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UNIVALENT FUNCTIONS FORMULATED BY THE SALAGEAN-DIFFERENCE OPERATOR

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Abstract. We present a class of univalent functions $T_m(\kappa, \alpha)$ formulated by a new differential-difference operator in the open unit disk. The operator is a generalization of the well known Salagean's differential operator. Based on this operator, we define a generalized class of bounded turning functions. Inequalities, extreme points of $T_m(\kappa, \alpha)$, some convolution properties of functions fitting to $T_m(\kappa, \alpha)$, and other properties are discussed.

1. Introduction

Let Λ be the class of analytic function formulated by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U = \{z : |z| < 1\}.$$

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We symbolize by $T(\alpha)$ the subclass of Λ for which $\Re\{f'(z)\} > \alpha$ in U. For a function $f \in \Lambda$, we present the following difference operator

$$\begin{aligned} &D_\kappa^0 f(z) = f(z) \\ &D_\kappa^1 f(z) = zf'(z) + \frac{\kappa}{2}\left(f(z) - f(-z) - 2z\right), \quad \kappa \in \mathbb{R} \\ &\vdots \\ &D_\kappa^m f(z) = D_\kappa(D_\kappa^{m-1} f(z)) \\ &\qquad\qquad = z + \sum_{n=2}^{\infty}[n + \frac{\kappa}{2}(1 + (-1)^{n+1})]^m a_n z^n. \end{aligned} \tag{1.1}$$

It is clear that when $\kappa = 0$, we have the Salagean's differential operator [1]. We call D_κ^m the Salagean-difference operator. Moreover, D_κ^m is a modified Dunkl operator of complex variables [2] and for recent work [3]. Dunkl operator describes a major generalization of partial derivatives and realizes the commutative law in $\mathbb{R}^n$. In geometry, it attains the reflexive relation, which is plotting the space into itself as a set of fixed points.

Example 1. *(see Figs 1 and 2)*

- *Let* $f(z) = z/(1-z)$ *then*

$$D_1^1 f(z) = z + 2z^2 + 4z^3 + 4z^4 + 6z^5 + 6z^6 + ...$$

- *Let* $f(z) = z/(1-z)^2$ *then*

$$D_1^1 f(z) = z + 4z^2 + 12z^3 + 16z^4 + 30z^5 + 36z^6 + ...$$

We proceed to define a generalized class of bounded turning utilizing the the Salagean-difference operator. Let $T_m(\kappa, \alpha)$ denote the class of functions $f \in \Lambda$ which achieve the condition

$$\Re\{(D_\kappa^m f(z))'\} > \alpha, \quad 0 \le \alpha \le 1, \ z \in U, \ m = 0, 1, 2, \dots .$$

Clearly, $T_0(\kappa, \alpha) = T(\alpha)$ (the bounded turning class of order α). The Hadamard product or convolution of two power series is denoted by $(*)$ achieving

$$\begin{aligned} f(z) * h(z) &= \Big(z + \sum_{n=2}^{\infty} a_n z^n\Big) * \Big(z + \sum_{n=2}^{\infty} \eta_n z^n\Big) \\ &= z + \sum_{n=2}^{\infty} a_n \eta_n z^n. \end{aligned} \tag{1.2}$$

The aim of this effort is to present several important properties of the class $T_m(\kappa, \alpha)$. For this purpose, we need the following auxiliary preliminaries.

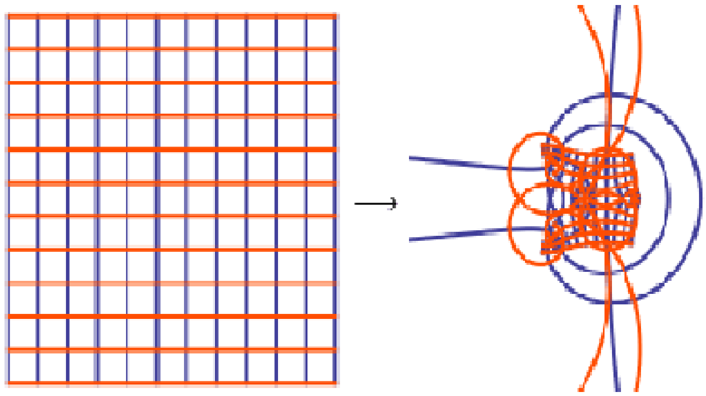

FIGURE 1. $D_1^1(z/(1-z))$

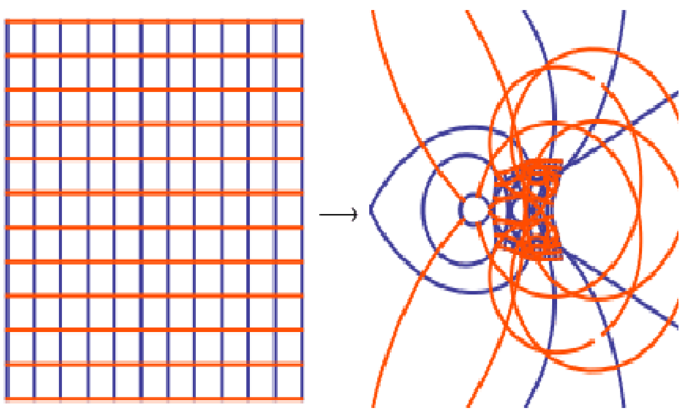

FIGURE 2. $D_1^1(z/(1-z)^2)$

Lemma 1. *Let $\{a_n\}_{n=0}^{\infty}$ be a convex null sequence $(a_0 - a_1 \geq a_1 - a_2, ... \geq 0)$. Then the function $\rho(z) = a_0/2 + \sum_{n=1}^{\infty} a_n z^n$, is analytic and $\Re\rho(z) > 0$ in U.*

Lemma 2. *If $\rho(z)$ is analytic in U, $\rho(0) = 1$ and $\Re\rho(z) > 1/2, z \in U$, then for any function ϱ analytic in U, the function $\rho * \varrho$ assigns its credits in the convex hull of $\varrho(U)$.*

Lemma 3. *[4] For all $z \in U$ the sum*

$$\Re\Big(\sum_{n=2} \frac{z^{n-1}}{n+1}\Big) > -\frac{1}{3}.$$

There are different techniques of studying the class of bounded turning functions, such as using partial sums or applying Jack Lemma [5]- [7].

2. Results

In this section, we illustrate our results.

Theorem 4. $T_{m+1}(\kappa,\alpha) \subset T_m(\kappa,\alpha)$.

Proof. Let $f \in T_{m+1}(\kappa,\alpha)$, then we have

$$\Re\{1+\sum_{n=2}^{\infty} n[n+\frac{\kappa}{2}(1+(-1)^{n+1})]^{m+1}\, a_n z^{n-1}\} > \alpha.$$

Dividing the last inequality by $1-\alpha$ and adding $+1$ we obtain the inequality

$$\Re\{1+\frac{1}{2(1-\alpha)}\sum_{n=2}^{\infty} n[n+\frac{\kappa}{2}(1+(-1)^{n+1})]^{m+1}\, a_n z^{n-1}\} > \frac{1}{2}.$$

By employing the definition of the convolution, we have the construction

$$\begin{aligned}(D_\kappa^m f(z))' &= 1+\sum_{n=2}^{\infty} n[n+\frac{\kappa}{2}(1+(-1)^{n+1})]^{m}\, a_n z^{n-1} \\ &= \Big(1+\frac{1}{2(1-\alpha)}\sum_{n=2}^{\infty} n[n+\frac{\kappa}{2}(1+(-1)^{n+1})]^{m+1}\, a_n z^{n-1}\Big) \\ &* \Big(1+2(1-\alpha)\sum_{n=2}^{\infty}\frac{z^{n-1}}{n+\frac{\kappa}{2}(1+(-1)^{n+1})}\Big).\end{aligned}$$

In view of Lemma 1, with $a_0 = 1$ and $a_n = 1/(n+\frac{\kappa}{2}(1+(-1)^{n+1}), n = 1,2,...,$ we have

$$\Re\Big(1+2(1-\alpha)\sum_{n=2}^{\infty}\frac{z^{n-1}}{n+\frac{\kappa}{2}(1+(-1)^{n+1})}\Big) > \alpha.$$

In virtue of Lemma 2, we arrive at the required result. □

Theorem 5. $T_{m+1}(\kappa,\alpha) \subset T_m(\kappa,\beta)$, $\beta \le \alpha$, $0 \le \kappa \le 1/2$.

Proof. Let $f \in T_{m+1}(\kappa,\alpha)$ then we have

$$\Re\{1+\sum_{n=2}^{\infty} n[n+\frac{\kappa}{2}(1+(-1)^{n+1})]^{m+1}\, a_n z^{n-1}\} > \alpha.$$

Also, we have the convolution

$$\begin{aligned}(D^m_\kappa f(z))' &= 1+\sum_{n=2}^{\infty} n[n+\frac{\kappa}{2}(1+(-1)^{n+1})]^m\, a_n z^{n-1}\\ &= \Big(1+\sum_{n=2}^{\infty} n[n+\frac{\kappa}{2}(1+(-1)^{n+1})]^{m+1}\, a_n z^{n-1}\Big)\\ &* \Big(1+\sum_{n=2}^{\infty}\frac{z^{n-1}}{n+\frac{\kappa}{2}(1+(-1)^{n+1})}\Big).\end{aligned}$$

It is clear that

$$n+\frac{\kappa}{2}(1+(-1)^{n+1}) \le n+2\kappa \le n+1, \quad 0\le\kappa\le 1/2.$$

By applying Lemma 3 on the second term of the above convolution, we obtain

$$\Re\Big(1+\sum_{n=2}^{\infty}\frac{z^{n-1}}{n+\frac{\kappa}{2}(1+(-1)^{n+1})}\Big) > 2/3.$$

Thus, we attain that

$$\Re(D^m_\kappa f(z))' > \frac{2}{3}\alpha.$$

By considering

$$\beta := \frac{2}{3}\alpha \le \alpha, \quad \alpha\in[0,1],$$

we attain the requested result. □

Theorem 6. *Let $f\in T_m(\kappa,\alpha)$ and $h\in C$, the set of convex univalent functions ($C\subset\Lambda$). Then $f*h\in T_m(\kappa,\alpha)$.*

Proof. By the Marx-Strohhacker theorem [8], if h is convex univalent in U, then

$$\Re\{\frac{h(z)}{z}\} > 1/2.$$

Utilizing convolution properties, we obtain

$$\Re(D^m_\kappa(f*h)(z))' = \Re\Big(\frac{h(z)}{z} * D^m_\kappa f(z)'\Big).$$

But $\Re(D^m_\kappa f(z)') > \alpha$; thus, in view of Lemma 2, we have the desire conclusion. □

Theorem 7. *Let $f,h\in T_m(\kappa,\alpha)$. Then $f*h\in T_m(\kappa,\beta)$, where*

$$\beta := \frac{\kappa(2\alpha+1)+4\alpha-1}{2(\kappa+1)}, \quad 0\le\kappa\le 1.$$

Proof. Define a function $h \in \Lambda$ as follows:

$$h(z) = z + \sum_{n=2}^{\infty} \vartheta_n z^n, \quad z \in U.$$

Since $h \in T_m(\kappa, \alpha)$ then

$$\Re\{1 + \sum_{n=2}^{\infty} n[n + \frac{\kappa}{2}(1 + (-1)^{n+1})]^m \, \vartheta_n z^{n-1}\} > \alpha.$$

Let $\varphi_0 = 1$, and in general, we have

$$\varphi_n = \frac{\kappa + 1}{[(n+1)(n + \frac{\kappa}{2}(1 + (-1)^{n+2}) + 1)]^m}, \quad n \geq 1, \; 0 \leq \kappa \leq 1, \; m = 1, 2, ...$$

Obviously, the sequence $\{\varphi_n\}_{n=0}^{\infty}$ is a convex null sequence. Therefore, by Lemma 1, we conclude that

$$\Re\{1 + \sum_{n=2}^{\infty} \frac{\kappa + 1}{[(n+1)(n + \frac{\kappa}{2}(1 + (-1)^{n+2}) + 1)]^m} z^{n-1}\} > \frac{1}{2}.$$

Now the convolution

$$\Big(1 + \sum_{n=2}^{\infty} n[n + \frac{\kappa}{2}(1 + (-1)^{n+1})]^m \, \vartheta_n z^{n-1}\Big) * \Big(1 + \sum_{n=2}^{\infty} \frac{\kappa + 1}{[(n)(n + \frac{\kappa}{2}(1 + (-1)^{n+1}))]^m} z^{n-1}\Big)$$
$$= 1 + \sum_{n=2}^{\infty} (\kappa + 1) \, \vartheta_n z^{n-1}$$

satisfies the real

$$\Re\{1 + \sum_{n=2}^{\infty} (\kappa + 1) \, \vartheta_n z^{n-1} z^{n-1}\} > \alpha.$$

In other words, we have

$$\Re\{\frac{h(z)}{z}\} = \Re\{1 + \sum_{n=2}^{\infty} \vartheta_n z^{n-1}\} > \frac{\kappa + \alpha}{\alpha + 1}.$$

Thus,

$$\Re\{\frac{h(z)}{z}\} = \Re\{1 + \sum_{n=2}^{\infty} \vartheta_n z^{n-1} - \frac{2\alpha + \kappa - 1}{2(\kappa + 1)}\} > \frac{1}{2}.$$

But $f, h \in T_m(\kappa, \alpha)$, this implies that

$$\Re\{\Big(\frac{h(z)}{z} - \frac{2\alpha + \kappa - 1}{2(\kappa + 1)}\Big) * D_\kappa^m(f)(z))'\} > \alpha.$$

Consequently, we conclude that

$$\Re\{\Big(\frac{h(z)}{z}\Big) * D_\kappa^m(f)(z))'\} > \frac{\kappa(2\alpha+1)+4\alpha-1}{2(\kappa+1)} := \beta.$$

Thus, by Lemma 2 and the fact

$$\Re(D_\kappa^m(f*h)(z))' = \Re\Big(\frac{h(z)}{z} * D_\kappa^m f(z)'\Big),$$

we realize the requested result. □

Note that some applications of the Dunkl operator in a complex domain can be found in [9].

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