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HADAMARD AND FEJÉR-HADAMARD TYPE INEQUALITIES FOR CONVEX AND RELATIVE CONVEX FUNCTIONS VIA AN EXTENDED GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract. In this paper, we will prove the Hadamard and the Fejér-Hadamard type integral inequalities for convex and relative convex functions due to an extended generalized Mittag-Leffler function. These results contain several fractional integral inequalities for the well known fractional integral operators.

1. Introduction

Convex functions are very useful in the field of mathematical inequalities.

Definition 1.1. *Let I be an interval of real numbers. Then a function $f : I \to \mathbb{R}$ is said to be convex function, if for all $x, y \in I$ and $0 \leq \lambda \leq 1$, the following inequality holds:*

$$f(x\lambda + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

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Convex functions are equivalently defined by the following inequality which is well known as the Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

where $f : I \to \mathbb{R}$ is a convex function on I and $a, b \in I, a < b$.

Following definitions are given in [8].

Definition 1.2. *Let T_g be a set of real numbers. This set T_g is said to be relative convex with respect to an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, if*

$$(1-t)x + tg(y) \in T_g$$

where $x, y \in \mathbb{R}$ such that $x, g(y) \in T_g$, $0 \leq t \leq 1$.

Definition 1.3. *A function $f : T_g \to \mathbb{R}$ is said to be relative convex, if there exists an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, such that*

$$f((1-t)x + tg(y)) \leq (1-t)f(x) + tf(g(y)),$$

holds, where $x, y \in \mathbb{R}$ such that $x, g(y) \in T_g$, $0 \leq t \leq 1$.

Noor et al. proved the following Hadamard type integral inequality in [8] for relative convex functions via Riemann-Liouville fractional integral operators.

Theorem 1.1. *Let f be a positive relative convex function and integralable on $[a, g(b)]$. Then for $\alpha > 0$, the following inequalities hold:*

$$\begin{aligned} f\left(\frac{a+g(b)}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(g(b)-a)^\alpha}[I^\alpha_{a^+}f(g(b)) + I^\alpha_{b_-}f(a)] \\ &\leq \frac{f(a)+f(g(b))}{2}. \end{aligned}$$

Now we define the extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(.;p)$ as follows:

Definition 1.4. *[2] Let $\mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$ is defined by:*

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n+\alpha)} \frac{t^n}{(l)_{n\delta}}, \tag{1.1}$$

where β_p is the generalized beta function defined by:

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}dt$$

and $(c)_{nk}$ is the Pochhammer symbol defined as $(c)_{nk} = \frac{\Gamma(c+nk)}{\Gamma(c)}$.

In [2], properties of the generalized Mittag-Leffler function are discussed and it is given that $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$ is absolutely convergent for $k < \delta + \Re(\mu)$. Let S be the sum of series of absolute terms of the Mittag-Leffler *function* $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$*, then we have* $\left|E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)\right| \leq S$*. We use this property of Mittag-Leffler function to prove the results of this paper.*

Remark 1.2. *Mittag-Leffler function (1.1) is the generalization of following functions:*

(i) By setting $p = 0$*, it reduces to the Salim-Faraj function* $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t)$ *defined in [12].*

(ii) By setting $l = \delta = 1$*, it reduces to the function* $E^{\gamma,k,c}_{\mu,\alpha}(t;p)$ *defined by Rahman et al. in [11].*

(iii) By setting $p = 0$ *and* $l = \delta = 1$*, it reduces to the Shukla-Prajapati function* $E^{\gamma,k}_{\mu,\alpha}(t)$ *defined in [13] (see also [14]).*

(iv) By setting $p = 0$ *and* $l = \delta = k = 1$*, it reduces to the Prabhakar function* $E^{\gamma}_{\mu,\alpha}(t)$ *defined in [10].*

The corresponding left and right sided generalized fractional integral operators $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}$ and $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}$ are defined as follows:

Definition 1.5. *[2] Let* $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$*,* $\Re(\mu), \Re(\alpha), \Re(l) > 0$*,* $\Re(c) > \Re(\gamma) > 0$ *with* $p \geq 0$*,* $\delta > 0$ *and* $0 < k \leq \delta + \Re(\mu)$*. Let* $f \in L_1[a,b]$ *and* $x \in [a,b]$*. Then the generalized fractional integral operators* $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f$ *and* $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}f$ *are defined by:*

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(x;p) = \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p)f(t)dt, \tag{1.2}$$

and

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}f\right)(x;p) = \int_x^b (t-x)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(t-x)^{\mu};p)f(t)dt. \tag{1.3}$$

Remark 1.3. *Operators in (1.2) and (1.3) are the generalization of the following fractional integral operators:*

(i) By setting $p = 0$*, they reduce to the fractional integral operators defined by Salim-Faraj in [12].*

(ii) By setting $l = \delta = 1$*, they reduce to the fractional integral operators defined by Rahman et al. in [11].*

(iii) By setting $p = 0$ *and* $l = \delta = 1$*, they reduce to the fractional integral operators defined by Srivastava-Tomovski in [14].*

(iv) By setting $p = 0$ *and* $l = \delta = k = 1$*, they reduce to the fractional integral operators defined by Prabhakar in [10].*

(v) By setting $p = \omega = 0$*, they reduces to the Riemann-Liouville fractional integrals.*

In [5] the Hadamard and the Fejér-Hadamard inequalities for convex functions via generalized fractional integral operator containing the Mittag-Leffler function defined in [12] have been proved.
In [1,6,8], the Hadamard and the Fejér-Hadamard type inequalities for convex and relative convex functions via Riemann-Liouville fractional integral operators and extended generalized fractional integral operators have been proved. In this paper, we give fractional integral inequalities of the Hadamard and the Fejér-Hadamard type for convex and relative convex functions by using the extended generalized Mittag-Leffler function. We also produce the results which are given in [1,6,8] by setting particular values of parameters.

2. **Main Results**

Following lemmas are useful to establish the main results.

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f \in L_1[a, b]$ and symmetric about $\frac{a+b}{2}$ with $a < b$. Then for extended generalized fractional integral operators (1.2) and (1.3), the following equality holds:*

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(b;p) = \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}f\right)(a;p) = \frac{\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(b;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}f\right)(a;p)}{2}. \tag{2.1}$$

Proof. Using symmetricity of f we have $f(a + b - t) = f(t)$, therefore by (1.2) of Definition 1.5, we have

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(b;p) = \int_a^b (b-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-t)^{\mu};p)f(t)dt, \tag{2.2}$$

putting $t = a + b - t$ in above, we get

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(b;p) = \int_a^b (t-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(t-a)^{\mu};p)f(t)dt.$$

By using Definition 1.5, we get

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(b;p) = \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}f\right)(a;p). \tag{2.3}$$

Therefore we get (2.1). □

Lemma 2.2. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f' \in L_1[a, b]$ with $a < b$. If $g : [a, b] \to \mathbb{R}$ is integrable and symmetric about $\frac{a+b}{2}$, then for extended generalized fractional integral operators (1.2) and (1.3), the following equality holds:*

$$\begin{aligned}
&\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}g\right)(b;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}g\right)(a;p)\right] \\
&-\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}fg\right)(b;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}fg\right)(a;p)\right] \\
&= \int_a^b \left[\int_a^t (b-s)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-s)^{\mu};p)g(s)ds \right.\\
&\left. - \int_t^b (s-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(s-a)^{\mu};p)g(s)ds\right] f'(t)dt.
\end{aligned} \tag{2.4}$$

Proof. One can note that

$$\begin{aligned}&\int_a^b\left[\int_a^t (b-s)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(b-s)^{\mu};p)g(s)ds\right.\\&\left.-\int_t^b (s-a)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(s-a)^{\mu};p)g(s)ds\right]f'(t)dt\\&=\int_a^b\left[\int_a^t (b-s)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(b-s)^{\mu};p)g(s)ds\right]f'(t)dt\\&+\int_a^b\left[-\int_t^b (s-a)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(s-a)^{\mu};p)g(s)ds\right]f'(t)dt.\end{aligned}$$

By simple calculation, we get

$$\begin{aligned}&\int_a^b\left[\int_a^t (b-s)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(b-s)^{\mu};p)g(s)ds\right]f'(t)dt\\&=f(b)\left(\int_a^b (b-s)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(b-s)^{\mu};p)g(s)ds\right)\\&\quad-\int_a^b\left((b-t)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(b-t)^{\mu};p\right)fg(t)dt.\end{aligned}$$

By using Definition 1.5, we get

$$f(b)\left(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c}g\right)(b;p)-\left(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c}fg\right)(b;p).$$

Now by using Lemma 2.1, we have

$$\begin{aligned}&\int_a^b\left[\int_a^t (b-s)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(b-s)^{\mu};p)g(s)ds\right]f'(t)dt\\&=\frac{f(b)}{2}\left[\left(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c}g\right)(b;p)+\left(\epsilon_{\mu,\alpha,l,\omega,b_-}^{\gamma,\delta,k,c}g\right)(a;p)\right]-\left(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c}fg\right)(b;p)\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}&\int_a^b\left[-\int_t^b (s-a)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(s-a)^{\mu};p)g(s)ds\right]f'(t)dt\\&=\frac{f(a)}{2}\left[\left(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c}g\right)(b;p)+\left(\epsilon_{\mu,\alpha,l,\omega,b_-}^{\gamma,\delta,k,c}g\right)(a;p)\right]-\left(\epsilon_{\mu,\alpha,l,\omega,b_-}^{\gamma,\delta,k,c}fg\right)(a;p).\end{aligned}\tag{2.6}$$

By adding (2.6) and (2.5), we get (2.4). □

In the following we give integral inequality of the Hadamard type.

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that $f'\in L_1[a,b]$ with $a<b$. If $|f'|$ is convex on $[a,b]$ and $g:I\to\mathbb{R}$ is continuous and symmetric function about $\frac{a+b}{2}$, then for extended generalized*

fractional integral operators (1.2) and (1.3), the following inequality holds:

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}g\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}g\right)(a;p)\right]\right.\\&\left.-\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}fg\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}fg\right)(a;p)\right]\right|\\&\leq\frac{\parallel g\parallel_\infty S(b-a)^{\alpha+1}}{\alpha(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)[|f'(a)+f'(b)|],\end{aligned}$$

for $k<\delta+\Re(\mu)$ and $\parallel g\parallel_\infty=\sup_{t\in[a,b]}|g(t)|$.

Proof. By using Lemma 2.2, we have

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}g\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}g\right)(a;p)\right]\right.\\&\left.-\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}fg\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}fg\right)(a;p)\right]\right|\\&\leq\int_a^b\left|\left[\int_a^t(b-s)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-s)^\mu;p)g(s)ds\right.\right.\\&\left.\left.-\int_t^b(s-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(s-a)^\mu;p)g(s)ds\right]\right||f'(t)|dt.\end{aligned}\tag{2.7}$$

Since $|f'|$ is convex, so we have

$$|f'(t)|\leq\frac{b-t}{b-a}|f'(a)|+\frac{t-a}{b-a}|f'(b)|\tag{2.8}$$

where $t\in[a,b]$.

From symmetricity of g, we have

$$\begin{aligned}&\int_t^b(s-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(s-a)^\mu;p)g(s)ds\\&=\int_a^{a+b-t}(b-s)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-s)^\mu;p)g(a+b-s)ds\\&=\int_a^{a+b-t}(b-s)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-s)^\mu;p)g(s)ds.\end{aligned}$$

This gives

$$\begin{aligned}&\left|\int_a^t(b-s)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-s)^\mu;p)g(s)ds\right.\\&\left.-\int_t^b(s-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(s-a)^\mu;p)g(s)ds\right|\\&=\left|\int_t^{a+b-t}(b-s)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-s)^\mu;p)g(s)ds\right|\\&\leq\begin{cases}\int_t^{a+b-t}|(b-s)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-s)^\mu;p)g(s)|ds, t\in[a,\frac{a+b}{2}]\\\int_{a+b-t}^t|(b-s)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-s)^\mu;p)g(s)|ds, t\in[\frac{a+b}{2},b].\end{cases}\end{aligned}\tag{2.9}$$

From (2.7), (2.8), (2.9) and absolute convergence of Mittag-Leffler function, we get

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}g\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}g\right)(a;p)\right]\right. \\
&\left.-\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}fg\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}fg\right)(a;p)\right]\right| \\
&\leq \int_a^{\frac{a+b}{2}}\left(\int_a^{a+b-t}|(b-s)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-s)^{\mu};p)g(s)|ds\right)\left(\frac{b-t}{b-a}|f'(a)|\right. \\
&\left.+\frac{t-a}{b-a}|f'(b)|\right)dt+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}|(b-s)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-s)^{\mu};p)g(s)|ds\right) \\
&\times\left(\frac{b-t}{b-a}|f'(a)|+\frac{t-a}{b-a}|f'(b)|\right)dt \\
&\leq \frac{\parallel g\parallel_\infty S}{\alpha(b-a)}\left[\int_a^{\frac{a+b}{2}}((b-t)^\alpha-(t-a)^\alpha(b-t)|f'(a)|)dt\right. \\
&+\int_a^{\frac{a+b}{2}}((b-t)^\alpha-(t-a)^\alpha(t-a)|f'(b)|)dt \\
&+\int_{\frac{a+b}{2}}^{b}((t-a)^\alpha-(b-t)^\alpha(b-t)|f'(a)|)dt \\
&\left.+\int_{\frac{a+b}{2}}^{b}((t-a)^\alpha-(b-t)^\alpha(t-a)|f'(b)|)dt\right].
\end{aligned} \tag{2.10}$$

As we have

$$\int_a^{\frac{a+b}{2}}\left((b-t)^\alpha-(t-a)^\alpha\right)(b-t)dt=\frac{(b-a)^{\alpha+2}}{\alpha+1}\left(\frac{\alpha+1}{\alpha+2}-\frac{1}{2^{\alpha+1}}\right)$$

and

$$\int_a^{\frac{a+b}{2}}\left((b-t)^\alpha-(t-a)^\alpha\right)(t-a)dt=\frac{(b-a)^{\alpha+2}}{\alpha+1}\left(\frac{1}{\alpha+2}-\frac{1}{2^{\alpha+1}}\right).$$

By using the values of above integrals in (2.10), we have

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}g\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}g\right)(a;p)\right]\right. \\
&\left.-\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}fg\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}fg\right)(a;p)\right]\right| \\
&\leq \frac{\parallel g\parallel_\infty S}{\alpha(b-a)}\frac{(b-a)^{\alpha+2}}{\alpha+1}\left[\left(\frac{\alpha+1}{\alpha+2}-\frac{1}{2^{\alpha+1}}\right)\right. \\
&\left.+\left(\frac{1}{\alpha+2}-\frac{1}{2^{\alpha+1}}\right)\right][|f'(a)|+|f'(b)|] \\
&=\frac{\parallel g\parallel_\infty S}{\alpha(\alpha+1)}(b-a)^{\alpha+1}\left(1-\frac{1}{2^\alpha}\right)[|f'(a)|+|f'(b)|].
\end{aligned}$$

□

Remark 2.4. *(i) If we put $p=0$ in Theorem 2.3, then we get [1, Theorem 2.3].*

(ii) If we put $\omega=p=0$ in Theorem 2.3, then we get [6, Theorem 2.6].

Theorem 2.5. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that $f'\in L_1[a,b]$ with $a<b$. If $|f'|^q$, $q\geq 1$ is convex on $[a,b]$ and $g:I\to\mathbb{R}$ is continuous and symmetric function about $\frac{a+b}{2}$, then for extended generalized fractional integral operators (1.2) and (1.3), the following inequality holds:*

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}g\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}g\right)(a;p)\right]\right.\\&\left.-\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}fg\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}fg\right)(a;p)\right]\right|\\&\leq\frac{2\parallel g\parallel_\infty S(b-a)^{\alpha+1}}{\alpha(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}},\end{aligned}\tag{2.11}$$

for $k<\delta+\Re(\mu)$ and $\parallel g\parallel_\infty=\sup_{t\in[a,b]}|g(t)|$.

Proof. By using Lemma 2.2, power mean inequality, the inequality (2.9) takes the following form:

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}g\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}g\right)(a;p)\right]\right.\\&\left.-\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}fg\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}fg\right)(a;p)\right]\right|\\&\leq\left[\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-s)^\mu;p)g(s)ds\right|dt\right]^{1-\frac{1}{q}}\\&\left[\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-s)^\mu;p)g(s)ds\right||f'(t)|^qdt\right]^{\frac{1}{q}}.\end{aligned}\tag{2.12}$$

Using absolute convergence of Mittag-Leffler function and $\parallel g\parallel_\infty=\sup_{t\in[a,b]}|g(t)|$, we have

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}g\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}g\right)(a;p)\right]\right.\\&\left.-\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}fg\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}fg\right)(a;p)\right]\right|\\&\leq\parallel g\parallel_\infty^{1-\frac{1}{q}}S^{1-\frac{1}{q}}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}(b-s)^{\alpha-1}ds\right)dt\right.\\&\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t(b-s)^{\alpha-1}ds\right)dt\right]^{1-\frac{1}{q}}\\&\times\parallel g\parallel_\infty^{\frac{1}{q}}S^{\frac{1}{q}}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}(b-s)^{\alpha-1}ds\right)|f'(t)|^qdt\right.\\&\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t(b-s)^{\alpha-1}ds\right)|f'(t)|^qdt\right]^{\frac{1}{q}}.\end{aligned}$$

By some calculation, we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}g\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}g\right)(a;p)\right]\right.$$
$$\left.-\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}fg\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}fg\right)(a;p)\right]\right|$$
$$\leq\| g \|_\infty S\left[\frac{(b-a)^{\alpha+1}}{\alpha(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)+\frac{(b-a)^{\alpha+1}}{\alpha(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_a^{\frac{a+b}{2}}((b-t)^\alpha-(t-a)^\alpha)|f'(t)|^q dt+\int_{\frac{a+b}{2}}^b((b-t)^\alpha-(t-a)^\alpha)|f'(t)|^q dt\right]^{\frac{1}{q}}.$$

Since $|f'|^q$ is convex, so we have

$$|f'(t)|^q\leq\frac{b-t}{b-a}|f'(a)|^q+\frac{t-a}{b-a}|f'(b)|^q. \tag{2.13}$$

Hence

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}g\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}g\right)(a;p)\right]\right.$$
$$\left.-\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}fg\right)(b;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b_-}fg\right)(a;p)\right]\right|$$
$$\leq\| g \|_\infty S\left[2\frac{(b-a)^{\alpha+1}}{\alpha(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_a^{\frac{a+b}{2}}((b-t)^\alpha-(t-a)^\alpha)\left(\frac{b-t}{b-a}|f'(a)|^q+\frac{t-a}{b-a}|f'(b)|^q\right)dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^b((b-t)^\alpha-(t-a)^\alpha)\left(\frac{b-t}{b-a}|f'(a)|^q+\frac{t-a}{b-a}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}.$$

From it (2.11) can be obtained. □

Remark 2.6. *(i) If we put $p=0$ in Theorem 2.5, then we get [1, Theorem 2.5].*
(ii) If we put $\omega=p=0$ in Theorem 2.5, then we get [6, Theorem 2.8].

In the following we give the Hadamard inequality for relative convex functions via generalized fractional integral operators.

Theorem 2.7. *Let $f:[a,g(b)]\to\mathbb{R}$ be a function such that $f\in L_1[a,g(b)]$ with $a<b$. If f is relative convex on $[a,g(b)]$, then for extended generalized fractional integral operators (1.2) and (1.3), the following inequalities hold:*

$$f\left(\frac{a+g(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',a^+}1\right)(g(b);p) \tag{2.14}$$
$$\leq\frac{1}{2}\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',a^+}f\right)(g(b);p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',g(b)_-}f\right)(a;p)\right]$$
$$\leq\frac{f(a)+f(g(b))}{2}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',g(b)_-}1\right)(a;p),$$

where $\omega' = \frac{\omega}{(g(b)-a)^{\mu}}$.

Proof. Since f is relative convex, so we have

$$f\left(\frac{a+g(b)}{2}\right) = f\left[\left(\frac{1}{2}(ta+(1-t)g(b)\right) + \left(1-\frac{1}{2}\right)((1-t)a+tg(b))\right] \tag{2.15}$$
$$\leq \frac{1}{2}f\left(ta+(1-t)g(b)\right) + \frac{1}{2}f\left((1-t)a+tg(b)\right).$$

Multiplying (2.15) by $2t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)$ on both sides and then integrating over $[0,1]$, we have

$$2f\left(\frac{a+g(b)}{2}\right)\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)dt \tag{2.16}$$
$$\leq \int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)f\left(ta+(1-t)g(b)\right)dt$$
$$+\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)f\left((1-t)a+tg(b)\right)dt.$$

Putting $x = ta+(1-t)g(b)$ and $y = (1-t)a+tg(b)$ in above, we have

$$2f\left(\frac{a+g(b)}{2}\right)\int_{g(b)}^{a}\left(\frac{g(b)-x}{g(b)-a}\right)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(\frac{g(b)-x}{g(b)-a}\right)^{\mu};p\right)\left(\frac{-dx}{g(b)-a}\right) \tag{2.17}$$
$$\leq \int_{g(b)}^{a}\left(\frac{g(b)-x}{g(b)-a}\right)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(\frac{g(b)-x}{g(b)-a}\right)^{\mu};p\right)f(x)\left(\frac{-dx}{g(b)-a}\right)$$
$$+\int_{a}^{g(b)}\left(\frac{y-a}{g(b)-a}\right)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega\left(\frac{y-a}{g(b)-a}\right)^{\mu};p\right)f(y)\left(\frac{dy}{g(b)-a}\right).$$

By using Definition 1.5, we get

$$2f\left(\frac{a+g(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',a^+}1\right)(g(b);p) \tag{2.18}$$
$$\leq \left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',a^+}f\right)(g(b);p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',g(b)_-}f\right)(a;p)\right].$$

Again by using the relative convexity of f, we have

$$f(ta+(1-t)g(b)) + f((1-t)a+tg(b)) \leq tf(a)+(1-t)f(g(b))+(1-t)f(a)+tf(g(b)). \tag{2.19}$$

Multiplying (2.19) by $t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)$ on both sides and then integrating over $[0,1]$, we have

$$\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)f(ta+(1-t)g(b))dt$$
$$+\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)f((1-t)a+tg(b))dt$$
$$\leq \int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)tf(a)+(1-t)f(g(b))dt$$
$$+\int_0^1 t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)(1-t)f(a)+tf(g(b))dt.$$

Putting $x = ta + (1-t)g(b)$ and $y = (1-t)a + tg(b)$ in above and then using Definition 1.5, we get

$$\begin{aligned}&\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',a^+}f\right)(g(b);p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',g(b)_-}f\right)(a;p)\right] \\ &\leq [f(a) + f(g(b))]\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',g(b)_-}1\right)(a;p).\end{aligned} \tag{2.20}$$

Combining it with (2.18), (2.14) is obtained. □

Remark 2.8. *(i) If we put $p = 0$ in Theorem 2.7, then we get [1, Theorem 2.8].*
(ii) If we put $\omega = p = 0$ and $k = 1$ in Theorem 2.7, then we get Theorem 1.1.

In the upcoming theorem we give the generalization of previous result.

Theorem 2.9. *Let $f : [g(a), g(b)] \to \mathbb{R}$ be a function such that $f \in L_1[g(a), g(b)]$ with $a < b$. If f is relative convex on $[g(a), g(b)]$, then for extended generalized fractional integral operators (1.2) and (1.3), the following inequalities hold:*

$$\begin{aligned}&f\left(\frac{g(a)+g(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',g(a)^+}1\right)(g(b);p) \\ &\leq \frac{1}{2}\left[\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',g(a)^+}f\right)(g(b);p) + \left(\epsilon^{\gamma,\delta,k}_{\mu,\alpha,l,\omega',g(b)_-}f\right)(a;p)\right] \\ &\leq \frac{f(g(a)) + f(g(b))}{2}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',g(b)_-}1\right)(g(a);p),\end{aligned}$$

where $\omega' = \frac{\omega}{(g(b)-g(a))^\mu}$.

Proof. Proof of this theorem is on the same lines of the proof of Theorem 2.7. □

Remark 2.10. *(i) If we put $p = 0$ in Theorem 2.9, then we get [1, Theorem 2.10].*
(ii) If we put $\omega = p = 0$ in Theorem 2.9, then we get [4, Corollary 1].

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