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RANDOM COMMON FIXED POINT THEOREMS FOR TWO PAIRS OF NONLINEAR CONTRACTIVE MAPS IN POLISH SPACES

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Abstract. This research work proves the random common fixed point theorem for two pairs of random weakly compatible mappings fulfilling certain generalized random nonlinear contractive conditions in Polish spaces. An example is given to support the validity of our results. Our results generalize and extend some works in literature.

1. Introduction

The random fixed point theory introduced in 1950 by Prague School of Probabilistic plays very important role in the theory of random integral, random differential equations and other areas of applied mathematics. Some classical fixed point theorems in different abstract spaces are proved in the context of random fixed point theory (see; Akewe et al.[1] , Rashwan and Albaqeri [2], Hans [3] and Nieto et al. [4]). The common fixed point of two pairs of weakly compatible mappings satisfying certain contractive conditions in G-partial metric spaces without assuming the continuity of any of the maps involved was proved by Eke and Akinlabi [8]. The random common fixed point of two pairs of random subsequentially continuous mappings with compatibility of type (E) satisfying certain generalized contractive conditions in Polish spaces (separable metric space) was established by Rashwan and Hammed [9]. In this paper, we prove the random version of

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the result of Eke and Akinlabi [8] in the context of Polish spaces by using the contractive maps of Rashwan and Hammed [9]. Our results are extension and an improvement on some related results in the literature.

2. Preliminaries

Let (Ω, ϕ) be a measurable space, A a separable metric space and $\sigma_n : \Omega \to A$ a measurable sequence. An operator $f : \Omega \times A \to A$ is random operator, if for every $x \in A$, the mapping $F(.,x) : \Omega \to A$ is measurable. A measurable mapping $\omega : \Omega \to A$ is a random fixed point of a random operator $F : \Omega \times A \to A$ if $F(v, \omega(v)) = \omega(v)$ for each $v \in \omega$, (details of these definitions can be found in Beg and Abbas [5], Choudhury and Ray [6] and Choudhury and Upadhyah [7]).

The following theorems are the results of Eke and Akinlabi [8] and Rashwan and Hammed [9] respectively.

Theorem 1.1 [8]: Let B, D, E and F be self-maps of a G-partial metric space A satisfying $B(A) \subset F(A)$, $D(A) \subset E(A)$ and

$G_p(Ba, Ba, Db) \leq hu_{a,a,b}(B, D, E, F)$,

and

$G_p(Ba, Db, Db) \leq hu_{a,b,b}(B, D, E, F)$,

where $h \in (0, 1)$ and

$$u_{a,a,b}(B, D, E, F) \in \{G_p(Ea, Ea, Fb), G_p(Ba, Ba, Ea), G_p(Db, Db, Fb), \frac{G_p(Ba, Ba, Fb) + G_p(Db, Db, Ea)}{2}\} \tag{1.1}$$

and

$$u_{a,b,b}(B, D, E, F) \in \{G_p(Ea, Fb, Fb), G_p(Ba, Ea, Ea), G_p(Db, Fb, Fb), \frac{G_p(Ba, Fb, Fb) + G_p(Db, Ea, Ea)}{2}\} \tag{1.2}$$

for all $a, b \in A$. If one of $B(A)$, $D(A)$, $E(A)$ or $F(A)$ is a complete subspace of A, then $\{B, E\}$ and $\{D, F\}$ have a unique point of coincidence in X. Moreover if $\{B, E\}$ and $\{D, F\}$ are weakly compatible, then B, D, E and F have a unique common fixed point.

Theorem 1.2 [9]: Let A be a Polish space and $B, D, E, F : \Omega \times A \to A$ are four random mappings satisfy

$d(B(v, a), F(v, b)) \leq$

$\phi(max\{d(E(v, a), F(v, a)), d(E(v, a), F(v, a)), d(F(v, b), D(v, b)), \frac{d(E(v,a),D(v,b))+d(F(v,y),B(v,x))}{2}\})$, for all $a, b \in A$, and $v \in \Omega$, where $\phi : [0, \infty) \to [0, \infty)$ is contractive modulus and non decreasing such that $\phi(0) = 0$. If the two pairs $\{B, E\}$ and $\{D, F\}$ are weakly random subsequentially continuous and random compatible of type (E). Then B, D, E and F have a unique common random fixed point in A.

3. Main Results

In this section we present our results as follow:

Theorem 2.1 : Let A be a Polish space and $B, D, E, F, : \phi \times A \to A$ are two pairs random operators fulfilling $B(v, A) \subset F(v, A)$, $D(v, A) \subset E(v, A)$ and

$$d(B(v,a), D(v,b)) \leq$$
$$\Omega(max\{d(E(v,a), F(v,b)), d(E(v,a), B(v,a)), d(F(v,b), D(v,b)), \tfrac{d(E(v,a),D(v,b))+d(F(v,b),B(v,a))}{2}\}), \quad (2.1)$$

for every $a, b \in A$, and $v \in \phi$ where $\Omega : [0, \infty) \to [0, \infty)$ is a comparison function. If one of B(v, A), D(v, A), E(v, A) or F(v, A) is a complete subspaces of A, then $\{B, E\}$ and $\{D, F\}$ have a unique random point of coincidence in A. Additionally, if $\{B, E\}$ and $\{D, F\}$ are random weakly compatible, then B, D, E and F have a unique random common fixed point in A.

Proof: Let $a_0(v) \in A$ be an arbitrary random variable in A. Consider $a_1(v) \in A$ such that $B(v, a_0) = F(v, a_1)$, since $B(A) \subset F(A)$. Suppose $D(A) \subset E(A)$ then there is $a_2(v) \in A$ such that $D(v, a_1) = E(v, a_2)$. Consequently, two sequences $a_n(v)$ and $b_n(v)$ in A can be generated such that;

$B(v, a_{2k}(v)) = F(v, a_{2k+1}(v)) = b_{2k+1}(v)$

$D(v, a_{2k+1}(v)) = E(v, a_{2k+2}(v)) = b_{2k+2}(v)$.

For a given $k \in N$ and employing (2.1) we get

$$\begin{aligned}
d(b_k(v), b_{k+1}(v)) &= d(B(v, a_{k-1}(v)), D(v, a_k(v))) \\
&\leq \Omega(max\{d(E(v, a_{k-1}(v)), F(v, a_k(v))), \\
&d(E(v, a_{k-1}(v)), B(v, a_{k-1}(v))), d(F(v, a_k(v)), D(v, a_k(v))), \\
&\frac{d(E(v, a_{k-1}(v)), D(v, a_k(v))) + d(F(v, a_k(v)), B(v, a_{k-1}(v)))}{2}\}) \\
&= \Omega(max\{d(b_{k-1}(v), b_k(v)), d(b_{k-1}(v), b_k(v)), d(b_k(v), b_{k+1}(v)), \\
&\frac{d(b_{k-1}(v), b_{k+1}(v)) + d(b_k(v), b_k(v))}{2}\}) \\
&\leq \Omega(max\{d(b_{k-1}(v), b_k(v)), d(b_k(v), b_{k+1}(v)), \frac{d(b_{k-1}(v), b_k(v)) + d(b_k(v), b_{k+1}(v))}{2}\}) \\
&\leq \Omega(d(b_{k-1}(v), b_k(v)))
\end{aligned}$$

Continuing the process and by induction we have

$$d(b_k(v), b_{k+1}(v)) \leq \Omega^k d(b_0(v), b_1(v)). \quad (2.2)$$

For $n > m$ and using the triangle inequality we have

$$\begin{aligned} d(b_m(v), b_n(v)) \quad &\leq \quad d(b_m(v), b_{m+1}(v)) + d(b_{m+1}(v), b_{m+2}(v)) + d(b_{m+2}(v), b_{m+3}(v)) \\ &+ \cdots + d(b_{n-1}(v), b_n(v)) \\ &\leq \Omega^m d(b_0(v), b_1(v)) + \Omega^{m+1} d(b_0(v), b_1(v)) + \Omega^{m+2} d(b_0(v), b_1(v)) \\ &+ \cdots + \Omega^{m+n-1} d(b_0(v), b_1(v)) \\ &< (\Omega^m + \Omega^{m+1} + \Omega^{m+2} + \cdots + \Omega^{m+n-1}) d(b_0(v), b_1(v)) \\ &\leq \frac{\Omega^m}{1 - \Omega^m} d(b_0(v), b_1(v)). \end{aligned}$$

With the condition of Ω we have that $\{b_k(v)\}$ is a Cauchy sequence in A. If E(v, A) is a complete subspace of A, then there is $b_0 \in A$ such that $E(v, a_k(v)) = b_k(v)$ converges to ω.

Likewise, $E(v, a_k(v)) = D(v, a_{k+1}(v)) = b_k \to \omega$

and $F(v, a_{k-1}(v)) = B(v, a_{k-2}(v)) = b_{k-1} \to \omega$

as $k \to \infty$. If there is z in A such that $E(v, z(v)) = \omega(v)$. Then we prove that $B(v, z(v)) = \omega(v)$. Otherwise, we show that $B(v, z(v)) \neq \omega(v)$.

$$\begin{aligned} d(B(v, z(v)), \omega(v)) \quad &\leq \quad d(B(v, z(v)), D(v, a_k(v))) + d(D(v, a_k(v), \omega(v)) \\ &\leq \Omega(max\{d(E(v, z(v)), F(v, a_k(v))), d(E(v, z(v)), B(v, z(v))), \\ &d(F(v, a_k(v)), D(v, a_k(v))), \\ &\frac{d(E(v, z(v)), D(v, a_k(v))) + d(F(v, a_k(v)), B(v, z(v)))}{2}\}) \\ &+ d(D(v, a_k(v), \omega(v)). \end{aligned}$$

As $k \to \infty$ we have

$$\begin{aligned} d(B(v, z(v)), \omega(v)) \quad &\leq \quad \Omega(max\{d(\omega(v), \omega(v)), d(\omega(v), B(v, z(v))), d(\omega(v), \omega(v)), \\ &\frac{d(\omega(v), \omega(v)) + d(\omega(v), B(v, z(v)))}{2}\}) \\ &\leq \Omega(d(B(v, z(v)), \omega(v))) < d(B(v, z(v)), \omega(v)), \end{aligned}$$

a contradiction, hence we have $d(B(v, z(v)) = \omega(v)$. This shows that

$B(v, z(v)) = E(v, z(v)) = \omega(v)$. Since $\omega(v) \in B(v, A) \subset F(v, A)$, then there is a $u(v) \in A$ such that $F(v, u(v)) = \omega(v)$. We claim that $D(v, u(v)) = \omega(v)$. On the other hand, we assume that $D(v, u(v)) \neq \omega(v)$.

So using (1) we obtain,

$$
\begin{aligned}
d(\omega(v), D(v,u(v))) \quad &\leq \quad d(\omega(v), B(v, a_k(v))) + (B(v, a_k(v)), D(v, u(v))) \\
&\leq \Omega(max\{d(E(v, a_k(v)), F(v, u(v))), d(E(v, u(v)), B(v, a_k(v))), \\
&d(D(v, u(v)), F(v, u(v))), \\
&\frac{d(E(v, a_k(v)), D(v, u(v))) + d(F(v, u(v)), B(v, a_k(v)))}{2}\}) \\
&+d(B(v, a_k(v), \omega(v)).
\end{aligned}
$$

As $k \to \infty$ we obtain

$$
\begin{aligned}
d(\omega(v), D(v, u(v))) \quad &\leq \quad \Omega(max\{d(\omega(v), D(v, u(v))), d(D(v, u(v)), \omega(v)), \\
&\frac{d(D(v, u(v)), \omega(v))}{2}\}) \\
&\leq \Omega(d(\omega(v), D(v, u(v))) < d(\omega(v), D(v, u(v)).
\end{aligned}
$$

This is a contradiction according to the condition of Ω. Therefore $D(v, u(v)) = \omega(v)$. This shows that $\{B, E\}$ and $\{D, F\}$ have a common point of coincidence. Consider $\{B, E\}$ and $\{D, F\}$ being random weakly compatible, then

$B(v, \omega(v)) = B(v, E(v, z(v))) = E(v, B(v, z(v))) = E(v, \omega(v)) = \omega_1(v)$ (say) and

$D(v, \omega(v)) = D(v, F(v, u(v))) = F(v, D(v, u(v))) = F(v, \omega(v)) = \omega_2(v)$ (say).

Now we prove that the points of coincidence are unique.

$$
\begin{aligned}
d(\omega_1(v), \omega_2(v)) \quad &= \quad d(B(v, \omega_1(v)), F(v, \omega_2(v)) \\
&\leq \Omega(max\{d(E(v, \omega_1(v)), F(v, \omega_2(v))), d(E(v, \omega_1(v)), B(v, \omega_1(v))), \\
&d(F(v, \omega_2(v)), D(v, \omega_2(v))), \\
&\frac{d(E(v, \omega_1(v))), D(v, \omega_2(v))) + d(F(v, \omega_2(v)), B(v, \omega_1(v)))}{2}\}) \\
&\leq \Omega(max\{d(\omega_1(v), \omega_2(v)), d(\omega_1(v), \omega_1(v)), d(\omega_2(v), \omega_2(v)), \\
&\frac{d(\omega_1(v), \omega_2(v)) + d(\omega_2(v), \omega_1(v))}{2}\}) \\
&\leq \Omega(d(\omega_1(v), \omega_2(v))) < d(\omega_1(v), \omega_2(v)).
\end{aligned}
$$

This shows that $\omega_1(v) = \omega_2(v)$ by the property of Ω. Therefore

$B(v, \omega(v)) = E(v, \omega(v)) = D(v, \omega(v)) = F(v, \omega(v))$

Now, we prove that $\omega(v)$ is the common fixed point of B, D, E and F in A. We claim that $\omega(v) = D(v, \omega(v))$.

Suppose $\omega(v) \neq D(v, \omega(v))$ then using (2.1) we have,

$$\begin{aligned}
d(\omega(v), D(v,\omega(v))) &= d(B(v,\omega(v)), D(v,\omega(v))) \\
&\leq \Omega(max\{d(E(v,\omega(v)), F(v,\omega(v))), d(E(v,\omega(v)), B(v,\omega(v))), \\
&d(F(v,\omega(v)), D(v,\omega(v))), \\
&\frac{d(E(v,\omega(v))), D(v,\omega(v))) + d(F(v,\omega(v)), B(v,\omega(v)))}{2}\}) \\
&\leq \Omega(max\{d(\omega(v), D(v,\omega(v))), d(D(v,\omega(v)), \omega(v)), \\
&\frac{d(D(v,\omega(v)), \omega(v))}{2}\}) \\
&\leq \Omega(d(\omega(v), D(v,\omega(v))) < d(\omega(v), D(v,\omega(v))
\end{aligned}$$

This contradict our assumption that $\omega(v) \neq D(v,\omega(v))$. Hence $\omega(v) = D(v,\omega(v))$. Thus $\omega(v)$ is the random common fixed point of B, D, E and F.

For the uniqueness of the random common fixed point of B, D, E and F, we assume a different random common fixed point of B, D, E and F say $\omega'(v)$ such that $\omega(v) = \omega'(v)$ .On the other hand, let $\omega(v) \neq \omega'(v)$ and using (2.1) we get

$$\begin{aligned}
d(\omega(v), \omega'(v)) &= d(B(v,\omega(v)), D(v,\omega'(v))) \\
&\leq \Omega(max\{d(E(v,\omega(v)), F(v,\omega'(v))), d(E(v,\omega(v)), B(v,\omega(v))), \\
&d(F(v,\omega'(v)), D(v,\omega'(v))), \\
&\frac{d(E(v,\omega(v))), D(v,\omega'(v))) + d(F(v,\omega'(v)), B(v,\omega(v)))}{2}\}) \\
&\leq \Omega(max\{d(\omega(v), \omega'(v)), d(\omega(v), \omega(v)), d(\omega'(v), \omega'(v)), \\
&\frac{d(\omega(v), \omega'(v)) + d(\omega'(v), \omega(v))}{2}\}) \\
&\leq \Omega(d(\omega(v), \omega'(v)) < d(\omega(v), \omega'(v)),
\end{aligned}$$

a contradiction, hence $\omega(v) = \omega'(v)$.

Remark : Theorem 2.1 gives an independent version of the result of Rashwan and Hammed [9](Theorem 9) because the result is proved without the assumption of weakly random subsequential continuity and compatibility of type (E). Theorem 2.1 proves the random version of the result of Eke and Akinlabi[8] (Theorem 2.1) with general contractive mappings in the context of Polish space.

If $\Omega(t) = k(t)$ in Theorem 2.1 then we obtain the following corollary.

Corollary 2.2 : Let A be a Polish space and $B, D, E, F, : \phi \times A \to A$ are two pairs random operators fulfilling

$d(B(v,a), D(v,b)) \leq$

$k(max\{d(E(v,a),F(v,b)),d(E(v,a),B(v,a)),d(F(v,b),D(v,b)),\frac{d(E(v,a),D(v,b))+d(F(v,b),B(v,a))}{2}\}),$

for every $a,b \in A$, and $v \in \phi$ where $k \in [0,1)$. If one of B(v, A), D(v, A), E(v, A) or F(v, A) is a complete subspaces of A, then $\{B,E\}$ and $\{D,F\}$ have a unique random point of coincidence in A. Additionally, if $\{B,E\}$ and $\{D,F\}$ are random weakly compatible, then B,D,E and F have a unique random common fixed point in A.

Example: Let $X = R$ with the usual metric d and $\phi = [0,1]$. Define $\Omega : (0,\infty) \to [0,\infty)$ by $\Omega(t) = t$. Let the random operator $B,D,E,F : \phi \times A \to A$ be defined by

$$E(v,a(v)) = F(v,a(v)) = \begin{cases} 0, & if\ a(v) \leq 1 \\ 2, & otherwise \end{cases} \tag{3.1}$$

$$B(v,a(v)) = D(v,a(v)) = \begin{cases} 0, & if\ a(v) \leq 1 \\ \frac{1}{2}, & otherwise \end{cases} \tag{3.2}$$

The pairs $\{B,E\}$ and $\{D,F\}$ are weakly compatible, $B(v,A) \subset F(v,A)$, and $D(v,A) \subset E(v,A)$. The contractive condition is satisfied and the unique random common fixed point of B,D,E and F is 0.

Conclusion: The random common fixed point of two pairs of generalized random nonlinear contractive mappings employing the property of random weakly compatible mappings is proved in the context of Polish space. Further work can be done using different contractive mappings in this space. We provided an example to support our result.

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