International Journal of Analysis and Applications
Volume 17, Number 3 (2019), 388-395
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-17-2019-388

SOME PROPERTIES OF GEODESIC STRONGLY E-B-VEX FUNCTIONS

WEDAD SALEH*

Department of Mathematics, Taibah University, Al- Medina, Saudi Arabia

* *Corresponding author:* wed_10_777@hotmail.com

Abstract. Geodesic E-b-vex sets and geodesic E-b-vex functions on a Riemannian manifold are extended to the so called geodesic strongly E-b-vex sets and geodesic strongly E-b-vex functions. Some basic properties of geodesic strongly E-b-vex sets are also studied.

1. Introduction

Convexity and its generalizations play an important role in optimization theory, convex anlysis and Minkowski space [3, 4, 6, 9, 10].

Youness [17] defined E-convex sets and E-convex functions by relaxing the definitions of convex sets and convex functions, which have some important applications in various branches of mathematical sciences [1, 12, 13]. Also, Youness [18] extended the definitions of E-convex sets and E-convex functions to strongly E-convex sets and strongly E-convex functions. The B-vex functions which shares many properties with convex functions was introduced by Bector and Singh [2]. Some reserchers studied some new generalizations of convex functions by relaxing definitions of E-convex functions and B-vex functions such as E-B-vex functions [15] and strongly E-B-vex functions [19]. Also, generalization of convexity on Riemannian manifolds were presented in ([5], [8] , [14], [16]).

Received 2019-03-08; accepted 2019-04-05; published 2019-05-01.

2010 *Mathematics Subject Classification.* 52A20, 52A41, 53C20, 53C22.

Key words and phrases. geodesic E-convex sets; geodesic E-convex functions; Riemannian manifolds.

In this paper, a new class of sets on Riemannian manifolds, called geodesic strongly E-b-vex sets, and a new class of functions defined on them, called geodesic strongly E-convex functions, have been proposed. Also, some of their properties have been discussed. This paper divides into three sections. In section 2, some of definities and properties which will be used throughout this work are presented that can be found in many books on differential geometry such as [16] . In section 3, a geodedic strongly E-b-vex set and geodesic strongly E-b-vex function are studied with some of their properties.

2. Preliminaries

Now, let $\aleph$ is a C^{∞} n-dimensional Riemannian manifold, also $\mu_1, \mu_2 \in \aleph$ and $\delta\colon [0,1] \longrightarrow \aleph$ be a geodesic joining the points μ_1 and μ_2 , which means that $\delta_{\mu_1,\mu_2}(0) = \mu_2$ and $\delta_{\mu_1,\mu_2}(1) = \mu_1$.

Strongly E-convex sets (SEC) and strongly E-convex (SEC) functions were introduced in [18] such as:

Definition 2.1. (1) *A subset $\Omega \subseteq \mathbb{R}^n$ is strongly E-convex (SEC) set if there is a map $\varepsilon\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that*

$$\delta(\alpha\mu_1 + \varepsilon(\mu_1)) + (1-\delta)(\alpha\mu_2 + \varepsilon(\mu_2)) \in B$$

for each $\mu_1, \mu_2 \in \Omega, \alpha \in [0,1]$ and $\delta \in [0,1]$.

(2) *A function $g\colon \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is strongly E-convex (SEC) function on Ω if there is a map $\varepsilon\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that Ω is a SEC set and*

$$g(\delta(\alpha\mu_1 + \varepsilon(\mu_1)) + (1-\delta)(\alpha\mu_2 + \varepsilon(\mu_2))) \leq \delta g(\varepsilon(\mu_1)) + (1-\delta) g(\varepsilon(\mu_2)),$$

$\forall \mu_1, \mu_2 \in \Omega, \alpha \in [0,1]$ and $\delta \in [0,1]$.

Definition 2.2. *[5]*

(1) *Considering $\varepsilon\colon \aleph \longrightarrow \aleph$ is a map. A subset $\Omega \subset \aleph$ is geodesic E-convex iff there exists a unique geodesic $\eta_{\varepsilon(\mu_1),\varepsilon(\mu_2)}(\delta)$ of length $d(\mu_1,\mu_2)$, which belongs to Ω, $\forall \mu_1, \mu_2 \in \Omega$ and $\delta \in [0,1]$.*

(2) *A functin $g\colon \Omega \subseteq \aleph \longrightarrow \mathbb{R}$ where Ω is a GEC set in $\aleph$ is geodesic E-convex if*

$$g(\eta_{\varepsilon(\mu_1),\varepsilon(\mu_2)}(\delta)) \leq \delta g(\varepsilon(\mu_1)) + (1-\delta) g(\varepsilon(\mu_2)),$$

$\forall \mu_1, \mu_2 \in \Omega$ and $\delta \in [0,1]$.

Definition 2.3. *[7]*

(1) *Considering $\varepsilon\colon \aleph \longrightarrow \aleph$ is a map. A subset $\Omega \subset \aleph$ is geodesic strongly E-convex(GSEC) iff there exists a unique geodesic*

$\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta)$ of length $d(\mu_1,\mu_2)$, which belongs to Ω, $\forall \mu_1, \mu_2 \in \Omega$, $\alpha \in [0,1]$ and $\delta \in [0,1]$ and .

(2) *A functin* $g\colon \Omega \subseteq \aleph \longrightarrow \mathbb{R}$*, where* Ω *is a GSEC set in* $\aleph$*, is geodesic strongly E-convex (GSEC) funtion if*

$$g(\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta)) \leq \delta g(\varepsilon(\mu_1)) + (1-\delta)g(\varepsilon(\mu_2)),$$

$\forall \mu_1, \mu_2 \in \Omega$ *and* $\delta \in [0,1]$.

3. Geodesic Strongly E-b-vex Sets and Geodesic Strongly E-b-vex Functions

In this part of work, a geodesic strongly E-b-vex (GSE-b-vex) set and a geodesic strongly E-b-convex (GSE-b-vex) function in a Riemannian manifold $\aleph$ are given and some of their properties are discussed.

Definition 3.1. *A subset* Ω *of* $\aleph$ *is called a geodesic strongly E-b-vex (GSE-b-vex) iff there exists a unique geodesic* $\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b)$ *of length* $d(\mu_1,\mu_2)$*, which belongs to* Ω*,* $\forall \mu_1, \mu_2 \in \Omega, \alpha \in [0,1]$ *and* $\delta \in [0,1]$.

Remark 3.1. (1) *Every GSE-b-vex set is a GSEC set when* $b(\mu_1,\mu_2,\delta) = 1$.

(2) *Every GSE-b-vex set is a GE-b-vex set when* $\alpha = 0$.

(3) *When*

$$\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b) = \delta b\left(\alpha\mu_1 + \varepsilon(\mu_1)\right) + (1-\delta b)\left(\alpha\mu_2 + \varepsilon(\mu_2)\right),$$

then we have strongly E-B-vex set.

Now, some propertie of GSE-b-vex sets are propoed.

Proposition 3.1. *Every convex set* $\Omega \subset \aleph$ *is a GSE-b-vex set.*

The proof of the above proposition is direct that by taking $\varepsilon\colon \aleph \longrightarrow \aleph$ as the identity map,$b(\mu_1,\mu_2,\delta) = 1$ and $\alpha = 0$.

Proposition 3.2. *Let* $\Omega \subset \aleph$ *be a GSE-b-vex set, then* $\varepsilon(\Omega) \subseteq \Omega$.

Proof. Since Ω is a GSE-b-vex set, then

$$\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b) \in \Omega,$$

$\mu_1, \mu_2 \in \Omega$, $\alpha \in [0,1]$ and $\delta \in [0,1]$. Let $\delta b = 1$ and $\alpha = 0$, then $\eta_{\varepsilon(\mu_1),\varepsilon(\mu_2)} = \varepsilon(\mu_2) \in \Omega$, then $\varepsilon(\Omega) \subseteq \Omega$. □

Theorem 3.1. *Suppose that a set* $\{\Omega_j\}_{j=1,2,\cdots,n}$ *is an arbitrary collection of GSE-v-vex subsets of* $\aleph$*, then* $\cap_{j=1,2,\cdots,n}\Omega_i$ *is a GSE-b-vex set.*

Proof. Considering $\{\Omega_j\}_{j=1,2,\cdots,n}$ is a collection of GSE-b-vex subsets of Ω. If $\cap_{j=1,2,\cdots,n}\Omega_j$ is an empty set, then the result is obvious. Assume that $\mu_1, \mu_2 \in \cap_{j=1,2,\cdots,n}\Omega_j$, then $\mu_1, \mu_2 \in \Omega_j$. Hence, $\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b) \in \Omega_j, \forall \alpha \in [0,1]$ and $\delta \in [0,1]$. Hence, $\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b) \in \cap_{j=1,2,\cdots,n}\Omega_j, \forall \alpha \in [0,1]$ and $\delta \in [0,1]$. □

Remark 3.2. *However, the above theorem is not true for the union of GSE-b-vex sets.*

Now, we introduce the definition of a geodesic E-b-vex (GSE-b-vex) function on $\aleph$.

Definition 3.2. *Assume that $\Omega \subset \aleph$ is a GSE-b-vex set . A function $g\colon \Omega \longrightarrow \mathbb{R}$ is called a geodesic strongly E-b-vex (GSE-b-vex) if*

$$g(\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b)) \leq \gamma g(\varepsilon(\mu_1)) + (1-\gamma)g(\varepsilon(\mu_2)), \tag{3.1}$$

$\forall \mu_1, \mu_2 \in \Omega, \alpha \in [0,1]$ *and* $\delta \in [0,1]$.

If the inequality (3.1) is strict, then g is called a strictly GSE-b-vex function.

Example 3.1. *Assume that $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that $g(\mu) = -|\mu|$. Aslo, assume that $\varepsilon\colon \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $\varepsilon(\mu) = \alpha\mu$ where $0 < \alpha \leq 1, \forall \mu \in \mathbb{R}$ and the geodesic η is given as*

$$\begin{aligned}
&\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b) \\
&\quad = \begin{cases} \frac{1}{2\alpha}\left[\alpha\mu_2 + \varepsilon(\mu_2) + \delta b(\alpha\mu_1 + \varepsilon(\mu_1) - \alpha\mu_2 - \varepsilon(\mu_2))\right] & ;\mu_1\mu_2 \geq 0, \\ \frac{1}{2\alpha}\left[\alpha\mu_2 + \varepsilon(\mu_2) + \delta b(\alpha\mu_2 + \varepsilon(\mu_2) - \alpha\mu_1 - \varepsilon(\mu_1))\right] & ;\mu_1\mu_2 < 0 \end{cases} \\
&\quad = \begin{cases} \mu_2 + \delta b(\mu_1 - \mu_2) & ;\mu_1\mu_2 \geq 0, \\ \mu_2 + \delta b(\mu_2 - \mu_1) & ;\mu_1\mu_2 < 0, \end{cases}
\end{aligned}$$

then g is GSE-b-vex function.

Proposition 3.3. *Let $g\colon \Omega \longrightarrow \mathbb{R}$ be a GSE-b-vex function on a GSE-b-vex set $\Omega \times \aleph$, then $g(\alpha\mu + \varepsilon(\mu)) \leq g(\varepsilon(\mu))$, $\mu \in \Omega$ and $\alpha \in [0,1]$.*

Proof. Since g is GSE-b-vex function on GSE-b-vex set Ω, then

$$g(\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b)) \leq \delta b g(\varepsilon(\mu_1)) + (1-\delta b)g(\varepsilon(\mu_2)),$$

then for $\delta b = 1$, we have

$$g(\alpha\mu_1 + \varepsilon(\mu_1)) \leq g(\varepsilon(\mu_1)).$$

□

Theorem 3.2. *If $g_1\colon \Omega \longrightarrow \mathbb{R}$ is a GSE-b-vex function on a GSE-b-vex set $\Omega \subset \aleph$ and $g_2\colon U \longrightarrow \mathbb{R}$ is a non-decreasing convex function such that $rang(g_1) \subset U$, then the composite function $g_2 o g_1$ is GSE-b-vex function on Ω.*

Proof. By using the hypothesis, we can write all $x_1, x_2 \in B, \alpha \in [0,1]$ and $\gamma \in [0,1]$,

$$g_1(\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b)) \leq \delta b g_1(\varepsilon(\mu_1)) + (1-\delta b) g_1(\varepsilon(\mu_2)),$$

$\forall \mu_1, \mu_2 \in \Omega, \alpha \in [0,1]$ and $\delta \in [0,1]$ and since g_2 is a non-decreasing convex function, then we get

$$\begin{aligned} g_2 o g_1(\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b)) &= g_2\left(g_2(\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b))\right) \\ &\leq g_2\left(\delta b g_1(\varepsilon(\mu_1)) + (1-\delta b) g_1(\varepsilon(\mu_2))\right) \\ &\leq \delta b g_2\left(g_1(\varepsilon(\mu_1))\right) + (1-\delta b) g_2\left(g_1(\varepsilon(\mu_2))\right) \\ &= \delta b (g_2 o g_1)(\varepsilon(\mu_1)) + (1-\delta b)(g_2 o g_1)(\varepsilon(\mu_2)) \end{aligned}$$

hence, $g_2 o g_1$ is GSE-b-vex on Ω. Moreover, $g_2 o g_1$ is a strictly GSE-b-vex function if g_2 is a strictly non-decreasing convex function. □

Theorem 3.3. *Considering $g_i \colon \Omega \longrightarrow \mathbb{R}, i = 1,2,...,n$ are GSE-b-vex functions. Then, the function*

$$g = \sum_{i=1}^{n} \xi_i g_i$$

is also GSE-b-vex geodesic on Ω, $\forall \xi_i \in \mathbb{R}, \xi_i \geq 0$.

Proof. Since $g_i, i = 1,2,...,n$ are GSE-b-vex functions, then

$$g_i(\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b)) \leq \delta b g_i(\varepsilon(\mu_1)) + (1-\delta b) g_i(\varepsilon(\mu_2)),$$

$\forall \mu_1, \mu_2 \in \Omega, \alpha \in [0,1]$ and $\delta \in [0,1]$, Hence,

$$\xi_i g_i(\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b)) \leq \delta b \xi_i g_i(\varepsilon(\mu_1)) + (1-\delta b) \xi_i g_i(\varepsilon(\mu_2)).$$

This implies to,

$$\begin{aligned} g(\eta_{\alpha\mu_1+\varepsilon(x_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b)) &= \sum_{i=1}^{n} \xi_i g_i(\eta_{\alpha\mu_1+\varepsilon(x_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b)) \\ &\leq \delta b \sum_{i=1}^{n} \xi_i g_i(\varepsilon(\mu_1)) + (1-\delta b) \sum_{i=1}^{n} \xi_i g_i(\varepsilon(\mu_2)) \\ &= \delta b g(\varepsilon(\mu_1)) + (1-\delta b) g(\varepsilon(\mu_2)). \end{aligned}$$

Then g is GSE-b-vex function. □

Next, we show that a funcion is GSE-b-vex iff its epigraph is a GSE-b-vex set.

Definition 3.3. *Assume that $\Omega \subset \aleph \times \mathbb{R}, E \colon \aleph \longrightarrow \aleph$, $b : \Omega \times \Omega \times [0,1] \longrightarrow \mathbb{R}_+$ and $F \colon \mathbb{R} \longrightarrow \mathbb{R}$. A set Ω is called a geodesic strongly $E \times F$-convex ($GSE \times F$-b-vex) if*

$$\left(\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b), \delta b F(\xi_1) + (1-\delta b) F(\xi_2)\right) \in \Omega,$$

$\forall (\mu_1, \xi_1), (\mu_2, \xi_2) \in \Omega$, $\alpha \in [0,1]$ *and* $\gamma \in [0,1]$.

Remark 3.3. *From Definition 3.3, we have found $\Omega \subseteq \aleph$ is a GSE-b-vex set iff $\Omega \times \mathbb{R}$ is a $GSE \times F$-b-vex set.*

Now , the epigraph of a function $g : \Omega \subset \aleph \longrightarrow \mathbb{R}$ is given as

$$E(g) = \{(\mu, a) \colon \mu \in \Omega, a \in \mathbb{R}, g(\mu) \leq a\}. \tag{3.2}$$

Theorem 3.4. *Suppose that $\Omega \subseteq \aleph$ is a GSE-b-vex set, $g \colon \Omega \longrightarrow \mathbb{R}$ is a function and $F \colon \mathbb{R} \longrightarrow \mathbb{R}$ is a map such that $F(g(\mu)+a) = g(\varepsilon(\mu))+a$, $\forall a \in \mathbb{R}$, $a \geqslant 0$. Then, g is a GSE-b-vex on Ω iff $E(g)$ is a $GSE \times F$-b-vex on $\Omega \times \mathbb{R}$.*

Proof. Let $(\mu_1, a_1), (\mu_2, a_2) \in E(g)$. Since Ω is GSE-b-vex, then

$$\eta_{\alpha\mu_1+\varepsilon(\mu_1), \alpha\mu_2+\varepsilon(\mu_2)}(\delta b) \in \Omega,$$

$\forall \alpha \in [0, 1]$ and $\delta \in [0, 1]$. When $\alpha = 0$ and $\delta b = 1$, we have $\varepsilon(\mu_1) \in \Omega$ also, when $\alpha = 0$ and $\delta b = 0$ we get $\varepsilon(\mu_2) \in \Omega$. Assume that $F(a_1)$ and $F(a_2)$ where $g(\varepsilon(\mu_1)) \leq F(a_1)$ and $g(\varepsilon(\mu_2)) \leq F(a_2)$. Then

$$(\varepsilon(\mu_1), F(a_1)), (\varepsilon(\mu_2), F(a_2)) \in E(g).$$

Considering g is a GSE-b-vex on Ω, then

$$\begin{aligned} g(\eta_{\alpha\mu_1+\varepsilon(\mu_1), \alpha\mu_2+\varepsilon(\mu_2)}(\delta b)) &\leq \delta b g(\varepsilon(\mu_1)) + (1 - \delta b) g(\varepsilon(\mu_2)) \\ &\leq \delta b F(a_1) + (1 - \delta b) F(a_2). \end{aligned}$$

This is leading to, $\left(\eta_{\alpha\mu_1+\varepsilon(\mu_1), \alpha\mu_2+\varepsilon(\mu_2)}(\delta b), \delta b F(a_1) + (1 - \delta b) F(a_2)\right) \in E(g)$, which means that $E(g)$ is GS$E \times \acute{E}$-b-vex on $\Omega \times \mathbb{R}$.

Conversely, let us take $E(g)$ is GS$E \times \acute{E}$-b-vex on $\Omega \times \mathbb{R}$. Assume that $\mu_1, \mu_2 \in \Omega, \alpha \in [0, 1]$ and $\delta \in [0, 1]$, then $(\mu_1, g(\mu_1)) \in E(g)$ and $(\mu_2, g(\mu_2)) \in E(g)$.

In addition, $\left(\eta_{\alpha\mu_1+\varepsilon(\mu_1), \alpha\mu_2+\varepsilon(\mu_2)}(\delta b), \delta b F(g(\mu_1)) + (1 - \delta b) F(g(\mu_2))\right) \in E(g) \Longrightarrow$

$$\begin{aligned} g(\eta_{\alpha\mu_1+\varepsilon(\mu_1), \alpha\mu_2+\varepsilon(\mu_2)}(\delta b)) &\leq \delta b F(g(\mu_1)) + (1 - \delta b) F(g(\mu_2)) \\ &= \delta b g(\varepsilon(\mu_1)) + (1 - \delta b) g(\varepsilon(\mu_2)). \end{aligned}$$

Hence, the result. □

Theorem 3.5. *Let $\{\Omega_j\}_{j=1,\cdots,n}$ be a family of $GSE \times F$-b-vex sets. Then $\cap_{j=1,\cdots,n}\Omega_j$ is also $GSE \times F$-b-vex set.*

Proof. Let $(\mu_1, a_1), (\mu_2, a_2) \in \cap_{j=1,\cdots,n}\Omega_j$, then $(\mu_1, a_1), (\mu_2, a_2) \in \Omega_j, \quad \forall j. \Longrightarrow$

$$\left(\eta_{\alpha\mu_1+\varepsilon(\mu_1), \alpha\mu_2+\varepsilon(\mu_2)}(\delta b), \delta b F(a_1) + (1 - \delta b) F(a_2)\right) \in \Omega_j,$$

$\forall \alpha \in [0,1]$ and $\delta \in [0,1]$. Hence,

$$\left(\eta_{\alpha\mu_1+\varepsilon(\mu_1),\alpha\mu_2+\varepsilon(\mu_2)}(\delta b), \delta b F(a_1) + (1-\delta b)F(a_2)\right) \in \cap_{j=1,\cdots,n}\Omega_j,$$

$\forall \alpha \in [0,1]$ and $\delta \in [0,1]$. This shows that , $\cap_{j=1,\cdots,n}\Omega_j$ is GS$E \times F$-b-vex set. □

Theorem 3.6. *Suppose that $G : \mathbb{R} \longrightarrow \mathbb{R}$ such that $G(g(x)+\mu) = g(\varepsilon(x)) + \mu, \forall \mu \in \mathbb{R}, \mu \geqslant 0$. Let $\{g_i\}_{i\in I}$ be a family of real valued functions that is defined on a GSE-b-vex set Ω and bounded from above. Then, $g(x) = sup_{i\in I} g_i(x), x \in \Omega$ is GSE-b-vex on Ω.*

Proof. Let $g_i, i \in I$ be a GSE-b-vex function on Ω, then

$$E(g_i) = \{(x,\mu) \colon x \in \Omega, \mu \in \mathbb{R}, g_i(x) \leq \mu\}$$

are GS$E \times F$-b-vex on $\Omega \times \mathbb{R}$. Hence,

$$\begin{aligned} \cap_{i\in I} E(g_i) &= \{(x,\mu) \colon x \in \Omega, \mu \in \mathbb{R}, g_i(x) \leq \mu, i \in I\} \\ &= \{(x,\mu) \colon x \in \Omega, \mu \in \mathbb{R}, g(x) \leq \mu\} \end{aligned}$$

is GS$E \times F$-b-vex set. Then, by Theorem 3.4 g is a GSE-b-vex function. □

References

[1] I. A. Abou-Tair and W. T. Sulaiman, Inequalities via convex functions, Int. J.Math. Sci. 22(1999), 543-546.

[2] C. R. Bector and C. Singh, B-vex functions, J. Optim. Theory Appl. 71(2) (1991), 237-253.

[3] V. Boltyanski, H. Martini and P.S. Soltan, Excursions Into Combinatorial Geometry, Springer, Berlin, 1997.

[4] L. Danzer, B. Grünbaum and V. Klee, Helly's theorem and its relatives. In: V. Klee (ed.) Convexity. Proc. Sympos. Pure Math., vol.7, pp.101-180. Amer. Math. Soc., Providence, 1963.

[5] A. Iqbal, S. Ali and I. Ahmad, On geodesic E-convex sets, geodesic E-convex functions and E-epigraphs, J. Optim. Theory Appl. 55(1)(2012), 239-251.

[6] M. A. Jiménez, G. R. Garzón and A. R. Lizana, Optimality conditions in vector optimization. Bentham Science Publishers, 2010.

[7] A. Kiliçman and W. Saleh, On geodesic strongly E-convex sets and geodesic strongly E-convex functions, J. Inequal. Appl. 2015 (2015), 297.

[8] A. Kiliçman and W. Saleh, On properties of geodesic semilocal E-preinvex functions, J. Inequal. Appl. 2018 (2018), 353.

[9] H.Martini and K.J. Swanepoel, Generalized Convexity notions and Combinatorial Geometry, Gongr. Numer. 164 (2003), 65-93.

[10] H. Martini and K.J. Swanepoel, The geometry of minkowski spaces- a survey, Part II. Expo. Math. 22 (2004), 14-93.

[11] F. Mirzapour, A. Mirzapour and M. Meghdadi, Generalization of some important theorems to E-midconvex functions, Appl. Math. Lett. 24(8) (2011),1384-1388.

[12] M. A. Noor, Fuzzy preinvex functions, Fuzzy Sets Syst. 64(1994), 95-104.

[13] M. A. Noor, K. I. Noor and M. U. Awan, Generalized convexity and integral inequalities, Appl. Math. Inf. Sci. 9(1)(2015), 233-243.

[14] T. Rapcsak, Smooth Nonlinear Optimizatio in $\mathbb{R}^n$, Kluwer Academic, 1997.

[15] Y. R. Syau, Lixing Jia, and E. Stanley Lee, Generalizations of E-convex and B-vex functions, Comput. Math. Appl. 58(4) (2009), 711-716.

[16] C. Udrist, Convex Funcions and Optimization Methods on Riemannian Manifolds, Kluwer Academic, 1994.

[17] E. A. Youness, On E-convex sets, E-convex functions and E-convex programming, J. Optim.Theory Appl. 102 (1999), 439-450.

[18] E. A. Youness and Tarek Emam, Strongly E-convex sets and strongly E-convex functions, J. Interdisciplinary Math. 8(1)(2005), 107-117.

[19] G.Y. Wang, Some Properties of strongly E-B-vex functions, Sustainable Development-Special track within SCET 2012(2012), 247.