, T) given in (4.4) and (4.5), the next result shows that there exists a (unique) solution of the equation (4.2) satisfying a certain integral equation.

Proposition 4.1. *Under the hypotheses of Theorem 4.1, if the pair of mappings (S, T) defined in (4.4)-(4.5) is non-trivially weakly compatible, then there is a unique solution ζ of the equation (4.2) satisfying the integral equation*

$$f_1(t) = -\lambda \int_0^T h_2(t, s)K_2(t, s, \zeta(s))ds.$$

Proof. Since the pair (S, T) is non-trivially weakly compatible, from Theorem 3.1 there is a unique solution ζ satisfying that $S\zeta(t) = T\zeta(t) = \zeta(t)$, moreover $ST\zeta(t) = TS\zeta(t)$, where

$$\begin{aligned} ST\zeta(t) &= -f_2(t) + \delta \int_0^t h_1(t, s)K_1(t, s, \zeta(s))ds, \\ TS\zeta(t) &= -f_2(t) + \delta \int_0^t h_1(t, s)K_1(t, s, \zeta(s))ds - f_1(t) - \lambda \int_0^T h_2(t, s)K_2(t, s, \zeta(s))ds. \end{aligned}$$

From this, we obtain

$$\begin{aligned} -f_2(t) + \delta \int_0^t h_1(t, s)K_1(t, s, \zeta(s))ds &= -f_2(t) + \delta \int_0^t h_1(t, s)K_1(t, s, \zeta(s))ds \\ &\quad -f_1(t) - \lambda \int_0^T h_2(t, s)K_2(t, s, \zeta(s))ds. \end{aligned}$$

This implies that

$$f_1(t) = -\lambda \int_0^T h_2(t, s)K_2(t, s, \zeta(s))ds.$$

This completes the proof. □

Remark 4.1. *In view of the proof of Proposition 4.1, one can observe that the only solution which satisfies the equation* $f_1(t) = -\lambda \int_0^T h_2(t,s)K_2(t,s,\zeta(s))ds$, *is a unique common fixed point of the pair* (S,T) *defined in (4.4) and (4.5).*

5. The equation (4.2) on a compact subspace of (X,d)

In that follows by (μ, d) we denote a compact subspace of X endowed with the induced uniform metric d defined in (4.1).

To establish the existence result in this case, we will use the operator S given in (4.4) and the next auxiliary mapping:

$$Hx(t) = \kappa x(t) - f_1(t) - \lambda \int_0^T h_2(t,s)K_2(t,s,\zeta(s))ds, \ 0 \leq \kappa < 1. \tag{5.1}$$

Theorem 5.1. *Under conditions (i)-(iii) of Theorem 4.1, if* S, H *defined in (4.4) and (5.1) are non-trivially weakly compatible self-mappings of* (μ, d), *then for all* $x \in \mu$ *the equation (4.2) has a unique solution* $\zeta \in \mu$ *satisfying*

$$f_1(t) = -\lambda \int_0^T h_2(t,s)K_2(t,s,\zeta(s))ds,$$

provided that

$$|\delta| M_1 + |\lambda| M_2 = \kappa.$$

holds.

Proof. We claim that (S,H) has the property (E. A) if it is non-trivially weakly compatible. In fact, let $\zeta_n \to \zeta$ a sequence of functions on μ converging to ζ, where the function ζ is a unique point of coincidence of the weakly compatible pair (S,H). From the continuity of the function $K_i(t,s)$ we have

$$\begin{aligned} \lim_{n\to\infty} S\zeta_n(t) &= -f_2(t) + \delta \int_0^t h_1(t,s)K_1(t,s,\lim_{n\to\infty}\zeta_n(s))ds = S\zeta(t), \\ \lim_{n\to\infty} H\zeta_n(t) &= q\zeta_n(t) - f_1(t) - \lambda \int_0^T h_2(t,s)K_2(t,s,\lim_{n\to\infty}\zeta_n(s))ds = H\zeta(t). \end{aligned}$$

Then, we conclude that (S,H) has the property (E. A).

On the other hand, it is easy to check that the operator H is continuous on (μ, d). Since μ is a compact and Hausdorff space, the *Closed Map Lemma* implies that $H(\mu)$ is a closed. Thus, from Theorem 3.2, H and S have a unique common fixed point $\zeta \in \mu$. The existence of a unique solution satisfying the above relation is obtained from the proof of Theorem 4.1, replacing the mapping T by H. The representation for the solution follows from the proof of Proposition 4.1, upon replacing T by H. □

6. The equation (4.2) on non-complete metric space

The existence Theorem 4.1 was proved by applying Theorem 3.3, since $\overline{S(X)}$ is a complete subspace. However, if equation (4.2) is posed in a non-complete metric subspace (χ, d) of (X, d), we are not able to apply such theorem. By imposing an extra condition we obtain the following existence result for this case.

Theorem 6.1. *Suppose the following conditions are satisfied:*

(i) $\int\limits_0^T \sup_{t\in[0,T]} |h_i(t,s)| \leq 1,\ i = \{1,2\}$,

(ii) for each $s \in [0,T]$ *and for all* $x, y \in \chi$, *there is* $M_i \geq 0$ *such that*

$$|K_i(t,s,x(s)) - K_i(t,s,y(s))| \leq M_i \left|x(s) - y(s)\right|,\ i = \{1,2\},$$

(iii) $\lambda \int_0^T h_2(t,s) K_2\left(t, s, \delta \int_0^s h_1(s,\kappa) K_1(\kappa, s, x(\kappa)) d\kappa + f_1(s) - f_2(s)\right) ds = 0$.

Then the integral equation (4.2) has a unique solution $\zeta \in \chi$, *satisfying*

$$f_1(t) = -\lambda \int_0^T h_2(t,s) K_2(t,s,\zeta(s)) ds,$$

provided that

$$|\delta|\, M_1 + |\lambda|\, M_2 = 1.$$

Proof. From the proof of Theorem 4.1, it is sufficient to show that the pair (S,T) defined in (4.4)-(4.5) has a POC in χ. To obtain this, we will apply Theorem 2.1, thus we prove that $S(\chi) \subseteq T(\chi)$.

In fact, adopting the same reasoning as in [4], by assumption (iii), for $x(t) \in \chi$ we have

$$\begin{aligned}
T(Sx(t) + f_1(t)) &= Sx(t) + f_1(t) - f_1(t) - \lambda \int_0^T h_2(t,s) K_2(t,s,Sx(s) + f_1(s)) ds \\
&= Sx(t) - \lambda \int_0^T h_2(t,s) K_2\left(t, s, \delta \int_0^s h_1(s,\kappa) K_1(\kappa,s,x(\kappa)) d\kappa + f_1(s) - f_2(s)\right) ds \\
&= Sx(t).
\end{aligned}$$

Thus, from Theorem 2.1 S and T have a unique POC, so all coincidence point related with the POC is a solution of the integral equation (4.2) in χ. As the proof of Proposition 4.1, the formula for the solution is a consequence of the non-trivially weakly compatibility and the existence of a unique common fixed point. □

7. Illustrative examples

In this section we are going to consider some nonlinear integral equations on $C([0,1], \mathbb{R})$ defined in (4.2). The existence of solutions will be established as an application of the previous results.

Example 7.1. *Let us consider the following nonlinear integral equation:*

$$\begin{aligned}
x(t) &= f_1(t) - f_2(t) + \frac{5}{4} \int_0^t (\frac{4s}{5})(\frac{5}{4} t x(s)) ds + \frac{5}{4} \int_0^1 (\frac{5ts}{6})(\frac{-6t^2}{5} x(s)) ds \\
&= f_1(t) - f_2(t) + \frac{5t}{4} \int_0^t s x(s) ds - \frac{5t^3}{4} \int_0^1 s x(s) ds, \quad t \in [0,1]. \qquad (7.1)
\end{aligned}$$

Taking $\delta = \lambda = \frac{5}{4}$, the kernels functions $K_i(t,s,x(s))$ and $h_i(t,s)$, $i \in \{1,2\}$ given by

$$K_1(t,s,x(s)) = \frac{5}{4}tx(s),\ K_2(t,s,x(s)) = \frac{-6t^2}{5}x(s),\ h_1(t,s) = \frac{4s}{5},\ h_2(t,s) = \frac{5ts}{6}.$$

Notice that the functions $K_i(t,s,x(s))$, $i \in \{1,2\}$ satisfy

$$|K_i(t,s,x(s)) - K_i(t,s,y(s))| \leq \frac{5}{4}|x(s) - y(s)|,$$

for all $x, y \in C([0,1],\mathbb{R})$, and the functions $h_i(t,s)$ satisfy

$$\int_0^1 \sup_{t\in[0,1]} |h_i(t,s)| \leq 1, \quad i = \{1,2\}.$$

Thus, Theorem 4.1 guarantees that the equation (7.1) has at least one solution, hence, the solution is the CP of the mappings S and T, which defined as follows:

$$\begin{aligned} Sx(t) &= -f_2(t) + \frac{5}{4}\int_0^t tsx(s)ds, \\ Tx(t) &= x(t) - f_1(t) + \frac{5}{4}\int_0^1 t^3 sx(s)ds. \end{aligned}$$

Now, let $\varkappa$ be a coincidence point of (S,T), and we assume that the following system is satisfied

$$\begin{cases} \varkappa(t) = f_1(t) - f_2(t), \\ \frac{5}{4}\int_0^1 t^3 s\varkappa(s)ds = \frac{5}{4}\int_0^t ts\varkappa(s)ds, \end{cases} \quad \text{for all } t \in [0,1]. \tag{7.2}$$

Since $t = 0$ obviously holds, we assume $t \neq 0$. Notice that the second equality of the system is equivalent to

$$t^2\int_0^1 s\varkappa(s)ds = \int_0^t s\varkappa(s)ds.$$

Differentiating with respect to t, equality above is equivalent to

$$2\int_0^1 s\varkappa(s)ds = \varkappa(t).$$

That means, the constant functions are the only coincidence point of (T,S) satisfying (7.2), provided $f_1(t) - f_2(t)$ is also constant. Let $\varkappa(t) = \zeta \in \mathbb{R}$, we obtain

$$\begin{aligned} S\zeta &= -f_2(t) + \frac{5}{4}\int_0^t ts\zeta ds = -f_2(t) + \frac{5}{8}t^3\zeta, \\ T\zeta &= \zeta - f_1(t) + \frac{5}{4}\int_0^1 t^3 s\zeta ds = \zeta - f_1(t) + \frac{5}{8}t^3\zeta. \end{aligned}$$

Therefore, the system (7.1) has a solution at the constant function $\varkappa(t) = \zeta$.

On the other hand, notice that the pair (S,T) is not weakly compatible. In fact,

$$ST\zeta = SS\zeta = -f_2(t) + \frac{5}{4}\int_0^t ts\left(-f_2(s) + \frac{5}{8}s^3\zeta\right)ds = -f_2(t) + \frac{5\zeta}{32}t^6 - \frac{5t}{4}\int_0^t sf_2(s)ds$$

and

$$TS\zeta = -f_2(t) - f_1(t) + \frac{45\zeta}{32}t^3 - \frac{4t^3}{5}\int_0^1 sf_2(s)ds = \zeta + \frac{45\zeta}{32}t^3 - \frac{4t^3}{5}\int_0^1 sf_2(s)ds.$$

Therefore, the solution $\varkappa(t)=\zeta$ does not satisfy the integral equation given in Proposition 4.1.

Example 7.2. *We will consider the following nonlinear integral equation:*

$$\begin{cases} x(t)=\frac{(\sin(e^2)-\sin(e))e^{-t}}{e+1}-\frac{e^{2t+2}-e^{t+2}}{e^2+e}+e^{t+1}+\frac{2}{(e^2+e)}\int_0^t e^{2t-2s+1}\frac{x(s)}{2}ds \\ \qquad +\frac{1}{e+1}\int_0^1 e^{s-t+1}\cos(x(s))ds, \quad t\in[0,1]. \end{cases} \tag{7.3}$$

Equation (7.3) is of the form (4.2), for

$$\begin{aligned} K_1(t,s,x(s)) &= e^{t+1}\frac{x(s)}{2},\ K_2(t,s,x(s))=e\cos(x(s)),\ h_1(t,s)=e^{t-2s},\ h_2(t,s)=e^{s-t}, \\ f_1(t) &= \frac{(\sin(e^2)-\sin(e))e^{-t}}{e+1},\ f_2(t)=\frac{e^{2t+2}-e^{t+2}}{e^2+e}-e^{t+1}, \end{aligned}$$

and $\delta=\frac{2}{e^2-e}$, $\lambda=-\frac{1}{e+1}$. Notice that $M_1=\frac{e}{2}$, $M_2=e$ and $|\delta|\,M_1+|\lambda|\,M_2=1$.

Let the mappings (S,T) given in this case by

$$\begin{aligned} Sx(t) &= e^{t+1}-\frac{e^{2t+2}-e^{t+2}}{e^2+e}+\frac{2}{(e^2+e)}\int_0^t e^{2t-2s+1}\frac{x(s)}{2}ds, \\ Tx(t) &= x(t)-\frac{(\sin(e^2)-\sin(e))e^{-t}}{e+1}+\frac{1}{e+1}\int_0^1 e^{s-t+1}\cos(x(s))ds. \end{aligned}$$

We are going to find the coincidence points of (S,T). A point $\varkappa(t)\in X$ is a CP of (S,T) if

$$\begin{aligned} & e^{2t}\left[e^{1-t}-\frac{e^2-e^{2-t}}{(e^2+e)}+\frac{2e}{(e^2+e)}\int_0^t e^{-2s}\frac{\varkappa(s)}{2}ds\right] \\ = \quad & e^{-t}\left[\varkappa(t)e^t-\frac{\sin(e^2)-\sin(e)}{e+1}+\frac{e}{e+1}\int_0^1 e^s\cos(\varkappa(s))ds\right], \end{aligned}$$

equivalently,

$$e^{3t}\left[e^{1-t}-\frac{e^2-e^{2-t}}{(e^2+e)}+\frac{2e}{(e^2+e)}\int_0^t e^{-2s}\frac{\varkappa(s)}{2}ds\right]=\varkappa(t)e^t-\frac{\sin(e^2)-\sin(e)}{e+1}+\frac{e}{e+1}\int_0^1 e^s\cos(\varkappa(s))ds. \tag{7.4}$$

Since the term

$$-\frac{\sin(e^2)-\sin(e)}{e+1}+\frac{e}{e+1}\int_0^1 e^s\cos(\varkappa(s))ds$$

is constant and the left side of equality (7.4) depends of t, necessarily we have that

$$\frac{\sin(e^2)-\sin(e)}{e+1}=\frac{e}{e+1}\int_0^1 e^s\cos(\varkappa(s))ds,$$

or equivalently,

$$\sin(e^2)-\sin(e)=\int_0^1 e^{s+1}\cos(\varkappa(s))ds,$$

whose solution is $\varkappa(t)=e^{t+1}$. Notice that equality (7.4) is satisfied for this function and

$$Se^{t+1}=Te^{t+1}=e^{t+1}, \qquad TSe^{t+1}=STe^{t+1}=e^{t+1}.$$

Therefore, $\varkappa(t) = e^{t+1}$ is a unique coincidence and fixed point of (T, S). Also the pair (T, S) is non-trivially weakly compatible, so from the Proposition 4.1, equation (7.3) has a solution satisfying the integral equation

$$\begin{aligned} f_1(t) &= -\lambda \int_0^1 h_2(t,s) K_2(t,s,\zeta(s)) ds \\ \frac{(\sin(e^2) - \sin(e))e^{-t}}{e+1} &= \frac{1}{e+1} \int_0^1 e^{s-t+1} \cos(x(s)) ds, \end{aligned}$$

whose solution is $x(t) = e^{t+1}$.

Question.

In Theorem 4.1, we consider ϕ is 2-superadditive i.e., $L = 2 > 0$. Are the results still true if we take $L < 0$?

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