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LINEAR FUNCTIONALS CONNECTED WITH STRONG DOUBLE CESARO SUMMABILITY

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Abstract. D. Borwein characterized linear functionals on the normed linear spaces w_p and W_p. In this paper we extend his results by presenting definitions for the double strong Cesaro mean. Using these new notions of strongly p-Cesaro summable double sequence and strongly p-Cesaro summable bivariate function we present extensions of D. Borwein's results.

1. Introduction

The first definitions and investigations of the convergence of double sequences are usually atributted to F. Pringsheim [12], who studied such sequences and series more than hundred years ago. Pringsheim defined what we call the P limit and gave examples of convergence (P convergence) of double sequences with and without the usual convergence of rows and columns. G. H. Hardy [4], considered in more details the case of convergence of double sequences where, besides the existence of the P limit, rows and columns converge. F. Moricz [6–8] discovered an alternative approach to the Hardy convergence, which significantly influenced the whole theory.

The following notion of convergence for double sequences was presented by Pringsheim in [11]. A double sequence $x = \{x_{nm}\}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$,

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there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{nm} - L| < \varepsilon$, whenever $n, m > N_\varepsilon$. In this case we denote such limit as follow:

$$P - \lim_{n,m\to\infty} x_{nm} = L.$$

A classical notion of sequence space is the following:

$$w_p = \{x = (x_n) : \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |x_n - \ell|^p = 0\}.$$

In [2], D. Borwein extended the sequence space w_p to the function space W_p, the space of real valued functions x, measurable (in the Lebesque sense) in the interval $(1,\infty)$ for which there is a number $\ell = \ell_x$ such that

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T |x(t) - \ell|^p = 0.$$

By a linear functional we mean one that is real-valued, additive, homogeneous and continuous. It is to be supposed throughout that $1 \leq p < \infty$ and that $\frac{1}{p} + \frac{1}{q} = 1$.

2. Main Results

We begin to the main results with following definitions:

Definition 2.1. *Let $x = \{x_{nm}\}$ be a real double sequence. Then the double sequence x is said to be strongly p-Cesaro summable to ℓ if*

$$P - \lim_{N,M\to\infty} \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M} |x_{nm} - \ell|^p = 0.$$

The space of all strongly p-Cesaro summable double sequences will be denote by w_p^2. Observe that this space is normed by

$$\|x\|_2 = \sup_{N,M\geq 1}\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M} |x_{nm} - \ell|^p\right)^{\frac{1}{p}}.$$

Definition 2.2. *Let x be a real valued bivariate function, measurable (in the Lebesque sense) in the $(1,\infty)\times(1,\infty)$. Then the bivariate function x is said to be strongly p-Cesaro summable to ℓ if*

$$\lim_{T,R\to\infty} \frac{1}{TR}\int_1^T\int_1^R |x(t,r) - \ell|^p drdt = 0.$$

The space of all strongly p-Cesaro summable bivariate functions will be denote by W_p^2. Observe that this space is normed by

$$\|x\|_2 = \sup_{T\geq 1, R\geq 1}\left(\frac{1}{TR}\int_1^T\int_1^R |x(t,r) - \ell|^p drdt\right)^{\frac{1}{p}}.$$

Given any real double sequence $\alpha = \{\alpha_{nm}\}$. We define a double sequence $\{m_{nm}(\alpha, p)\}$ by

$$m_{nm}(\alpha,p)=\begin{cases}\sup\limits_{2^n\le v<2^{n+1};\,2^m\le u<2^{m+1}}\{|vu\alpha_{vu}|\}, & \text{if} \quad p=1\\[2ex] \left(\frac{1}{2^{n+m}}\sum_{v=2n}^{2^{n+1}-1}\sum_{u=2m}^{2^{m+1}-1}|vu\alpha_{vu}|^q\right)^{\frac{1}{q}}, & \text{if} \quad p>1.\end{cases}$$

Given any real real valued bivariate function $\alpha(t,r)$ measurable in $(1,\infty)\times(1,\infty)$. We define a double sequence $\{M_{nm}(\alpha,p)\}$ by

$$M_{nm}(\alpha,p)=\begin{cases}\underset{2^n\le t<2^{n+1};\,2^m\le r<2^{m+1}}{ess.sup}\{|tr\alpha(t,r)|\}, & \text{if} \quad p=1\\[2ex] \left(\frac{1}{2^{n+m}}\int_{2n}^{2^{n+1}}\int_{2m}^{2^{m+1}}|tr\alpha(t,r)|^q\right)^{\frac{1}{q}}, & \text{if} \quad p>1.\end{cases}$$

Theorem 2.1. (i) *If f is a linear functional on W_p^2, then there is a real number a and a real valued bivariate function α, measurable in $(1,\infty)\times(1,\infty)$ such that*

$$f(x)=a\ell+\int_1^\infty\int_1^\infty\alpha(t,r)x(t,r)drdt \tag{2.1}$$

for every $x\in W_p^2$ and

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}M_{nm}(\alpha,p)<\infty. \tag{2.2}$$

(ii) *If a is a real number and α is a real valued bivariate function, measurable in $(1,\infty)\times(1,\infty)$, satisfying (2.2), then (2.1) defines a linear function on W_p^2 with*

$$\|f\|_2\le|a|+2^{\frac{2}{p}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}M_{nm}(\alpha,p)$$

and the integral in (2.1) is absolutely convergent for every $x\in W_p^2$.

Proof. Let L_p^2 be the linear space of real valued bivariate functions x measurable in $(1,\infty)\times(1,\infty)$ for which

$$\int_1^\infty\int_1^\infty|x(t,r)|^p drdt<\infty,$$

with norm

$$\|x\|_{L_p^2}=\left(\int_1^\infty\int_1^\infty|x(t,r)|^p drdt\right)^{\frac{1}{p}}.$$

Clearly, if $x\in L_p^2$, then $x\in W_p^2$, $\ell=0$ and $\|x\|_2=\|x\|_{W_p^2}\le\|x\|_{L_p^2}$. Consequently the restriction to L_p^2 of the given linear functional f on W_p^2 is linear on L_p^2. It follows from standard results that there is a real valued bivariate function α, measurable in $(1,\infty)\times(1,\infty)$, such that

$$f(x)=\int_1^\infty\int_1^\infty\alpha(t,r)x(t,r)drdt \tag{2.3}$$

for all $x \in L_p^2$ and either

$$\begin{aligned} &ess.sup \quad \{|\alpha(t,r)|\} < \infty \qquad \text{if} \quad p = 1 \\ &1 \leq t < \infty \\ &1 \leq r < \infty \end{aligned}$$

or

$$\int_1^\infty \int_1^\infty |\alpha(t,r)|^q drdt < \infty \qquad \text{if } p > 1.$$

To show that α must necessarily satisfy (2.2) we consider the cases $p = 1$ and $p > 1$ separately. If $p = 1$, let $M_{nm} = M_{nm}(\alpha, 1)$. There is a measurable set e_{nm} of positive measure $|e_{nm}|$ in the $(2^n, 2^{n+1}) \times (2^m, 2^{m+1})$ such that

$$|tr\alpha(t,r)| > M_{nm} - \frac{1}{2^{n+m}}$$

for all $(t,r) \in e_{nm}$.

Let

$$x(t,r) = \begin{cases} \frac{2^{n+m}}{e_{nm}} sign(\alpha(t,r)), & \text{if} \quad (t,r) \in e_{nm}, n \leq s, m \leq u \\ \\ 0, & otherwise. \end{cases}$$

Then $x \in L_1^2$ and so, by (2.3),

$$\begin{aligned} \|f\|_2 \|x\|_2 \geq f(x) &= \int_1^\infty \int_1^\infty x(t,r)\alpha(t,r) drdt \\ &= \sum_{n=0}^{s} \sum_{m=0}^{u} \int\int_{e_{nm}} \frac{2^{n+m}}{|e_{nm}|} |\alpha(t,r)| drdt \\ &\geq \frac{1}{4} \sum_{n=0}^{s} \sum_{m=0}^{u} \frac{1}{|e_{nm}|} \int\int_{e_{nm}} |tr\alpha(t,r)| drdt \qquad (2.4) \\ &\geq \frac{1}{4} \sum_{n=0}^{s} \sum_{m=0}^{u} (M_{nm} - \frac{1}{2^{n+m}}). \end{aligned}$$

Furtermore, for $2^z \leq T < 2^{z+1} \leq 2^{s+1}$, $2^h \leq R < 2^{h+1} \leq 2^{u+1}$,

$$\begin{aligned} \frac{1}{TR} \int_1^T \int_1^R |x(t,r)| drdt &\leq \frac{1}{2^{z+h}} \int_1^{2^{z+1}} \int_1^{2^{h+1}} |x(t,r)| drdt \\ &= \frac{1}{2^{z+h}} \sum_{n=0}^{z} \sum_{m=0}^{h} \int\int_{enm} x(t,r)| drdt \\ &\leq \frac{1}{2^{z+h}} \sum_{n=0}^{z} \sum_{m=0}^{h} 2^{n+m} < 4, \end{aligned}$$

and for $T > 2^{s+1}$, $R > 2^{u+1}$

$$\frac{1}{TR} \int_1^T \int_1^R |x(t,r)| drdt \leq \frac{1}{2^{s+1} 2^{u+1}} \int_1^{2^{s+1}} \int_1^{2^{u+1}} |x(t,r)| drdt < 1.$$

Hence $\|x\|_2 < 4$ and so, by (2.4),

$$4\|f\|_2 + \frac{1}{4}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{2^{n+m}} = 4\|f\|_2 + 1 \geq \frac{1}{4}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} M_{nm},$$

which establishes (2.2) in this case. If $p > 1$, let $M_{nm} = M_{nm}(\alpha, p)$ and let

$$x(t,r) = \begin{cases} \frac{(tr)^q}{2^{n+m}}\left|\frac{\alpha(t,r)}{M_{nm}}\right|^{\frac{q}{p}} sign(\alpha(t,r)), \quad \text{if} & \begin{array}{c} 2^n \leq t < 2^{n+1} \leq 2^{z+1}; \\ 2^m \leq r < 2^{m+1} \leq 2^{u+1} \text{ and } M_{nm} \neq 0 \end{array} \\ \\ 0, & otherwise. \end{cases}$$

Then $x \in L_p^2$ and so, by (2.3),

$$\begin{aligned} f(x) = \int_1^{2^{z+1}}\int_1^{2^{u+1}} |\alpha(t,r)x(t,r)| drdt &= \sum_{n=0}^{z}\sum_{m=0}^{u}\int_{2^n}^{2^{n+1}}\int_{2^m}^{2^{m+1}} |\alpha(t,r)x(t,r)| drdt \\ &= \sum_{n=0}^{z}\sum_{m=0}^{u} M_{nm}. \end{aligned} \tag{2.5}$$

Furtermore, for $2^z \leq T < 2^{z+1} \leq 2^{s+1}$, $2^h \leq R < 2^{h+1} \leq 2^{u+1}$,

$$\begin{aligned} \frac{1}{TR}\int_1^T\int_1^R |x(t,r)|^p drdt &\leq \frac{1}{2^{z+h}}\int_1^{2^{z+1}}\int_1^{2^{h+1}} |x(t,r)|^p drdt \\ &= \frac{1}{2^{z+h}}\sum_{n=0}^{z}\sum_{m=0}^{h}\int\int_{e_{nm}} |x(t,r)|^p drdt \\ &\leq \frac{2^{2p}}{2^{z+h}}\sum_{n=0}^{z}\sum_{m=0}^{h} 2^{n+m} < 2^{2p+2}, \end{aligned}$$

and for $T > 2^{z+1}$, $R > 2^{h+1}$

$$\frac{1}{TR}\int_1^T\int_1^R |x(t,r)|^p drdt \leq \frac{1}{2^{z+1}2^{h+1}}\int_1^{2^{z+1}}\int_1^{2^{h+1}} |x(t,r)|^p drdt < 4^p.$$

Hence $\|x\|_2 < 2^{2+\frac{2}{p}}$ and so, by (2.5),

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} M_{nm} \leq 2^{2+\frac{2}{p}}\|f\|_2,$$

which established (2.2) in this case.

Suppose now $p \geq 1$, $M_{nm} = M_{nm}(\alpha, p)$ and $x \in W_p^2$. Then by Hölder inequality

$$\begin{aligned} \int_1^{\infty}\int_1^{\infty} |\alpha(t,r)x(t,r)| drdt &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_{2^n}^{2^{n+1}}\int_{2^m}^{2^{m+1}} |\alpha(t,r)x(t,r)| drdt \\ &\leq \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} M_{nm}\left(2^{p(1-\frac{1}{p})(n+m)}\int_{2^n}^{2^{n+1}}\int_{2^m}^{2^{m+1}}\left|\frac{x(t,r)}{tr}\right|^p drdt\right)^{\frac{1}{p}} \\ &\leq \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} M_{nm}\left(2^{-(n+m)}\int_{2^n}^{2^{n+1}}\int_{2^m}^{2^{m+1}} |x(t,r)|^p drdt\right)^{\frac{1}{p}} \end{aligned}$$

$$\leq 2^{\frac{2}{p}} \|x\|_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm}. \tag{2.6}$$

It follows that

$$\int_1^\infty \int_1^\infty |\alpha(t,r)x(t,r)| drdt < \infty$$

whenever $x \in W_p^2$, and in particular since the characteristic function of $(1,\infty)\times(1,\infty)$ is in W_p^2, that

$$\int_1^\infty \int_1^\infty |\alpha(t,r)| drdt < \infty.$$

Suppose next that $x \in W_p^2$ and $\ell = \ell_x$. Let

$$y(t,r) = x(t,r) - \ell$$

$$y_{nm}(t,r) = \begin{cases} y(t,r), & \text{if } \ 1 \leq t \leq n, \ 1 \leq r \leq m; \\ 0, & \text{if } \quad t \geq n \text{ and } r \geq m. \end{cases}$$

Then $y \in W_p^2$, $y_{nm} \in L_p^2$ and

$$\|y_{nm} - y\|_2 = \sup_{T \geq n, R \geq m} \left(\frac{1}{TR} \int_n^T \int_m^R |x(t,r) - \ell|^p \right)^{\frac{1}{p}} = o(1) \text{ as } n,m \to \infty.$$

But

$$|f(y_{nm} - y)| = |f(y_{nm}) - f(y)| \leq \|y_{nm} - y\|_2 \|f\|_2,$$

and so, by (2.3),

$$f(y) = P - \lim_{n,m\to\infty} f(y_{nm}) = P - \lim_{n,m\to\infty} \int_1^n \int_1^m y(t,r)\alpha(t,r) drdt$$
$$= \int_1^\infty \int_1^\infty x(t,r)\alpha(t,r)drdt - \ell \int_1^\infty \int_1^\infty \alpha(t,r)drdt.$$

Since both integrals on the right hand side have been shown to be absolutely convergent. Taking δ to be characteristic function of $(1,\infty)\times(1,\infty)$ we see that

$$f(x) = f(y + \ell\delta) f(y) + \ell f(\delta) = \int_1^\infty \int_1^\infty x(t,r)\alpha(t,r)drdt + a\ell$$

where $a = f(\delta) - \int_1^\infty \int_1^\infty \alpha(t,r)$. This completes the proof of part (i.

(ii) It follows from (2.6) that if $x \in W_p^2$, $\ell = \ell_x$ and $M_{nm} = M_{nm}(\alpha, p)$, then

$$|f(x)| = \left| \int_1^\infty \int_1^\infty x(t,r)\alpha(t,r)drdt + a\ell \right| \leq \|x\|_2 2^{\frac{2}{p}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm} + |a\ell|. \tag{2.7}$$

Further, by Minkowski's inequality

$$\left(1 - \frac{1}{TR}\right)^{\frac{1}{p}} |\ell| \leq \left(\frac{1}{TR} \int_1^T \int_1^R |x(t,r) - \ell|^p drdt \right)^{\frac{1}{p}} + \left(\frac{1}{TR} \int_1^T \int_1^R |x(t,r)|^p drdt \right)^{\frac{1}{p}}$$

and the first term on the right hand side is $o(1)$. Hence $|\ell| \leq \|x\|_2$ and consequently, by (2.7),

$$|f(x)| \leq \|x\|_2 \left(|a| + 2^{\frac{2}{p}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm} \right)$$

for every $x \in W_p^2$. The additive and homogenous functional f defined by (2.1) is therefore also continuous on W_p^2 and

$$|f(x)| \leq |a| + 2^{\frac{2}{p}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm}.$$

Finally, by (2.6), the integral in (2.1) is absolutely convergent. Thus the proof is completed. □

Theorem 2.2. (i) *If f is a linear functional on w_p^2, then there is a real number a and a real double sequence $\alpha = \{\alpha_{nm}\}$ such that*

$$f(x) = a\ell + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{nm} x_{nm} \tag{2.8}$$

for every $x = \{x_{nm}\} \in w_p^2$ and

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m_{nm}(\alpha, p) < \infty. \tag{2.9}$$

(ii) *If a is a real number and $\alpha = \{\alpha_{nm}\}$ is a real double sequence satisfying (2.9), then (2.8) defines a linear function on w_p^2 with*

$$\|f\|_2 \leq |a| + 2^{\frac{2}{p}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m_{nm}(\alpha, p)$$

and the series in (2.8) is absolutely convergent for every $x = \{x_{nm}\} \in w_p^2$.

Proof. Given any real double sequence $x = \{x_{nm}\}$, define a bivariate function x^* by

$$x^*(t, r) = x_{nm} \text{ for } n < t \leq n+1; m < r \leq m+1, n = 1, 2, 3, ..., m = 1, 2, 3, ...$$

It is easily verified that this defines a one to one correspondence between w_p^2 and a linear subspace $(W_p^2)^*$ of W_p^2 such that

$$\ell_{x^*} = \ell_x \text{ and } \|x^*\|_2 \leq \|x\|_2 \leq 2^{\frac{2}{p}} \|x^*\|_2.$$

Hence given a linear functional on W_p^2, the functional f^* defined by

$$f^*(x^*) = f(x)$$

is linear on $(W_p^2)^*$. Consequently, by the Hahn-Banach theorem and Theorem2.1, there is a real number a and a real valued bivariate function α^*, integrable over $(1, \infty) \times 1, \infty)$, such that

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm}(\alpha^*, p) < \infty$$

and, for every $x \in w_p^2$,

$$f(x) = f^*(x^*) = a\ell_{x^*} + \int_1^{\infty} \int_1^{\infty} \alpha^*(t, r) x^*(t, r) dr dt = a\ell_x + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{nm} x_{nm}$$

where $\alpha_{nm} = \int_n^{n+1} \int_m^{m+1} \alpha^*(t,r)drdt$. Furthermore, for $\alpha = \{\alpha_{nm}\}$,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} m_{nm}(\alpha, p) \leq \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} M_{nm}(\alpha^*, p);$$

and this completes the proof of (i).

(ii) If $x = \{x_{nm}\} \in w_p^2$ $m_{nm} = m_{nm}(\alpha, p)$ and $\ell = \ell_x$ then by Hölder's and Minkowski's inequalities, as in the proof of (ii) of Theorem2.1,

$$f(x) = a\ell + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_{nm}x_{nm} \leq |a\ell| + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\alpha_{nm}x_{nm}|$$

$$\leq |a\ell| + 2^{\frac{2}{p}}\|x\|_2 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} m_{nm} \leq \|x\|_2 \left(|a| + 2^{\frac{2}{p}} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} m_{nm}\right).$$

The functional f defined by (2.8) is therefore linear on w_p^2,

$$\|f\|_2 \leq |a| + 2^{\frac{2}{p}} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} m_{nm}$$

and the series in (2.8) absolutely convergent. This completes the proof. □

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