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GENERALIZED STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES IN PARANORMED SPACES

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Abstract. We introduce the notion of (λ, μ)-statistical convergence of double sequences in a setting of paranormed space and prove that every convergent sequence is (λ, μ)-statistically convergent but not conversely by supporting an illustrative example. We also define the notions of (λ, μ)-statistical Cauchy and strongly $(\lambda, \mu)_p$-summable double sequences in a paranormed space and obtain their relationship with (λ, μ)-statistical convergence.

1. Introduction and preliminaries

The notion of statistical convergence, which is an extension of the idea of common convergence, was first appeared, under the name of almost convergence, in the first edition of the celebrated monograph of Zygmund [32]. This idea was introduced by Fast [11] as follows: The sequence $x = (x_k)$ is statistically convergent to ℓ if for every $\varepsilon > 0$, $\lim_n n^{-1}|\{k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0$. Some basic properties of statistical convergence were studied by Schoenberg [30], Šalát [31] and Connor [9]. An interesting notion of statistically

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Cauchy sequence was first defined by Fridy [12] and he also showed that it is equivalent to statistical convergence. Thereafter, this notion turned out to be one of the most active areas of research in summability theory. The statistical convergence was studied in various setup such as topological Hausdorff groups [8], normed spaces [15], locally convex Hausdorff topological spaces [16], paranormed spaces [2], random 2-normed spaces [19] and many others. Mursaleen [24] presented a generalization of statistical convergence with the help of non-decreasing sequence $\lambda = (\lambda_k)$ such that $\lambda_{k+1} \leq \lambda_k + 1$ and $\lambda_1 = 0$ of positive numbers tending to ∞ and called it λ-statistical convergence. We also refer to the recent work in [1,3,4,6,7,10,14,17,18,21,22] for some applications of convergence methods to approximation theorems. Pringsheim [29] extended the notion of usual convergence from single sequences of real numbers to double sequences as follows: A double sequence $x = (x_{jk})$ has a Pringsheim limit ξ (convergent to ξ in Pringsheim's sense), in symbols, we shall write $(P)\lim x = \xi$, provided that given an $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{jk} - \xi| < \epsilon$ whenever $j, k > N$. Also, $x = (x_{jk})$ is bounded, denoted by $\mathcal{L}_\infty$, if

$$\|x\| = \sup_{j,k} |x_{jk}| < \infty.$$

It is well known that every convergent single sequence is bounded but this fact need not be true for double sequences. Statistical convergence extended to double sequences by Mursaleen and Edely [26] with the help of two dimensional analogue of natural density of subsets of $\mathbb{N} \times \mathbb{N}$ and further it was defined and studied in intuitionistic fuzzy normed spaces, locally solid Riesz spaces and paranormed spaces by Mursaleen and Mohiuddine [27,28], Mohiuddine et al. [20] and Arani et al. [5], respectively. Also, we refer to [13,23]. Let $K \subset \mathbb{N} \times \mathbb{N}$. Then, the double natural density of K is defined by

$$\delta_2(K) = (P)\lim_{m,n} \frac{|K(m,n)|}{mn}$$

provided that the limit exists, where $|K(m,n)|$ be the numbers of (j,k) in K such that $j \leq m$ and $k \leq n$. $x = (x_{jk})$ is said to be statistically convergent to ξ if for each $\epsilon > 0$,

$$(P)\lim_{m,n} (mn)^{-1} |\{(j,k), j \leq m, k \leq n : |x_{jk} - \xi| \geq \epsilon\}| = 0.$$

Mursaleen et al. [25] defined and studied the notion of (λ, μ)-statistical convergence for double sequences where $\lambda = \lambda_m$ and $\mu = \mu_n$ are two non-decreasing sequences of positive real numbers each tending to ∞ such that $\lambda_1 = 0$, $\lambda_{m+1} \leq \lambda_m + 1$ and $\mu_1 = 0$, $\mu_{n+1} \leq \mu_n + 1$ for all m, n. The (λ, μ)-density of the set $K \subseteq \mathbb{N} \times \mathbb{N}$ is given by

$$\delta_{\lambda,\mu}(K) = (P)\lim_{m,n} \frac{1}{\lambda_m \mu_n} |\{m - \lambda_m + 1 \leq j \leq m, n - \mu_n + 1 \leq k \leq n : (j,k) \in K\}|$$

provided that the limit exists. We remark that $\lambda_m = m$ and $\mu_n = n$, the (λ, μ)-density reduces to the natural double density. A double sequence $x = (x_{jk})$ is (λ, μ)-statistically convergent to ξ if for every $\epsilon > 0$,

$$(P) \lim_{m,n} \frac{1}{\lambda_m \mu_n} \left|\{(j,k), j \in I_m, k \in J_n : |x_{jk} - \xi| \geq \epsilon\}\right| = 0$$

where $I_m = [m - \lambda_m + 1, m]$ and $J_n = [n - \mu_n + 1, n]$.

If X is a linear space and $g : X \to \mathbb{R}$ such that (i) $x = 0 \Rightarrow g(x) = 0$, (ii) $g(x + y) \leq g(x) + g(y)$, (iii) $g(-x) = g(x)$ and (iv) if $t_k \to t$ $(k \to \infty)$ and $x_k \to x$ $(k \to \infty)$ in the sense that $g(x_k - x) \to 0$ $(k \to \infty)$ for scalars t_k, t and the vectors $x_k, x \in X$, then $t_k x_k \to tx$ $(k \to \infty)$ in the sense that $g(t_k x_k - tx) \to 0$ $(k \to \infty)$, then g is said to be a paranorm on X and the pair (X, g) is called a paranormed space. Note that if $g(x) = 0$ implies $x = 0$, then paranorm g is called a total paranorm on X and the pair (X, g) is called a total paranormed space. It is to further note that each seminorm (norm) on X is a paranorm (total) but not conversely.

2. (λ, μ)-Statistical convergence in paranormed spaces

In this section, we introduce the notion of convergence and (λ, μ)-statistical convergence in the framework of paranormed space and prove various interesting results and display an illustrative example is support of our result.

Definition 2.1. Let (X, g) be a paranormed space. We say that a double sequence $x = (x_{jk})$ is convergent, shortly, g_2-convergent, to ξ in (X, g) if for each $\epsilon > 0$, there is $k_0 \in \mathbb{N}$ such that $g(x_{jk} - \xi) < \epsilon$ whenever $j, k \geq k_0$. In symbols, one writes $g_2\text{-}\lim x = \xi$, and ξ is called the g_2-limit of x.

Definition 2.2. Let (X, g) be a paranormed space. We say that a double sequence $x = (x_{jk})$ is (λ, μ)-statistically convergent, shortly, $g(S_{\lambda,\mu})$-convergent, to ξ in (X, g), if for each $\epsilon > 0$, the set $\{(j,k) \in \mathbb{N} \times \mathbb{N} : g(x_{jk} - \xi) \geq \epsilon\}$ has (λ, μ)-density zero, equivalently, one writes

$$(P) \lim_{m,n} \frac{1}{\lambda_m \mu_n} \left|\{(j,k), j \in I_m, k \in J_n : g(x_{jk} - \xi) \geq \epsilon\}\right| = 0.$$

In this case, one writes $g(S_{\lambda,\mu})\text{-}\lim x = \xi$. If we choose $\lambda_m = m$ and $\mu_n = n$ then the notion of $g(S_{\lambda,\mu})$-convergence is reduced to statistically convergence for double sequence in (X, g) due to Arani et al. [5]. We denote this by $g(S_2)$-convergence and write $g(S_2)\text{-}\lim x = \xi$.

Theorem 2.3. If a double sequence $x = (x_{jk})$ is $g(S_{\lambda,\mu})$-convergent then $g(S_{\lambda,\mu})$-limit is unique.

Proof. Assume that $g(S_{\lambda,\mu})\text{-}\lim x = \xi'$ and $g(S_{\lambda,\mu})\text{-}\lim x = \xi''$. Let $\epsilon > 0$ be given. We now define the following two sets $B'(\epsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : g(x_{jk} - \xi') \geq \epsilon/2\}$ and $B''(\epsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : g(x_{jk} - \xi'') \geq \epsilon/2\}$.

Since $g(S_{\lambda,\mu})$- $\lim x = \xi'$, one obtains

$$\delta_{\lambda,\mu}(B'(\epsilon)) = (P)\lim_{m,n} \frac{1}{\lambda_m \mu_n} |\{(j,k), j \in I_m, k \in J_n : g(x_{jk} - \xi') \geq \epsilon/2\}| = 0.$$

Similarly, the assumption $g(S_{\lambda,\mu})$- $\lim x = \xi''$ gives $\delta_{\lambda,\mu}(B''(\epsilon)) = 0$. Now, let $B(\epsilon) = B'(\epsilon) \cup B''(\epsilon)$. Then $\delta_{\lambda,\mu}(B(\epsilon)) = 0$ and so $\delta_{\lambda,\mu}(B^c(\epsilon)) = 1$ since $B^c(\epsilon)$ is a nonempty set. Now if $(j,k) \in \mathbb{N} \times \mathbb{N} \setminus B(\epsilon)$, then

$$g(\xi' - \xi'') \leq g(x_{jk} - \xi') + g(x_{jk} - \xi'') < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, one obtains $g(\xi' - \xi'') = 0$ which yields $\xi' = \xi''$. □

Theorem 2.4. If a double sequence $x = (x_{jk})$ is g_2-convergent to ξ, then it is $g(S_{\lambda,\mu})$-convergent to the same limit.

Proof. Assume that (x_{jk}) is g_2-convergent to ξ, that is, g_2- $\lim x = \xi$. Let $\epsilon > 0$ be given. Then, there is $N \in \mathbb{N}$ such that $g(x_{jk} - \xi) < \epsilon$ for all $j, k \geq N$. Since, the set $K(\epsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : g(x_{jk} - \xi) \geq \epsilon\}$ is contained in $\mathbb{N} \times \mathbb{N}$, hence $\delta_{\lambda,\mu}(K(\epsilon)) = 0$, that is, (x_{jk}) is $g(S_{\lambda,\mu})$-convergent to ξ.

Example 2.5. The present example proves that the converse of last Theorem is not true in general. Let

$$X = \ell(1/jk) = \left\{ x = (x_{jk}) : \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} | x_{jk} |^{1/jk} < \infty \right\}$$

with the paranorm

$$g(x) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} | x_{jk} |^{1/jk} .$$

Let us define $x = (x_{jk})$ by

$$x_{jk} = \begin{cases} jk \text{ for } m - [\sqrt{\lambda_m}] + 1 \leq j \leq m, n - [\sqrt{\mu_n}] + 1 \leq k \leq n, \\ 0 \text{ otherwise.} \end{cases}$$

For $0 < \epsilon < 1$, one writes

$$B(\epsilon) := \{(j,k) \in \mathbb{N} \times \mathbb{N} : g(x_{jk}) \geq \epsilon\}.$$

It is easy to see that

$$g(x_{jk}) = \begin{cases} (jk)^{1/jk} \text{ for } m - [\sqrt{\lambda_m}] + 1 \leq j \leq m, n - [\sqrt{\mu_n}] + 1 \leq k \leq n, \\ 0 \text{ otherwise;} \end{cases}$$

and hence we obtain

$$(P)\lim_{jk} g(x_{jk}) := \begin{cases} 1 \text{ for } m - [\sqrt{\lambda_m}] + 1 \leq j \leq m, n - [\sqrt{\mu_n}] + 1 \leq k \leq n, \\ 0 \text{ otherwise.} \end{cases}$$

We obtain that (x_{jk}) is not convergent in (X, g) (g_2- $\lim x$ does not exist) but $\delta_{\lambda,\mu}(B(\epsilon)) = 0$, that is, $g(S_{\lambda,\mu})$- $\lim x = 0$. Hereby, we conclude that the converse of above Theorem 2.4 need not be true in general. □

The proof of the following theorem is straightforward and hence omitted.

Theorem 2.6. Let (X, g) be a paranormed space and assume that $g(S_{\lambda,\mu})$- $\lim x' = \xi'$ and $g(S_{\lambda,\mu})$- $\lim x'' = \xi''$. Then

(i) $g(S_{\lambda,\mu})$- $\lim(x' \pm x'') = \xi' \pm \xi''$,

(ii) $g(S_{\lambda,\mu})$- $\lim \alpha x' = c\xi'$ for $c \in \mathbb{R}$.

Theorem 2.7. Let (X, g) be a paranormed space. Then, a double sequence $x = (x_{jk})$ is (λ, μ)-statistically convergent to ξ in (X, g) if and only if there exists a set $B = \{(j_m, k_n) : j_1 < j_2 < ... < j_m < ...; k_1 < k_2 < ... < k_n < ...\} \subseteq \mathbb{N} \times \mathbb{N}$ with $\delta(B) = 1$ such that g- $\lim_{m,n} x_{j_m k_n} = \xi$.

Proof. Suppose that $x = (x_{jk})$ is $g(S_{\lambda,\mu})$-convergent to ξ, that is, $g(S_{\lambda,\mu})$- $\lim x = \xi$. For $s = 1, 2, ...,$ one writes

$$B(s) = \left\{(m, n) \in \mathbb{N} \times \mathbb{N} : g(x_{j_m k_n} - \xi) \geq \frac{1}{s}\right\}$$

and

$$D(s) = \left\{(m, n) \in \mathbb{N} \times \mathbb{N} : g(x_{j_m k_n} - \xi) < \frac{1}{s}\right\}.$$

Then $\delta_{\lambda,\mu}(B(s)) = 0$,

$$D(1) \supset D(2) \supset ... \supset D(i) \supset D(i + 1) \supset ..., \tag{1}$$

and

$$\delta_{\lambda,\mu}(D(s)) = 1 \ (s = 1, 2, ...). \tag{2}$$

We need to prove that $(x_{j_m k_n})$ is g_2-convergent to ξ for $(m, n) \in D(s)$. Let us assume, on contrary, that $(x_{j_m k_n})$ is not g_2-convergent to ξ. Consequently, there is $\epsilon > 0$ such that $g(x_{k_n} - \xi) \geq \epsilon$ for infinitely many terms. Let us write

$$D(\epsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : g(x_{j_m k_n} - \xi) < \epsilon\}$$

and $\epsilon > \frac{1}{s}$, $s \in \mathbb{N}$. Then

$$\delta_{\lambda,\mu}(D(\epsilon)) = 0,$$

and by (1), $D(s) \subset D(\epsilon)$. Hence $\delta_{\lambda,\mu}(D(s)) = 0$, which contradicts (2) and therefore $(x_{j_m k_n})$ is g_2-convergent to ξ.

Conversely, let us assume there exists a set $B = \{(j_m, k_n) : j_1 < j_2 < ... < j_m < ...; k_1 < k_2 < k_3 < ... < k_n < ...\} \subseteq \mathbb{N} \times \mathbb{N}$ with $\delta(B) = 1$ such that $g\text{-}\lim_{n\to\infty} x_{j_m k_n} = \xi$. Then there is a positive integer N such that $g(x_{mn} - \xi) < \epsilon$ for $m, n > N$. Put

$$B(\epsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : g(x_{mn} - \xi) \geq \epsilon\}$$

and $B' = \{(j_{N+1}, k_{N+1}), (j_{N+2}, k_{N+2}), ...\}$. Then $\delta_{\lambda,\mu}(B') = 1$ and $B(\epsilon) \subseteq \mathbb{N} \times \mathbb{N} - B'$ which implies that $\delta_{\lambda,\mu}(B(\epsilon)) = 0$. Hence $g(S_{\lambda,\mu})\text{-}\lim x = \xi$. □

We are now defining the notion of (λ, μ)-statistically Cauchy double sequence in a paranormed space and prove that it is equivalent to the notion of (λ, μ)-statistically convergence double sequence.

Definition 2.8. Let (X, g) be a paranormed space. We say that $x = (x_{jk})$ is (λ, μ)-statistically Cauchy double sequence in (X, g), denoted by $g(S_{\lambda,\mu})$-Cauchy, if for every $\epsilon > 0$ there exist $M, N \in \mathbb{N}$ such that, for all $j, m \geq M$, $k, n \geq N$, we have

$$(P) \lim_{m,n} \frac{1}{\lambda_m \mu_n} |\{(j, k), j \in I_m, k \in J_n : g(x_{jk} - x_{mn}) \geq \epsilon\}| = 0.$$

Theorem 2.9. Let $x = (x_{jk})$ be a double sequence in a complete paranormed space (X, g). Then, x is $g(S_{\lambda,\mu})$-convergent iff it is $g(S_{\lambda,\mu})$-Cauchy.

Proof. Assume that $g(S_{\lambda,\mu})\text{-}\lim x = \xi$. Then, the set

$$G(\epsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : g(x_{jk} - \xi) \geq \epsilon/2\}$$

has (λ, μ)-density zero which yields

$$\delta_{\lambda,\mu}(G^c(\epsilon)) = \delta_{\lambda,\mu}(\{(j, k) \in \mathbb{N} \times \mathbb{N} : g(x_{jk} - \xi) < \epsilon\}) = 1.$$

Suppose $(m, n) \in G^c(\epsilon)$. Therefore, $g(x_{mn} - \xi) < \epsilon/2$. Now, let

$$H(\epsilon) := \{(j, k) \in \mathbb{N} \times \mathbb{N} : g(x_{jk} - x_{mn}) \geq \epsilon\}.$$

We need to show that $H(\epsilon) \subset G(\epsilon)$. Let $(j, k) \in H(\epsilon)$. Then $g(x_{jk} - x_{mn}) \geq \epsilon$ and hence $g(x_{jk} - \xi) \geq \epsilon/2$, i.e., $(j, k) \in G(\epsilon)$. Otherwise, if $g(x_{jk} - \xi) < \varepsilon$ then

$$\epsilon \leq g(x_{jk} - x_{mn}) \leq g(x_{jk} - \xi) + g(x_{mn} - \xi) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which is not possible. Thus $H(\epsilon) \subset G(\epsilon)$ and hence

$$\delta_{\lambda,\mu}(H(\epsilon)) = \delta_{\lambda,\mu} (\{(j, k) \in \mathbb{N} \times \mathbb{N} : g(x_{jk} - x_{mn}) \geq \epsilon\}) = 0.$$

Therefore (x_{jk}) is (λ, μ)-statistically Cauchy in (X, g).

Conversely, assume that $x = (x_{jk})$ is $g(S_{\lambda,\mu})$-Cauchy but not $g(S_{\lambda,\mu})$-convergent. Then there exist $M, N \in \mathbb{N}$ such that for all $j, m \geq M$, $k, n \geq N$, the set

$$A(\epsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : g(x_{jk} - x_{mn}) \geq \epsilon\},$$

has (λ, μ)-density zero, that is, $\delta_{\lambda,\mu}(A(\epsilon) = 0$ and $\delta_{\lambda,\mu}(E(\epsilon)) = 0$, where

$$E(\epsilon) = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : g(x_{jk} - \xi) < \frac{\epsilon}{2}\right\},$$

that is,

$$(P)\lim_{m,n} \frac{1}{\lambda_m \mu_n} \left|\left\{(j,k), j \in I_m, k \in J_n : g(x_{jk} - \xi) < \frac{\epsilon}{2}\right\}\right| = 0,$$

which yields $\delta_{\lambda,\mu}(E^c(\epsilon)) = 1$. Since $g(x_{jk} - x_{mn}) \leq 2g(x_{jk} - \xi) < \epsilon$, if $g(x_{jk} - \xi) < \epsilon/2$. Thus, $\delta_{\lambda,\mu}(A^C(\epsilon)) = 0$ and so $\delta_{\lambda,\mu}(A(\epsilon) = 1$. This is a contradiction to our assumption that x is $g(S_{\lambda,\mu})$-Cauchy. Hence x is $g(S_{\lambda,\mu})$-convergent.

3. Strong summability for double sequences in (X, g)

We give the idea of strong $(\lambda, \mu)_p$-summability in the setting of paranormed space (X, g) and obtain its relation with $g(S_{\lambda,\mu})$-convergence.

Definition 3.1. Let (X, g) be a paranormed space and let p be a positive real number. The double sequence $x = (x_{jk})$ is said to be strongly $(\lambda, \mu)_p$-summable to the limit ξ in (X, g), denoted by $x_{jk} \longrightarrow \xi[V_{\lambda,\mu}, g]_p$, if

$$(P)\lim_{m,n} \frac{1}{\lambda_m \mu_n} \sum_{j \in I_m} \sum_{k \in J_n} \left(g\left(x_{jk} - \xi\right)\right)^p = 0 \ (0 < p < \infty).$$

Theorem 3.2. One has the following:

(i) If $0 < p < \infty$ and $x_{jk} \longrightarrow \xi[V_{\lambda,\mu}, g]_p$, then $x = (x_{jk})$ is $g(S_{\lambda,\mu})$-convergent to ξ.

(ii) If $x = (x_{jk}) \in \mathcal{L}_\infty$ and $g(S_{\lambda,\mu})$-convergent to ξ then $x_{jk} \longrightarrow \xi[V_{\lambda,\mu}, g]_p$, where $p \in (0, \infty)$.

Proof. (i) Assume that $x_{jk} \longrightarrow \xi[V_{\lambda,\mu}, g]_p$. Then, as $m, n \to \infty$, one obtains

$$0 \longleftarrow \frac{1}{\lambda_m \mu_n} \sum_{j \in I_m} \sum_{k \in J_n} (g(x_{jk} - \xi))^p \geq \frac{1}{\lambda_m \mu_n} \sum_{\substack{j \in I_m \\ (g(x_k - \xi))^p \geq \epsilon}} \sum_{\substack{k \in J_n \\ (g(x_k - \xi))^p \geq \epsilon}} (g(x_k - \xi))^p$$

$$\geq \frac{\epsilon^p}{\lambda_m \mu_n} |F(\epsilon)|,$$

where $F(\epsilon) = \{j \in I_m, k \in J_n : (g(x_{jk} - \xi))^p \geq \epsilon\}$. That is, $(P)\lim_{m,n\to\infty} \frac{1}{\lambda_m \mu_n} |F(\epsilon)| = 0$ and so

$$\delta_{\lambda,\mu}(F(\epsilon)) = (P)\lim_{m,n} \frac{1}{\lambda_m \mu_n} |\{(j,k), j \in I_m, k \in J_n : (g(x_{jk} - \xi))^p \geq \epsilon\}| = 0.$$

Thus $x = (x_{jk})$ is $g(S_{\lambda,\mu})$-convergent to ξ.

(ii) Assume that a double sequence $x = (x_{jk})$ is bounded and $g(S_{\lambda,\mu})$-convergent to ξ. Let $\epsilon > 0$ be given. Then, we have $\delta_{\lambda,\mu}(F(\epsilon)) = 0$. Since $x \in \mathcal{L}_\infty$, there is an $M > 0$ such that $g(x_{jk} - \xi) \leq M$. We have

$$\frac{1}{\lambda_m \mu_n} \sum_{j \in I_m} \sum_{k \in J_n} (g(x_{jk} - \xi))^p$$

$$= \frac{1}{\lambda_m \mu_n} \sum_{\substack{j \in I_m \\ (j,k) \notin F(\epsilon)}} \sum_{\substack{k \in J_n \\ (j,k) \notin F(\epsilon)}} (g(x_{jk} - \xi))^p + \frac{1}{\lambda_m \mu_n} \sum_{\substack{j \in I_m \\ (j,k) \in F(\epsilon)}} \sum_{\substack{k \in J_n \\ (j,k) \in F(\epsilon)}} (g(x_{jk} - \xi))^p.$$

If we take $(j,k) \notin F(\epsilon)$ then

$$\frac{1}{\lambda_m \mu_n} \sum_{\substack{j \in I_m \\ (j,k) \notin F(\epsilon)}} \sum_{\substack{k \in J_n \\ (j,k) \notin F(\epsilon)}} (g(x_{jk} - \xi))^p < \epsilon.$$

On the other and, if $(j,k) \in F(\epsilon)$, we have

$$\frac{1}{\lambda_m \mu_n} \sum_{\substack{j \in I_m \\ (j,k) \in F(\epsilon)}} \sum_{\substack{k \in J_n \\ (j,k) \in F(\epsilon)}} (g(x_{jk} - \xi))^p$$

$$\leq (\sup g(x_{jk} - \xi)) \frac{1}{\lambda_m \mu_n} (|\{j \in I_m, k \in J_n : (g(x_{jk} - \xi))^p \geq \epsilon\}|)$$

$$\leq \frac{M}{\lambda_m \mu_n} |\{j \in I_m, k \in J_n : (g(x_{jk} - \xi))^p \geq \epsilon\}|.$$

We see that the right of above inequality tends to zero as $m, n \to \infty$, since $\delta_{\lambda,\mu}(F(\epsilon)) = 0$. Hence, we conclude that $x_k \longrightarrow \xi[V_{\lambda,\mu}, g]_p$. $\square$

Remark 3.3. If we choose $\lambda_m = m$ and $\mu_n = n$, then strong $(\lambda, \mu)_p$-summablity in a paranormed space is reduced to the notion of strong p-Cesàro summablity for double sequences in the same setup, denoted by $[C_{1,1}, g]_p$. Then, we have the following corollary from Theorem 3.3.

Corollary 3.4. Let (X, g) be a paranormed space.

(i) If $p \in (0, \infty)$ and $x_{jk} \longrightarrow \xi[C_{1,1}, g]_p$, then $g(S_2)$- $\lim x = \xi$.

(ii) If $x = (x_{jk}) \in \mathcal{L}_\infty$ and $g(S_2)$- $\lim x = \xi$ then $x_{jk} \longrightarrow \xi[C_{1,1}, g]_p$ $(p \in (0, \infty))$.

Theorem 3.5. If a double $x = (x_{jk})$ is strongly $(\lambda, \mu)_p$-summable or (λ, μ)-statistically convergent to ξ in (X, g), then there is a convergent double sequence $y = (y_{jk})$ and a (λ, μ)-statistically null double sequence $z = (z_{jk})$ such that $y = (y_{jk})$ is convergent to ξ in Pringsheim's sense, $x = y + z$ and

$$(P) \lim_{m,n} \frac{1}{\lambda_m \mu_n} |\{(j,k), j \in I_m, k \in J_n : z_{jk} \neq 0\}| = 0. \tag{3}$$

Moreover, if a double sequence $x = (x_{jk})$ is bounded, then both (y_{jk}) and (z_{jk}) are bounded.

Proof. It is clear from Theorem 3.2 that if a double sequence $x_{jk} \longrightarrow \xi[V_{\lambda,\mu}, g]_p$, then it is (λ, μ)-statistically convergent to ξ. Let us take $S(0) = 0$ and choose a strictly increasing sequence $S(1) < S(2) < S(3) < ...$ of positive integers such that

$$\frac{1}{\lambda_m \mu_n} |\{(j,k), j \in I_m, k \in J_n : |x_{jk} - \xi| \geq l\}| < l^{-1},$$

for $m, n > S(l)$. We are defining $y = (y_{jk})$ and $z = (z_{jk})$ as follows: Choose $z_{jk} = 0$ and $y_{jk} = x_{jk}$ if $S(0) < j, k < S(1)$. Suppose $l \geq 1$ and $S(l) < j, k < S(l+1)$. We now set

$$y_{jk} = \begin{cases} x_{jk},\ z_{jk} = 0 \text{ for } |x_{jk} - \xi| < l^{-1}, \\ \xi,\ z_{jk} = x_{jk} - \xi \text{ for } |x_{jk} - \xi| \geq l^{-1}. \end{cases}$$

Clearly, $x = y + z$ and double sequences y and z are bounded if a double sequence x is bounded. We have to show that $y = (y_{jk})$ is convergent to ξ in the Pringsheim's sense. For given $\epsilon > 0$, let us choose l such that $\epsilon > 1/l$. We can see that for $j, k > S(l)$, one obtains

$$|y_{jk} - \xi| < \epsilon \text{ (since } |y_{jk} - \xi| = |x_{jk} - \xi| < \epsilon) \text{ if } |x_{jk} - \xi| < l^{-1}$$

and

$$|y_{jk} - \xi| = |\xi - \xi| = 0 \text{ if } |x_{jk} - \xi| > l^{-1}.$$

If follows that (y_{jk}) is convergent to ξ in the Pringsheim's sense. It remains to prove that (3) holds. It is enough to prove that if $\delta > 0$ and $l \in \mathbb{N}$ such that $1/l < \delta$, then

$$|\{(j,k), j \in I_m, k \in J_n : z_{jk} \neq 0\}| < \delta \ \forall\ m, n > S(l).$$

As we have seen from the construction that if $S(l) < j, k \leq S(l+1)$ then $z_{jk} = 0$ only if $|x_{jk} - \xi| > 1/l$. It follows that if $S(r) < j, k \leq S(r+1)$, then

$$\{(j,k), j \in I_m, k \in J_n : z_{jk} \neq 0\} \subseteq \{(j,k), j \in I_m, k \in J_n : |x_{jk} - \xi| > 1/r\}.$$

Consequently, if $S(r) < j, k \leq S(r+1)$ and $r > l$, one obtains

$$\frac{1}{\lambda_m \mu_n} |\{(j,k), j \in I_m, k \in J_n : z_{jk} \neq 0\}|$$

$$\subseteq \frac{1}{\lambda_m \mu_n} |\{(j,k), j \in I_m, k \in J_n : |x_{jk} - \xi| > 1/r\}| < 1/r < 1/l < \delta.$$

Thus, we have the following

$$(P) \lim_{m,n} \frac{1}{\lambda_m \mu_n} |\{(j,k), j \in I_m, k \in J_n : z_{jk} \neq 0\}| = 0.$$

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