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# FIXED POINT THEOREMS FOR GENERALIZED F-CONTRACTIONS AND GENERALIZED F-SUZUKI-CONTRACTIONS IN COMPLETE DISLOCATED $S_b$-METRIC SPACES

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Abstract. In this paper, first we describe the notion of dislocated $S_b$-metric space and then we introduce the new notions of generalized $F$-contraction and generalized $F$-Suzuki-contraction in the setup of dislocated $S_b$-metric spaces. We establish some fixed point theorems involving these contractions in complete dislocated $S_b$-metric spaces. We also furnish some examples to verify the effectiveness and applicability of our results.

## 1. Introduction and Preliminaries

Bakhtin [1] and Czerwik [2] introduced $b$-metric spaces and proved the contraction principle in this framework. In recent times, many authors obtained fixed point results for single-valued or set-valued functions, in the setting of $b$-metric spaces.

In 2012, Sedghi et al. [11] introduced the concept of $S$-metric space by modifying $D$-metric and $G$-metric spaces and proved some fixed point theorems for a self-mapping on a complete $S$-metric space. After that Özgür and TaŞ studied some generalizations of the Banach contraction principle on $S$-metric spaces in [8]. They also obtained some fixed point theorems for the Rhoades' contractive condition on $S$-metric spaces [7]. Sedghi et al. [10] introduced the concept of $S_b$-metric space as a generalization of $S$-metric space and proved some coupled common fixed point theorems in $S_b$-metric space. Kishore et al. [4] proved some fixed point

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theorems for generalized contractive conditions in partially ordered complete $S_b$-metric spaces and gave some applications to integral equations and homotopy theory.

On the other hand, Wardowski [12] introduced a new contraction, the so-called $F$-contraction, and obtained a fixed point result as a generalization of the Banach contraction principle. Thereafter, Dung and Hang [3] studied the notion of a generalized $F$-contraction and established certain fixed point theorems for such mappings. Recently, Piri and Kumam [6] extended the fixed point results of [12] by introducing a generalized $F$-Suzuki-contraction in $b$-metric spaces.

Motivated by the aforementioned works, in this paper, we first introduce the notion of dislocated $S_b$- metric space and then we describe some fixed point results of [3], [6] by introducing generalized $F$-contractions and generalized $F$-Suzuki-contractions in dislocated $S_b$-metric spaces. We begin with some basic well-known definitions and results which will be used further on.

Throughout this paper $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ denote the set of all real numbers, the set of all nonnegative real numbers and the set of all positive integers, respectively.

**Definition 1.1.** *[11] Let $X$ be a nonempty set. An $S$-metric on $X$ is a function $S : X^3 \to \mathbb{R}_+$ that satisfies the following conditions:*

- $(S1)$ $0 < S(x, y, z)$ *for each* $x, y, z \in X$ *with* $x \neq y \neq z \neq x$,
- $(S2)$ $S(x, y, z) = 0$ *if and only if* $x = y = z$,
- $(S3)$ $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ *for each* $x, y, z, a \in X$.

*Then the pair $(X, S)$ is called an $S$-metric space.*

**Definition 1.2.** *[10] Let $X$ be a nonempty set and $b \geq 1$ be a given real number. Suppose that a mapping $S_b : X^3 \to \mathbb{R}_+$ satisfies:*

- $(S_b1)$ $0 < S_b(x, y, z)$ *for all* $x, y, z \in X$ *with* $x \neq y \neq z \neq x$,
- $(S_b2)$ $S_b(x, y, z) = 0$ *if and only if* $x = y = z$,
- $(S_b3)$ $S_b(x, y, z) \leq b\big(S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)\big)$ *for all* $x, y, z, a \in X$.

*Then $S_b$ is called an $S_b$-metric on $X$ and the pair $(X, S_b)$ is called an $S_b$-metric space.*

**Definition 1.3.** *[10] If $(X, S_b)$ is an $S_b$-metric space, a sequence $\{x_n\}$ in $X$ is said to be:*

1. *Cauchy sequence if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S_b(x_n, x_n, x_m) < \varepsilon$ for all $m, n \geq n_0$.*
2. *convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer $n_0$ such that $S_b(x_n, x_n, x) < \varepsilon$ or $S_b(x, x, x_n) < \varepsilon$ for all $n \geq n_0$, and we denote by $\lim\limits_{n\to\infty} x_n = x$.*

**Definition 1.4.** *[10] An $S_b$-metric space $(X, S_b)$ is called complete if every Cauchy sequence is convergent in $X$.*

**Example 1.1.** *[9] Let $X = \mathbb{R}$. Define $S_b : X^3 \to \mathbb{R}_+$ by $S_b(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in X$. Then $(X, S_b)$ is a complete $S_b$-metric space with $b = 2$.*

**Definition 1.5.** *Let $(X, S_b)$ be an $S_b$-metric space. Then $S_b$ is called symmetric if*

$$S_b(x, x, y) = S_b(y, y, x) \tag{1.1}$$

*for all $x, y \in X$.*

It is easy to see that the symmetry condition (1.1) is automatically satisfied by an $S$-metric [11]. We conclude this section recalling the following fixed point theorems of Dung and Hang [3] and Piri and Kumam [6]. For this, we need some preliminaries.

**Definition 1.6.** *[12] Let $\mathcal{F}$ be the family of all functions $F : (0, +\infty) \to \mathbb{R}$ such that:*

- $(F1)$ *$F$ is strictly increasing, that is for all $\alpha, \beta \in (0, +\infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$,*
- $(F2)$ *for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to+\infty} \alpha_n = 0$ if and only if $\lim_{n\to+\infty} F(\alpha_n) = -\infty$,*
- $(F3)$ *there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

In 2014, Piri and Kumam [5] described a large class of functions by replacing the condition $(F3)$ in the above definition with the following one:

- $(F3')$ $F$ is continuous on $(0, +\infty)$.

They denote by $\mathfrak{F}$ the family of all functions $F : (0, +\infty) \to \mathbb{R}$ which satisfy conditions $(F1)$, $(F2)$, and $(F3')$.

**Example 1.2.** *(see [5], [13]) The following functions $F : (0, +\infty) \to \mathbb{R}$ are the elements of $\mathfrak{F}$.*
*(1)* $F(\alpha) = -\frac{1}{\sqrt{\alpha}}$,
*(2)* $F(\alpha) = -\frac{1}{\alpha} + \alpha$,
*(3)* $F(\alpha) = \frac{1}{1-e^{\alpha}}$,
*(4)* $F(\alpha) = \ln \alpha$,
*(5)* $F(\alpha) = \ln \alpha + \alpha$.

**Definition 1.7.** *[3] Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is said to be a generalized $F$-contraction on $(X, d)$ if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F\big(d(Tx, Ty)\big) \le F\big(N(x, y)\big),$$

*in which*

$$N(x, y) = \max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}, \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty)\Big\}.$$

**Theorem 1.1.** *[3] Let $(X,d)$ be a complete metric space and let $T : X \to X$ be a generalized $F$-contraction mapping. If $T$ or $F$ is continuous, then $T$ has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}$ converges to $x^*$.*

We use $\mathfrak{F}_G$ to denote the set of all functions $F : (0,+\infty) \to \mathbb{R}$ which satisfy conditions $(F1)$ and $(F3')$ and $\Psi$ to denote the set of all functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi$ is continuous and $\psi(t) = 0$ if and only if $t = 0$ (see [6]).

**Definition 1.8.** *[6] Let $(X,d)$ be a b-metric space. A self-mapping $T : X \to X$ is said to be a generalized $F$-Suzuki-contraction if there exists $F \in \mathfrak{F}_G$ such that, for all $x, y \in X$ with $x \neq y$,*

$$\frac{1}{2s} d(x,Tx) < d(x,y) \Rightarrow F\big(s^5 d(Tx,Ty)\big) \leq F\big(M_T(x,y)\big) - \psi\big(M_T(x,y)\big),$$

*in which $\psi \in \Psi$ and*

$$M_T(x,y) = \max\Big\{d(x,y), d(T^2x,y), \frac{d(Tx,y)+d(x,Ty)}{2s}, \frac{d(T^2x,x)+d(T^2x,Ty)}{2s},$$
$$d(T^2x,Ty)+d(T^2x,Tx), d(T^2x,Ty)+d(Tx,x), d(Tx,y)+d(y,Ty)\Big\}.$$

**Theorem 1.2.** *[6] Let $(X,d)$ be a complete b-metric space and $T : X \to X$ be a generalized $F$-Suzuki-contraction. Then $T$ has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}$ converges to $x^*$.*

## 2. Main results

In this section, we first introduce the concept of dislocated $S_b$-metric space and then we demonstrate some fixed point results for generalized $F$-contractions and generalized $F$-Suzuki-contractions in such spaces. Our results are remarkable for two reasons: first dislocated $S_b$- metric is more general, second the contractivity condition involves auxiliary functions form a wider class.

**Definition 2.1.** *Let $X$ be a nonempty set and $b \geq 1$ be a given real number. A mapping $S_b : X^3 \to \mathbb{R}_+$ is a dislocated $S_b$-metric if, for all $x, y, z, a \in X$, the following conditions are satisfied:*

$(dS_b1)$ *$S_b(x,y,z) = 0$ implies $x = y = z$,*

$(dS_b2)$ *$S_b(x,y,z) \leq b\big(S_b(x,x,a) + S_b(y,y,a) + S_b(z,z,a)\big)$.*

*A dislocated $S_b$-metric space is a pair $(X,S_b)$ such that $X$ is a nonempty set and $S_b$ is a dislocated $S_b$-metric on $X$. In the case that $b = 1$, $S_b$ is denoted by $S$ and it is called dislocated $S$-metric, and the pair $(X,S)$ is called dislocated $S$-metric space.*

**Definition 2.2.** *Let $(X,S_b)$ be a dislocated $S_b$-metric space, $\{x_n\}$ be any sequence in $X$ and $x \in X$. Then:*

- $(i)$ *The sequence $\{x_n\}$ is said to be a Cauchy sequence in $(X,S_b)$ if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S_b(x_n,x_n,x_m) < \varepsilon$ for each $m,n \geq n_0$.*

(ii) *The sequence* $\{x_n\}$ *is said to be convergent to* $x$ *if, for each* $\varepsilon > 0$*, there exists a positive integer* $n_0$ *such that* $S_b(x, x, x_n) < \varepsilon$ *for all* $n \geq n_0$ *and we denote it by* $\lim_{n\to\infty} x_n = x$.

(iii) $(X, S_b)$ *is said to be complete if every Cauchy sequence is convergent.*

The following example shows that a dislocated $S_b$-metric need not be a dislocated $S$-metric.

**Example 2.1.** *Let* $X = \mathbb{R}_+$*, then the mapping* $S_b : X^3 \to \mathbb{R}_+$ *defined by*
$S_b(x, y, z) = x + y + 4z$ *is a complete dislocated* $S_b$*-metric on* $X$ *with* $b = 2$*. However, it is not a dislocated* $S$*-metric space. Indeed, we have*

$$4 = S_b(0, 0, 1) \nleq 2S_b(0, 0, 0) + S_b(1, 1, 0) = 2.$$

**Definition 2.3.** *Suppose that* $(X, S_b)$ *is a dislocated* $S_b$*-metric space. A mapping* $T : X \to X$ *is said to be a generalized* $F$*-contraction on* $(X, S_b)$ *if there exist* $F \in \mathfrak{F}$ *and* $\tau > 0$ *such that for all* $x, y \in X$,

$$S_b(Tx, Tx, Ty) > 0 \Rightarrow \tau + F\big(b^2 S_b(Tx, Tx, Ty)\big) \leq F\big(N(x, y)\big), \tag{2.1}$$

*where*

$$N(x, y) = \max\big\{S_b(x, x, y), S_b(Tx, Tx, Ty), \frac{S_b(y, y, Tx)}{10b^8}, \frac{S_b(x, x, Ty)}{10b^9}, \frac{S_b(y, y, T^2x)}{10b^4}\big\}.$$

Our first main result is the following.

**Theorem 2.1.** *Let* $(X, S_b)$ *be a complete dislocated* $S_b$*-metric space and* $T : X \to X$ *be a generalized* $F$*-contraction mapping satisfying the following condition:*

$$\max\big\{\frac{S_b(y, y, Ty)}{5b^7} + \frac{S_b(Tx, Tx, Ty)}{10b^7}, \frac{S_b(x, x, Ty)}{10b^9}, \frac{S_b(y, y, T^2x)}{10b^4}\big\} \leq S_b(Tx, Tx, Ty)$$

*for all* $x, y \in X$.

*Then* $T$ *has a unique fixed point* $v \in X$.

*Proof.* Let $x_0$ be an arbitrary point in $X$ and let $\{x_n\}$ be the Picard sequence of T based on $x_0$, that is, $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$. If there exists $n_0 \in \mathbb{N}$ such that $S_b(x_{n_0}, x_{n_0}, x_{n_0+1}) = 0$, then $x_{n_0}$ is a fixed point of $T$ and the existence part of the proof is finished. On the contrary case, assume that $S_b(x_n, x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Applying the contractivity condition (2.1), we get

$$F\big(b^2 S_b(Tx_{n-1}, Tx_{n-1}, Tx_n)\big) \leq F\big(N(x_{n-1}, x_n)\big) - \tau. \tag{2.2}$$

Using the definition of $N(x, y)$ and the property $(dS_b2)$, we obtain that

$$\max\big\{S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_n, x_{n+1})\big\} \leq N(x_{n-1}, x_n) \tag{2.3}$$

$$\begin{aligned}
&= \max\big\{S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_n, Tx_n), \frac{S_b(x_n, x_n, Tx_{n-1})}{10b^8},\\
&\qquad \frac{S_b(x_{n-1}, x_{n-1}, Tx_n)}{10b^9}, \frac{S_b(x_n, x_n, Tx_n)}{10b^4}\big\}\\
&\leq \max\{S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_n, Tx_n), \frac{3S_b(x_n, x_n, x_{n+1})}{10b^7},\\
&\qquad \frac{S_b(x_{n-1}, x_{n-1}, Tx_n)}{10b^9}, \frac{S_b(x_n, x_n, Tx_n)}{10b^4}\big\}\\
&= \max\big\{S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_n, x_{n+1})\big\}.
\end{aligned}$$

Then $N(x_{n-1},x_n)=\max\{S_b(x_{n-1},x_{n-1},x_n),S_b(x_n,x_n,x_{n+1})\}$ and so (2.2), becomes

$$F\big(b^2S_b(Tx_{n-1},Tx_{n-1},Tx_n)\big)\le F\big(\max\{S_b(x_{n-1},x_{n-1},x_n),S_b(x_n,x_n,x_{n+1})\}\big)-\tau.$$

If we assume that

$$\max\{S_b(x_{n-1},x_{n-1},x_n),S_b(Tx_{n-1},Tx_{n-1},Tx_n)\}=S_b(Tx_{n-1},Tx_{n-1},Tx_n)$$

for some $n$, then we have

$$\begin{aligned}F\big(b^2S_b(Tx_{n-1},Tx_{n-1},Tx_n)\big) &\le F\big(S_b(Tx_{n-1},Tx_{n-1},Tx_n)\big)-\tau\\ &< F\big(S_b(Tx_{n-1},Tx_{n-1},Tx_n)\big).\end{aligned}$$

Using condition $(F1)$ we conclude that $S_b(x_n,x_n,x_{n+1})<S_b(x_n,x_n,x_{n+1})$, which is a contradiction. Therefore, for each $n\in\mathbb{N}$ we have

$$\max\{S_b(x_{n-1},x_{n-1},x_n),S_b(x_n,x_n,x_{n+1})\}=S_b(x_{n-1},x_{n-1},x_n).$$

Applying again (2.2) and condition $(F1)$, we deduce that

$$S_b(x_n,x_n,x_{n+1})<S_b(x_{n-1},x_{n-1},x_n)$$

for each $n$. Thus $\{S_b(x_n,x_n,x_{n+1})\}$ is a nonnegative decreasing sequence of real numbers. Then there exists $A\ge 0$ such that

$$\lim_{n\to+\infty}S_b(x_n,x_n,x_{n+1})=\inf_{n\in\mathbb{N}}S_b(x_n,x_n,x_{n+1})=A.$$

We claim that $A=0$. To support the claim, let it be untrue and $A>0$. Then, for any $\varepsilon>0$, it is possible to find a positive integer $m$ so that

$$S_b(x_m,x_m,Tx_m)<A+\varepsilon.$$

So, from $(F1)$, we get

$$F\big(S_b(x_m,x_m,Tx_m)\big)<F(A+\varepsilon). \tag{2.4}$$

It follows from (2.1) that

$$\tau+F\big(b^2S_b(Tx_m,Tx_m,T^2x_m)\big)\le F\big(N(x_m,Tx_m)\big). \tag{2.5}$$

By a similar argument as (2.3), it yields that

$$N(x_m,Tx_m)=\max\{S_b(x_m,x_m,Tx_m),S_b(Tx_m,Tx_m,T^2x_m)\}.$$

Hence (2.5), becomes

$$F\big(b^2S_b(Tx_m,Tx_m,T^2x_m)\big)\le F\big(\max\{S_b(x_m,x_m,Tx_m),S_b(Tx_m,Tx_m,T^2x_m)\}\big)-\tau. \tag{2.6}$$

Now if, $\max\{S_b(x_m,x_m,Tx_m),S_b(Tx_m,Tx_m,T^2x_m)\}=S_b(Tx_m,Tx_m,T^2x_m)$ for some $m$, then (2.6) gives us a contradiction. Thus, we infer that

$$\max\{S_b(x_m,x_m,Tx_m),S_b(Tx_m,Tx_m,T^2x_m)\}=S_b(x_m,x_m,Tx_m),$$

and therefore, we have

$$F\big(b^2 S_b(Tx_m, Tx_m, T^2x_m)\big) \le F\big(S_b(x_m, x_m, Tx_m)\big) - \tau.$$

It implies that

$$\begin{aligned} F\big(b^2 S_b(T^2x_m, T^2x_m, T^3x_m)\big) &\le F\big(S_b(Tx_m, Tx_m, T^2x_m)\big) - \tau \\ &\le F\big(b^2 S_b(Tx_m, Tx_m, T^2x_m)\big) - \tau \\ &\le F\big(S_b(x_m, x_m, Tx_m)\big) - 2\tau. \end{aligned}$$

Continuing the above process and taking (2.4) into account, we deduce that

$$\begin{aligned} F\big(b^2 S_b(T^n x_m, T^n x_m, T^{n+1}x_m)\big) &\le F\big(S_b(T^{n-1}x_m, T^{n-1}x_m, T^n x_m)\big) - \tau \\ &\le F\big(b^2 S_b(T^{n-1}x_m, T^{n-1}x_m, T^n x_m)\big) - \tau \\ &\le F\big(S_b(T^{n-2}x_m, T^{n-2}x_m, T^{n-1}x_m)\big) - 2\tau \\ &\vdots \\ &\le F\big(S_b(x_m, x_m, Tx_m)\big) - n\tau \\ &< F(A+\varepsilon) - n\tau, \end{aligned}$$

and by passing to the limit as $n \to +\infty$ we obtain

$$\lim_{n\to+\infty} F\big(b^2 S_b(T^n x_m, T^n x_m . T^{n+1}x_m)\big) = -\infty.$$

This fact together with the condition $(F2)$ implies that

$$\lim_{n\to+\infty} S_b(T^n x_m, T^n x_m, T^{n+1}x_m) = 0.$$

Thus $S_b(T^n x_m, T^n x_m, T^{n+1}x_m) < A$ for $n$ sufficiently large, which is a contradiction with the definition of $A$. Then,

$$\lim_{n\to+\infty} S_b(x_n, x_n, x_{n+1}) = 0. \tag{2.7}$$

Next, we intend to show that the sequence $\{x_n\}$ is a Cauchy sequence in $X$. Arguing by contradiction, we assume that there exist $\varepsilon > 0$, and subsequences $\{x_{q(n)}\}$ and $\{x_{p(n)}\}$ of $\{x_n\}$ with $n < q(n) < p(n)$ such that

$$S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}) \ge \varepsilon \tag{2.8}$$

for each $n \in \mathbb{N}$. Further, corresponding to $q(n)$, we can choose $p(n)$ in such a way that it is the smallest integer with $q(n) < p(n)$ satisfying the above inequality, then

$$S_b(x_{q(n)}, x_{q(n)}, x_{p(n)-1}) < \varepsilon \tag{2.9}$$

for all $n \in \mathbb{N}$.

In the light of (2.8) and the condition (2.1), we conclude that

$$F\big(b^2 S_b(Tx_{q(n)-1}, Tx_{q(n)-1}, Tx_{p(n)-1})\big) \le F\big(N(x_{q(n)-1}, x_{p(n)-1})\big) - \tau. \tag{2.10}$$

By our hypothesis and in view of $(dS_b2)$, we get

$$\begin{aligned}
&\max\{S_b(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}),S_b(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1})\}\\
&\leq N(x_{q(n)-1},x_{p(n)-1})\\
&= \max\{S_b(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}),S_b(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1}),\\
&\quad \frac{S_b(x_{p(n)-1},x_{p(n)-1},Tx_{q(n)-1})}{10b^8},\frac{S_b(x_{q(n)-1},x_{q(n)-1},Tx_{P(n)-1})}{10b^9},\\
&\quad \frac{S_b(x_{p(n)-1},x_{p(n)-1},Tx_{q(n)})}{10b^4}\}\\
&\leq \max\{S_b(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}),S_b(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1}),\\
&\quad \frac{S_b(x_{p(n)-1},x_{p(n)-1},Tx_{p(n)-1})}{5b^7}+\frac{S_b(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1})}{10b^7},\\
&\quad \frac{S_b(x_{q(n)-1},x_{q(n)-1},Tx_{P(n)-1})}{10b^9},\frac{S_b(x_{p(n)-1},x_{p(n)-1},Tx_{q(n)})}{10b^4}\}\\
&\leq \max\{S_b(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}),S_b(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1})\}.
\end{aligned}$$

It enforces that

$$N(x_{q(n)-1},x_{p(n)-1})=\max\{S_b(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}),S_b(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1})\}.$$

Suppose that the maximum on the right-hand side is equal to $S_b(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1})$ for some $n$, then from relation (2.10) together with the condition $(F1)$ we get

$$S_b(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1})<S_b(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1})$$

which is a contradiction. Thus, we find that

$$\max\Big\{x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}),S_b(x_{q(n)},x_{q(n)},x_{p(n)})\Big\}=S_b(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1})$$

for all $n$. Accordingly, (2.10) becomes

$$F\big(b^2S_b(Tx_{q(n)-1},Tx_{q(n)-1},Tx_{p(n)-1})\big)\leq F\big(S_b(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1})\big)-\tau \tag{2.11}$$

and so using $(F1)$ we get

$$S_b\big(x_{q(n)},x_{q(n)},x_{p(n)}\big)<S_b(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1}). \tag{2.12}$$

Regarding to (2.8), (2.12) and employing $(dS_b2)$ we observe that

$$\begin{aligned}
\varepsilon &\leq S_b(x_{q(n)},x_{q(n)},x_{p(n)})\\
&< S_b(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1})\\
&\leq 2bS_b(x_{q(n)-1},x_{q(n)-1},x_{q(n)})+bS_b(x_{p(n)-1},x_{p(n)-1},x_{q(n)})\\
&\leq 2bS_b(x_{q(n)-1},x_{q(n)-1},x_{q(n)})+2b^2S_b(x_{p(n)-1},x_{p(n)-1},x_{p(n)-1})\\
&\quad +b^2S_b(x_{q(n)},x_{q(n)},x_{p(n)-1})\\
&\leq 2bS_b(x_{q(n)-1},x_{q(n)-1},x_{q(n)})+6b^3S_b(x_{p(n)-1},x_{p(n)-1},x_{p(n)})\\
&\quad +b^2S_b(x_{q(n)},x_{q(n)},x_{p(n)-1}).
\end{aligned}$$

Combining this result with (2.7) and (2.9) we get

$$\varepsilon\leq\limsup_{n\to+\infty}S_b(x_{q(n)},x_{q(n)},x_{p(n)})\leq\limsup_{n\to+\infty}S_b(x_{q(n)-1},x_{q(n)-1},x_{p(n)-1})\leq b^2\varepsilon. \tag{2.13}$$

In view of (2.13) and (2.11) and applying the conditions $(F1)$ and $(F3')$, we have

$$\begin{aligned} F(b^2\varepsilon) &\leq F\big(b^2 \limsup_{n\to+\infty} S_b(x_{q((n)}, x_{q(n)}, x_{p(n)})\big) \\ &\leq F\big(\limsup_{n\to+\infty} S_b(x_{q(n)-1}, x_{q(n)-1}, x_{p(n)-1})\big) - \tau \\ &\leq F(b^2\varepsilon) - \tau. \end{aligned}$$

It is a contradiction with $\tau > 0$, and therefore it follows that $\{x_n\}$ is a Cauchy sequence in $X$. By completeness of $(X, S_b)$, $\{x_n\}$ converges to some point $v \in X$. Then, for each $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$S_b(v, v, x_n) < \varepsilon, \tag{2.14}$$

for all $n \geq N_1$. We are going to show that $v$ is a fixed point of $T$. For this aim, we consider two following cases:

**Case 1.** If $S_b(Tv, Tv, Tx_n) = 0$ for some $n \geq N_1$, then from $(dS_b2)$ we find that

$$S_b(Tv, Tv, v) \leq 2bS_b(Tv, Tv, Tx_n) + bS_b(v, v, Tx_n) \leq b\varepsilon.$$

**Case 2.** If $S_b(Tv, Tv, Tx_n) > 0$ for all $n \geq N_1$, then using (2.1), we get

$$F\big(b^2 S_b(Tv, Tv, Tx_n)\big) \leq F\big(N(v, x_n)\big) - \tau. \tag{2.15}$$

From our assumptions, and using $(dS_b2)$, it follows that

$$\begin{aligned} &\max\{S_b(v, v, x_n), S_b(Tv, Tv, Tx_n)\} \\ &\leq N(v, x_n) \\ &= \max\Big\{S_b(v, v, x_n), S_b(Tv, Tv, Tx_n), \frac{S_b(x_n, x_n, Tv)}{10b^8}, \frac{S_b(v, v, Tx_n)}{10b^9}, \frac{S_b(x_n, x_n, T^2v)}{10b^4}\Big\} \\ &\leq \max\Big\{S_b(v, v, x_n), S_b(Tv, Tv, Tx_n), \frac{S_b(x_n, x_n, Tx_n)}{5b^7} + \frac{S_b(Tv, Tv, Tx_n)}{10b^7}, \\ &\qquad \frac{S_b(v, v, Tx_n)}{10b^9}, \frac{S_b(x_n, x_n, T^2v)}{10b^4}\Big\} \\ &= \max\{S_b(v, v, x_n), S_b(Tv, Tv, Tx_n)\}. \end{aligned}$$

It enforces that $N(v, x_n) = \max\{S_b(v, v, x_n), S_b(Tv, Tv, Tx_n)\}$. Now, if we assume that the maximum on the right-hand side of this equality is equal to $S_b(Tv, Tv, Tx_n)$, then by replacing it in (2.15), we obtain $S_b(Tv, Tv, Tx_n) < S_b(Tv, Tv, Tx_n)$ which is a contradiction. Consequently, for each $n \in \mathbb{N}$ we have

$$\max\{S_b(v, v, x_n), S_b(Tv, Tv, Tx_n)\} = S_b(v, v, x_n).$$

Hence, (2.15) turns into

$$\begin{aligned} F\big(b^2 S_b(Tv, Tv, Tx_n)\big) &\leq F\big(S_b(v, v, x_n)\big) - \tau \\ &< F\big(S_b(v, v, x_n)\big). \end{aligned}$$

Employing the condition $(F1)$, we get

$$S_b(Tv, Tv, Tx_n) < S_b(v, v, x_n). \tag{2.16}$$

From $(dS_b2)$, (2.16) and (2.14), we deduce that

$$S_b(Tv,Tv,v) \leq 2bS_b(Tv,Tv,Tx_{N_1}) + bS_b(v,v,Tx_{N_1}) < 3b\varepsilon.$$

From the arbitrariness of $\varepsilon$ in each case, it follows that $S_b(Tv,Tv,v) = 0$ which implies that $Tv = v$. Hence, $v$ is a fixed point of $T$.

Finally, we show that $T$ has at most one fixed point. Indeed, if $v_1, v_2 \in X$ are two fixed points of $T$, such that $v_1 \neq v_2$, then we obtain

$$F\big(b^2 S_b(Tv_1,Tv_1,Tv_2)\big) \leq F\big(N(v_1,v_2)\big) - \tau, \tag{2.17}$$

From our hypothesis and by using $(dS_b2)$, it follows that

$$\begin{aligned} S_b(v_1,v_1,v_2) &\leq N(v_1,v_2) \\ &\leq \max\big\{S_b(v_1,v_1,v_2), S_b(Tv_1,Tv_1,Tv_2), \frac{S_b(v_2,v_2,Tv_2)}{5b^7} + \frac{S_b(Tv_1,Tv_1,Tv_2)}{10b^7}, \\ &\quad \frac{S_b(v_1,v_1,Tv_2)}{10b^9}, \frac{S_b(Tv_2,Tv_2,T^2v_1)}{10b^4}\big\} \\ &= \max\big\{S_b(v_1,v_1,v_2), S_b(Tv_1,Tv_1,Tv_2)\big\} \\ &= S_b(v_1,v_1,v_2). \end{aligned}$$

Then (2.17) becomes

$$F\big(b^2 S_b(v_1,v_1,v_2)\big) \leq F\big(S_b(v_1,v_1,v_2)\big) - \tau.$$

It gives us a contradiction. Therefore, $v_1 = v_2$ and the fixed point is unique. $\square$

Now we illustrate our result contained in Theorem 2.1 with help of two examples.

**Example 2.2.** *Let $(X, S_b)$ be as in Example 2.1 and let $\tau > 0$ be an arbitrary fixed number. Define the mapping $T : X \to X$ by $T(x) = e^{-\tau}\frac{x}{8}$ and take $F(\alpha) = \ln\alpha + \alpha$ $(\alpha > 0)$. It is easily verified that $N(x,y) = S_b(x,x,y) = 2x + 4y$. Assume that $x$ or $y$ is nonzero, then $S_b(Tx,Tx,Ty) > 0$ and we have*

$$\begin{aligned} \tau + F\big(b^2 S_b(Tx,Tx,Ty)\big) &= \tau + \ln(e^{-\tau}(x+2y)) + e^{-\tau}(x+2y) \\ &= \ln(x+2y) + e^{-\tau}(x+2y) \\ &\leq \ln(2x+4y) + 2x + 4y \\ &= F\big(S_b(x,x,y)\big) \\ &= F\big(N(x,y)\big). \end{aligned}$$

*Hence, $T$ is a generalized $F$-contraction. On the other hand, if we assume that $0 < \tau \leq 0.0250587314$, then the following estimate holds:*

$$\begin{aligned} &\max\big\{\frac{S_b(y,y,Ty)}{5b^7} + \frac{S_b(Tx,Tx,Ty)}{10b^7}, \frac{S_b(x,x,Ty)}{10b^9}, \frac{S_b(y,y,T^2x)}{10b^4}\big\} \\ &= \max\big\{\frac{2y + e^{-\tau}\frac{y}{2}}{5\times 2^7} + \frac{e^{-\tau}(\frac{x}{4}+\frac{y}{2})}{10\times 2^7}, \frac{2x + e^{-\tau}\frac{y}{2}}{10\times 2^9}, \frac{2y + e^{-2\tau}\frac{x}{16}}{10\times 2^4}\big\} \\ &\leq e^{-\tau}(\frac{x}{4}+\frac{y}{2}) = S_b(Tx,Tx,Ty). \end{aligned}$$

Thus all conditions of Theorem 2.1 hold and 0 is a unique fixed point of $T$.

**Example 2.3.** *Let $X = \mathbb{R}$, and $S_b : X^3 \to \mathbb{R}_+$ be a mapping defined by $S_b(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2} + 2z^2$. Then $(X, S_b)$ is a complete dislocated $S_b$-metric with $b = 2$. Define the mapping $T : X \to X$ by $T(x) = \frac{x}{3}$ and take $F(\alpha) = \ln \alpha$ $(\alpha > 0)$. It is easily checked that $N(x, y) = S_b(x, x, y) = x^2 + 2y^2$. Assume that $x$ or $y$ is nonzero, then $S_b(Tx, Tx, Ty) > 0$ and we have*

$$\tau + F\big(b^2 S_b(Tx, Tx, Ty)\big) \leq F\big(N(x, y)\big) \Leftrightarrow \ln(\frac{9}{4}) \geq \tau.$$

*Also, we observe that*

$$\begin{aligned}
&\max\left\{\tfrac{S_b(y,y,Ty)}{5b^7} + \tfrac{S_b(Tx,Tx,Ty)}{10b^7}, \tfrac{S_b(x,x,Ty)}{10b^9}, \tfrac{S_b(y,y,T^2x)}{10b^4}\right\}\\
&\qquad = \max\left\{\frac{48y^2 + 2x^2}{23040}, \frac{18x^2 + 4y^2}{92160}, \frac{162y^2 + 4x^2}{25920}\right\}\\
&\qquad \leq \frac{10240x^2 + 20480y^2}{92160}\\
&\qquad = S_b(Tx, Tx, Ty)
\end{aligned}$$

*for all $x, y \in X$. Now, if we assume that $0 < \tau \leq \ln(\frac{9}{4})$, then all the conditions of Theorem 2.1 hold and $0$ is a unique fixed point of $T$.*

Now, we describe the concept of generalized $F$-Suzuki-contraction in the framework of dislocated $S_b$-metric spaces.

**Definition 2.4.** *Let $(X, S_b)$ be a dislocated $S_b$-metric space. A mapping $T : X \to X$ is said to be a generalized $F$-Suzuki-contraction if there exists $F \in \mathfrak{F}$ such that for all $x, y \in X$*

$$\frac{1}{2b} S_b(x, x, Tx) < S_b(x, x, y) \Rightarrow F\big(2b^3 S_b(Tx, Tx, Ty)\big) \leq F\big(M_T(x, y)\big) - \psi\big(M_T(x, y)\big), \tag{2.18}$$

*where $\psi \in \Psi$ and*

$$\begin{aligned}
M_T(x, y) = \max\Big\{&S_b(x, x, y), \frac{S_b(y, y, Ty)}{10}, \frac{S_b(x, x, Tx)}{10}, S_b(Tx, Tx, Ty),\\
&\frac{S_b(y, y, Tx)}{18b}, \frac{S_b(Tx, Tx, T^2x)}{2}\Big\}.
\end{aligned}$$

Our second main result is the following.

**Theorem 2.2.** *Let $(X, S_b)$ be a complete dislocated $S_b$-metric space and $T : X \to X$ be a generalized $F$-Suzuki-contraction satisfying the following condition:*

$$\max\Big\{\frac{S_b(y, y, Ty)}{10}, \frac{S_b(x, x, Tx)}{10}, \frac{S_b(y, y, Ty)}{9} + \frac{S_b(Tx, Tx, Ty)}{18}, \frac{S_b(Tx, Tx, T^2x)}{2}\Big\} \leq S_b(Tx, Tx, Ty)$$

*for all $x, y$ in $X$. Then $T$ has a unique fixed point in $X$.*

*Proof.* Let $x_0$ be arbitrary. Define $x_n = Tx_{n-1}$ for each $n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $S_b(x_n, x_n, Tx_n) = 0$, then $x_n = Tx_n$ and $x_n$ becomes a fixed point of $T$, which completes the proof. Therefore, we assume that $S_b(x_n, x_n, Tx_n) > 0$ for all $n \in \mathbb{N}$. Taking into account (2.18), we deduce

$$F\big(2b^3 S_b(Tx_n, Tx_n, Tx_{n+1})\big) \leq F\big(M_T(x_n, x_{n+1})\big) - \psi\big(M_T(x_n, x_{n+1})\big). \tag{2.19}$$

Using $(dS_b2)$ we get

$$
\begin{aligned}
\max\Big\{S_b(x_n,x_n,x_{n+1}),S_b(x_{n+1},x_{n+1},Tx_{n+1})\Big\} & \\
&\leq M_T(x_n,x_{n+1})\\
&\leq \max\Big\{S_b(x_n,x_n,x_{n+1}),\frac{S_b(x_{n+1},x_{n+1},Tx_{n+1})}{10},\frac{S_b(x_n,x_n,Tx_n)}{10},\\
&\quad S_b(Tx_n,Tx_n,Tx_{n+1}),\frac{S_b(Tx_n,Tx_n,Tx_{n+1})}{2},\frac{S_b(x_{n+1},x_{n+1},x_{n+2})}{6}\Big\}\\
&= \max\Big\{S_b(x_n,x_n,x_{n+1}),S_b(x_{n+1},x_{n+1},x_{n+2})\Big\}
\end{aligned}
$$

and combining it with the relation (2.19) we derive

$$
\begin{aligned}
F\big(2b^3S_b(Tx_n,Tx_n,Tx_{n+1})\big) &\leq F\big(\max\big\{S_b(x_n,x_n,x_{n+1}),S_b(x_{n+1},x_{n+1},x_{n+2}\Big\}\big)\\
&\quad -\psi\big(\max\Big\{S_b(x_n,x_n,x_{n+1}),S_b(x_{n+1},x_{n+1},x_{n+2}\Big\}\big). \qquad (2.20)
\end{aligned}
$$

If $\max\{S_b(x_n,x_n,x_{n+1}),S_b(x_{n+1},x_{n+1},x_{n+2})\}=S_b(x_{n+1},x_{n+1},x_{n+2})$, then (2.20) becomes

$$F\big(2b^3S_b(Tx_n,Tx_n,Tx_{n+1})\big)\leq F\big(S_b(x_{n+1},x_{n+1},x_{n+2})\big)-\psi\big(S(x_{n+1},x_{n+1},x_{n+2})\big).$$

By the property of $\psi$ and using condition $(F1)$, we obtain

$$2b^3S_b(Tx_n,Tx_n,Tx_{n+1})<S_b(Tx_n,Tx_n,Tx_{n+1}),$$

which is a contradiction. Hence $\max\{S_b(x_n,x_n,x_{n+1}),S_b(x_{n+1},x_{n+1},x_{n+2})\}=S_b(x_n,x_n,x_{n+1})$, then (2.20) becomes

$$F\big(2b^3S_b(Tx_n,Tx_n,Tx_{n+1})\big)\leq F\big(S_b(x_n,x_n,x_{n+1})\big)-\psi\big(S(x_n,x_n,x_{n+1})\big). \qquad (2.21)$$

This together with condition $(F1)$ implies that $S_b(Tx_n,Tx_n,Tx_{n+1})<S_b(x_n,x_n,x_{n+1})$ for each $n\in\mathbb{N}$. Then $\{S_b(x_n,x_n,x_{n+1})\}$ is a nonnegative decreasing sequence of real numbers. Therefore, there exists $A\geq 0$ such that $\lim\limits_{n\to+\infty}S_b(x_n,x_n,x_{n+1})=A$.
Letting $n\to+\infty$ in (2.21) and using $(F3')$ and continuity of $\psi$, we get

$$F(2b^3A)\leq F(A)-\psi(A).$$

It gives us $\psi(A)=0$. By property of $\psi$ we deduce that $A=0$. Consequently, we have

$$\lim_{n\to+\infty}S_b(x_n,x_n,x_{n+1})=0. \qquad (2.22)$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence in $X$. If it is not true, then there exist $\varepsilon>0$ and increasing sequences of natural numbers $\{p(n)\}$ and $\{q(n)\}$ such that

$$
\begin{aligned}
&n<q(n)<p(n),\\
&S_b(x_{q(n)},x_{q(n)},x_{p(n)})\geq\varepsilon,\\
&S_b(x_{q(n)},x_{q(n)},x_{p(n)-1})<\varepsilon \qquad (2.23)
\end{aligned}
$$

for all $n\in\mathbb{N}$.

Owing to (2.22), there exists $N_1\in\mathbb{N}$ such that

$$S_b(x_{q(n)},x_{q(n)},Tx_{q(n)})<\varepsilon \qquad (2.24)$$

for all $n \geq N_1$. Hence, from (2.23) and (2.24) it follows that

$$\frac{1}{2b} S_b(x_{q(n)}, x_{q(n)}, Tx_{q(n)}) < \frac{1}{2b}\varepsilon < S_b(x_{q(n)}, x_{q(n)}, x_{p(n)})$$

for all $n \geq N_1$. By using (2.18) we obtain

$$F\big(2b^3 S_b(Tx_{q(n)}, Tx_{q(n)}, Tx_{p(n)})\big) \leq F\big(M_T(x_{q(n)}, x_{p(n)})\big) - \psi\big(M_T(x_{q(n)}, x_{p(n)})\big). \tag{2.25}$$

From our assumptions and regarding $(dS_b2)$, we get

$$\begin{aligned}
&\max\Big\{S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}), S_b(Tx_{q(n)}, Tx_{q(n)}, Tx_{p(n)})\Big\} \\
\leq\ & M_T(x_{q(n)}, x_{p(n)}) \\
\leq\ & \max\Big\{S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}), \frac{S_b(x_{p(n)}, x_{p(n)}, x_{p(n)+1})}{10}, S_b(Tx_{q(n)}, Tx_{q(n)}, Tx_{p(n)}) \\
& \frac{S_b(x_{q(n)}, x_{q(n)}, x_{q(n)+1})}{10}, \frac{S_b(x_{q(n)+1}, x_{q(n)+1}, x_{q(n)+2})}{2}, \\
& \frac{S_b(x_{p(n)}, x_{p(n)}, x_{p(n)+1})}{9} + \frac{S_b(x_{q(n)+1}, x_{q(n)+1}, x_{p(n)+1})}{18}\Big\} \\
\leq\ & \max\Big\{S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}), S_b(x_{q(n)+1}, x_{q(n)+1}, x_{p(n)+1})\Big\}.
\end{aligned}$$

Then (2.25) becomes

$$\begin{aligned}
&F\big(2b^3 S_b(Tx_{q(n)}, Tx_{q(n)}, Tx_{p(n)})\big) \\
\leq\ & F\Big(\max\Big\{S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}), S_b(x_{q(n)+1}, x_{q(n)+1}, x_{p(n)+1})\Big\}\Big) \\
-\ & \psi\Big(\max\Big\{S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}), S_b(x_{q(n)+1}, x_{q(n)+1}, x_{p(n)+1})\Big\}\Big).
\end{aligned}$$

If $\max\Big\{S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}), S_b(x_{q(n)+1}, x_{q(n)+1}, x_{p(n)+1})\Big\} = S_b(x_{q(n)+1}, x_{q(n)+1}, x_{p(n)+1})$ for some $n$, then we have

$$\begin{aligned}
F\big(2b^3 S_b(Tx_{q(n)}, Tx_{q(n)}, Tx_{p(n)})\big) \quad &\leq \quad F\big(S_b(x_{q(n)+1}, x_{q(n)+1}, x_{p(n)+1})\big) \\
&- \quad \psi\big(S_b(x_{q(n)+1}, x_{q(n)+1}, x_{p(n)+1})\big).
\end{aligned}$$

Obviously, $S_b(x_{q(n)+1}, x_{q(n)+1}, x_{p(n)+1}) > 0$ and by the property of $\psi$ and $(F1)$, we get

$$S_b(Tx_{q(n)}, Tx_{q(n)}, Tx_{p(n)}) < S_b(x_{q(n)+1}, x_{q(n)+1}, x_{p(n)+1}),$$

which is a contradiction. Duo to this fact, we find that

$$\max\Big\{S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}), S_b(x_{q(n)+1}, x_{q(n)+1}, x_{p(n)+1})\Big\} = S_b(x_{q(n)}, x_{q(n)}, x_{p(n)})$$

for all $n$. Therefore

$$F\big(2b^3 S_b(Tx_{q(n)}, Tx_{q(n)}, Tx_{p(n)})\big) \leq F\big(S_b(x_{q(n)}, x_{q(n)}, x_{p(n)})\big) - \psi\big(S_b(x_{q(n)}, x_{q(n)}, x_{p(n)})\big), \tag{2.26}$$

and by $(F1)$, it follows that

$$S_b(x_{q(n)+1}, x_{q(n)+1}, x_{p(n)+1}) < S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}).$$

In view of (2.23) and $(dS_b2)$, we infer that

$$\begin{aligned}\varepsilon &\leq S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}) \leq 2bS_b(x_{q(n)}, x_{q(n)}, x_{p(n)-1}) + bS_b(x_{p(n)}, x_{p(n)}, x_{p(n)-1}) \\ &\leq 2bS_b(x_{q(n)}, x_{q(n)}, x_{p(n)-1}) + 2b^2S_b(x_{p(n)}, x_{p(n)}, x_{p(n)}) + b^2S_b(x_{p(n)-1}, x_{p(n)-1}, x_{p(n)}) \\ &\leq 2bS_b(x_{q(n)}, x_{q(n)}, x_{p(n)-1}) + 6b^3S_b(x_{p(n)}, x_{p(n)}, x_{p(n)+1}) \\ &\quad + b^2S_b(x_{p(n)-1}, x_{p(n)-1}, x_{p(n)}).\end{aligned}$$

Taking the limit as $n \to +\infty$ in the above inequality and regarding (2.22) and (2.23), we deduce that

$$\varepsilon \leq \lim_{n\to+\infty} S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}) \leq 2b\varepsilon. \tag{2.27}$$

On the other hand, we have

$$\begin{aligned}\varepsilon &\leq S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}) \leq 2bS_b(x_{q(n)}, x_{q(n)}, x_{q(n)+1}) + bS_b(x_{p(n)}, x_{p(n)}, x_{q(n)+1}) \\ &\leq 2bS_b(x_{q(n)}, x_{q(n)}, x_{q(n)+1}) + 2b^2S_b(x_{p(n)}, x_{p(n)}, x_{p(n)+1}) \\ &\quad + b^2S_b(x_{q(n)+1}, x_{q(n)+1}, x_{p(n)+1}).\end{aligned}$$

Taking the limit supremum as $n \to +\infty$ in the above inequality. By using (2.22) we obtain

$$\frac{\varepsilon}{b^2} \leq \limsup_{n\to+\infty} S_b(x_{q(n)+1}, x_{q(n)+1}, x_{p(n)+1}). \tag{2.28}$$

Taking the limit supremum as $n \to +\infty$ on each side of (2.26) and using conditions (2.27) and (2.28) together with $(F1)$ and $(F3')$, we deduce that

$$\begin{aligned}F(2b\varepsilon) = F(2b^3\frac{\varepsilon}{b^2}) &\leq F\big(2b^3 \limsup_{n\to+\infty} S_b(x_{q(n)+1}, S_b(x_{q(n)+1}, x_{p(n)+1})\big) \\ &\leq F\big(\limsup_{n\to+\infty} S_b(x_{q(n)}, S_b(x_{q(n)}, x_{p(n)})\big) \\ &\quad -\psi\big(\liminf_{n\to+\infty} S_b(x_{q(n)}, S_b(x_{q(n)}, x_{p(n)})\big) \\ &\leq F(2b\varepsilon) - \psi(\varepsilon).\end{aligned}$$

It enforces that $\psi(\varepsilon) = 0$, which leads to a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is a complete dislocated $S_b$-metric space, it follows that there exists $v \in X$ in which for each $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$S_b(v, v, x_n) < \varepsilon \tag{2.29}$$

for all $n > N_2$. Now, we prove that $v$ is a fixed point of $T$. To this end, we show that $S_b(Tv, Tv, v) = 0$. We consider the following cases:

**Case 1.** If $S_b(v, v, x_n) = 0$ for sufficiently large $n$, then $v = x_n$. Thus, for sufficiently large $n$, we can write

$$S_b(Tv, Tv, v) = S_b(Tx_n, Tx_n, v) \leq 2bS_b(x_{n+1}, x_{n+1}, x_{n+2}) + bS_b(v, v, x_{n+2}).$$

Letting $n \to +\infty$ in the above inequality. From (2.22) and (2.29) we get $S_b(Tv, Tv, v) = 0$. Thus $Tv = v$ and $v$ is a fixed point of $T$.

**Case 2.** If there exists $n \geq N_2$ such that $S_b(v, v, x_n) > 0$ and $S_b(Tv, Tv, Tx_n) = 0$, then from $(dS_b2)$ we have

$$S_b(Tv, Tv, v) \leq 2bS_b(Tv, Tv, Tx_n) + bS_b(v, v, x_{n+1}) \leq b\varepsilon,$$

which implies that $Tv = v$ by virtue of the arbitrariness of $\varepsilon$.

**Case 3.** If $S_b(v, v, x_n) > 0$ and $S_b(Tv, Tv, Tx_n) > 0$ for all $n \geq N_2$, then using (2.18) we obtain

$$F\big(2b^3 S_b(Tv, Tv, Tx_n)\big) \leq F\big(M_T(v, x_n)\big) - \psi\big(M_T(v, x_n)\big). \tag{2.30}$$

Thus, by using the hypothesis and taking into account $(dS_b2)$, it yields

$$\begin{aligned}
\max\Big\{S_b(v, v, x_n), S_b(Tv, Tv, Tx_n)\Big\} \\
&\leq M_T(v, x_n) \\
&\leq \max\Big\{S_b(v, v, x_n), S_b(Tv, Tv, Tx_n), \frac{S_b(x_n, x_n, Tx_n)}{10} \\
&\quad \frac{S_b(v, v, Tv)}{10}, \frac{S_b(x_n, x_n, Tx_n)}{9} + \frac{S_b(Tv, Tv, Tx_n)}{18}, \frac{S_b(Tv, Tv, T^2v)}{2}\Big\} \\
&\leq \max\Big\{S_b(v, v, x_n), S_b(Tv, Tv, Tx_n)\Big\}.
\end{aligned}$$

Then (2.30) becomes

$$\begin{aligned}
F\big(2b^3 S_b(Tv, Tv, Tx_n)\big) &\leq F\big(\max\Big\{S_b(v, v, x_n), S_b(Tv, Tv, Tx_n)\Big\}\big) \\
&- \psi\big(\max\Big\{S_b(v, v, x_n), S_b(Tv, Tv, Tx_n)\Big\}\big).
\end{aligned}$$

If $\max\{S_b(v, v, x_n), S_b(Tv, Tv, Tx_n)\} = S_b(Tv, Tv, Tx_n)$, then we have

$$F\big(2b^3 S_b(Tv, Tv, Tx_n)\big) \leq F\big(S_b(Tv, Tv, Tx_n)\big) - \psi\big(S_b(Tv, Tv, Tx_n)\big).$$

From this it follows that $2b^3 S_b(Tv, Tv, Tx_n) < S_b(Tv, Tv, Tx_n)$, which is a contradiction. Therefore,

$$\max\{S_b(v, v, x_n), S_b(Tv, Tv, Tx_n)\} = S_b(v, v, x_n)$$

and (2.30) becomes

$$\begin{aligned}
F\big(2b^3 S_b(Tv, Tv, Tx_n)\big) &\leq F\big(S_b(v, v, x_n)\big) - \psi\big(S_b(v, v, x_n)\big) \\
&< F\big(S_b(v, v, x_n)\big).
\end{aligned}$$

Thus, from $(F1)$ we get

$$S_b(Tv, Tv, Tx_n) < S_b(v, v, x_n). \tag{2.31}$$

Applying (2.29), (2.31) and $(dS_b2)$ we get

$$S_b(Tv, Tv, v) \leq 2bS_b(Tv, Tv, Tx_n) + bS_b(v, v, x_{n+1}) < 3b\varepsilon$$

for sufficiently large $n$. It enforces that $Tv = v$ by virtue of the arbitrariness of $\varepsilon$. Then $v$ is a fixed point of $T$.

Next, we show the uniqueness. Indeed, if $v_1$, $v_2$ are two fixed points of $T$ such that $v_1 \neq v_2$, then in view of (2.18) we get

$$F\big(2b^3 S_b(Tv_1, Tv_1, Tv_2)\big) \leq F\big(M_T(v_1, v_2)\big) - \psi\big(M_T(v_1, v_2)\big). \tag{2.32}$$

According to our assumptions and by using ($dS_b2$), we find that

$$\begin{aligned} S_b(v_1, v_1, v_2) &\leq M_T(v_1, v_2) \\ &\leq \max\Big\{S_b(v_1, v_1, v_2), S_b(Tv_1, Tv_1, Tv_2), \\ &\quad \frac{S_b(v_2, v_2, Tv_2)}{10}, \frac{S_b(v_1, v_1, Tv_1)}{10}, \frac{S_b(v_2, v_2, Tv_2)}{9} \\ &\quad + \frac{S_b(Tv_1, Tv_1, Tv_2)}{18}\Big\} \\ &\leq \max\big\{S_b(v_1, v_1, v_2), S_b(Tv_1, Tv_1, Tv_2)\big\} \\ &= S_b(v_1, v_1, v_2). \end{aligned}$$

Then (2.32) becomes

$$F\big(2b^3 S_b(v_1, v_1, v_2)\big) \leq F\big(S_b(v_1, v_1, v_2)\big) - \psi\big(S_b(v_1, v_1, v_2)\big).$$

From this it follows that $2b^3 S_b(v_1, v_1, v_2) < S_b(v_1, v_1, v_2)$, which is a contradiction. Then $v_1 = v_2$ and so $T$ has a unique fixed point in $X$. □

**Example 2.4.** *Let $X = \{-1, 0, 1\}$. Define the mapping $S_b : X^3 \to \mathbb{R}_+$ by*

$$S_b(x, y, z) = \begin{cases} \frac{3}{2}, & 0 = x = y \neq z = 1 \ \text{or} \ -1 = x = y \neq z = 1 \\ \frac{10}{6}, & 1 = x = y \neq z \\ 0, & x = y = z = -1 \ \text{or} \ 1 \\ \frac{1}{5}, & \text{otherwise} \end{cases}$$

*for all $x, y, z \in X$. It is easy to show that $(X, S_b)$ is a complete dislocated $S_b$-metric space with $b = \frac{3}{2}$. Put $F(\alpha) = \ln \alpha$ $(\alpha > 0)$ and $\psi(t) = t$ $(t \geq 0)$. Define $T : X \to X$ by*

$$T(x) = \begin{cases} 0, & x = 1 \\ -1, & x = -1, 0. \end{cases}$$

*Note that $S_b(x, x, y) > 0$ and $S_b(T(x), T(x), T(y)) > 0$ if and only if $x \in \{-1, 0\}$, $y = 1$ or $x = 1$, $y \in \{-1, 0\}$. Also, for each $x, y \in X$ we have $M_T(x, y) = S_b(x, x, y)$ and we find that*

$$F\big(2b^3 S_b(T(x), T(x), T(y))\big) \leq F\big(S_b(x, x, y)\big) - \psi\big(S_b(x, x, y)\big) \Leftrightarrow \ln \frac{S_b(x, x, y)}{2b^3 S_b(T(x), T(x), T(y))} \geq S_b(x, x, y).$$

*Now, we consider two cases:*

**Case 2.1.** *Case 1. Let $x \in \{-1, 0\}$ and $y = 1$, then*

$$S_b(x,x,y) = \frac{3}{2},$$

$$S_b(T(x),T(x),T(y)) = S_b(-1,-1,0) = \frac{1}{5},$$

$$S_b(0,0,T(0)) = \frac{1}{5},\ \ S_b(-1,-1,T(-1)) = 0.$$

*So, we have*

$$\ln \frac{S_b(x,x,y)}{2b^3 S_b(T(x),T(x),T(y))} = \ln \frac{\frac{3}{2}}{\frac{54}{40}} = \ln \frac{120}{108} = 3.0377 \geq S_b(x,x,y) = \frac{3}{2}.$$

*Case 2. Let $x = 1$ and $y \in \{-1, 0\}$, then*

$$S_b(x,x,y) = \frac{10}{6},$$

$$S_b(T(x),T(x),T(y)) = \frac{1}{5},$$

$$S_b(x,x,T(x)) = \frac{10}{6}.$$

*So, we have*

$$\ln \frac{S_b(x,x,y)}{2b^3 S_b(T(x),T(x),T(y))} = \ln \frac{\frac{10}{6}}{\frac{54}{40}} = \ln \frac{400}{324} = 3.4369 \geq S_b(x,x,y) = \frac{10}{6}.$$

*On the other hand, for all $x, y \in X$ we have*

$$\max\Big\{\frac{S_b(y,y,T(y))}{10}, \frac{S_b(x,x,T(x))}{10}, \frac{S_b(y,y,T(y))}{9} + \frac{S_b(T(x),T(x),T(y))}{18}, \frac{S_b(T(x),T(x),T^2(x))}{2}\Big\}$$
$$\leq \frac{1}{5} = S_b(T(x),T(x),T(y)).$$

*Hence, $T$ is a generalized $F$-Suzuki-contraction which satisfies the assumption of Theorem 2.2 and so it has a unique fixed point $-1$.*

## References


[1] I. A. Bakhtin, The contraction mapping principle in almost metric spaces, Funct. Anal. Unianowsk Gos. Ped. Inst. 30 (1989), 26–37.

[2] S. Czerwik, Contraction mappings in $b$-metric spaces, Acta Math. Inform. Univ. Ostra. 1 (1993), 5–11.

[3] N.V. Dung and V.L. Hang, A fixed point theorem for generalized $F$-contractions on complete metric spaces, Vietnam J. Math. 43 (2015), 743–753.

[4] G.N.V. Kishore, K.P.R. Rao, D. Panthi, B. Srinuvasa Rao and S. Satyanaraya, Some applications via fixed point results in partially ordered $S_b$-metric spaces, Fixed Point Theory Appl. 2017 (2017), 10.

[5] H. Piri and P. Kumam, Some fixed point theorems concerning $F$-contraction in complete metric spaces, Fixed Point Theory Appl. 2014 (2014), 210.

[6] H. Piri and P. Kumam, Fixed point theorems for generalized $F$-Suzuki-contraction mappings in complete $b$-metric spaces, Fixed Point Theory Appl. 2016 (2016), 90 .

[7] N.Y. Özgür and N. TaŞ, Some fixed point theorems on S-metric spaces, Mat. Vesnik 69 (1) (2017), 39-52.

[8] N.Y. Özgür, N. TaŞ and U. Celik, New fixed point-circle results on $S$-metric spaces, Bull. Math. Anal. Appl. 9 (2) (2017), 10–23.

[9] Y. Rohen*á*, *T*. Došenovi*ć* and S. Radenovi*ć*, A note on the paper "A Fixed point Theorems in $S_b$-Metric Spaces", Filomal 31 (11) (2017), 3335–3346.

[10] Sh. Sedghi, A. Gholidahne, T. Došenovic, J. Esfahani and S. Radenovic, Common fixed point of four maps in $S_b$-metric spaces, J. Linear Topol. Alg. 5 (2) (2016), 93–104.

[11] Sh. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorem in $S$-metric spaces, Mat. Vesn. 64 (2012), 258–266.

[12] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012 (2012), 94.

[13] D. Wardowski and N.V. Dung, Fixed points of $F$-weak contractions on complete metric spaces, Demonstr. Math. 47 (1) (2014), 146–155.