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COMMON FIXED POINT THEOREM FOR ĆIRIĆ TYPE QUASI-CONTRACTIONS IN RECTANGULAR b-METRIC SPACES

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Abstract. The purpose of this paper is to give positive answers to questions concerning Ćirić type quasi-contractions in rectangular b-metric spaces proposed in George et al. (J. Nonlinear Sci. Appl. 8 (2015), 1005-1013).

1. Introduction and preliminaries

In [1], George et al. introduced the concept of rectangular b-metric spaces as a generalization of metric space, rectangular metric space and b-metric space (see also [2, 3]). Since then many fixed point theorems for various contractions were established in rectangular b-metric spaces (see [4–12]).

Definition 1.1. *([1]) Let X be a nonempty set and the mapping $d : X \times X \to [0, \infty)$ satisfies:*

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ for all $x, y \in X$ and all distinct points $u, v \in X \backslash \{x, y\}$.

Then d is called a rectangular b-metric on X and (X, d) is called a rectangular b-metric space (in short RbMS) with coefficient s.

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Definition 1.2. *([1]) Let (X, d) be a RbMS, $\{x_n\}$ be a sequence in X and $x \in X$. Then*

(1) The sequence $\{x_n\}$ is said to be convergent in (X, d) and converges to x, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}^+$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(2) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, d) if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}^+$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0$ and $p > 0$.

(3) (X, d) is said to be a complete RbMS if every Cauchy sequence in X converges to some $x \in X$.

In the setting of $RbMS$, limit of a convergent sequence is not necessarily unique and also every convergent sequence is not necessarily a Cauchy sequence. For details, we can see [1]. However, we have that the following result.

Lemma 1.1. *([3]) Let (X, d) be a RbMS with $s \geq 1$, and let $\{x_n\}$ be a Cauchy sequence in X such that $x_n \neq x_m$ whenever $n \neq m$. Then $\{x_n\}$ can converge to at most one point.*

George et al. [1] raised the following problems.

Problem 1.1. *([1]) In [1, Theorem 2.1], can we extent the range of λ to the case $\frac{1}{s} < \lambda < 1$?*

Problem 1.2. *([1]) Prove analogue of Chatterjea contraction, Reich contraction, Ćirić contraction and Hardy-Rogers contraction in RbMS.*

In [6], Mitrović has given a positive answer to Problem 1.1. In [7], Mitrović et al. obtained an analogue of Reich's contraction principle in RbMS and thus give a partial solution to Problem 1.2. For further results, the reader can refer to [13, 14].

In this paper, we proved a common fixed point theorem for Ćirić type quasi-contractions in RbMS. It is well known that Ćirić contraction is more general than other contractions in Problem 1.2. Thus, we give a complete solution to the above Problem 1.2.

2. Main Results

The following lemma is crucial in this paper.

Lemma 2.1. *Let (X, d) be a RbMS with coefficient $s \geq 1$ and $f, g : X \to X$ be two self maps such that $f(X) \subseteq g(X)$. Assume that there exists $\lambda \in [0, \frac{1}{s})$ such that*

$$d(fx, fy) \leq \lambda \max\{d(gx, gy), d(gx, fx), d(gy, fy), d(gy, fx), d(gx, fy)\}. \tag{2.1}$$

Taking $x_0 \in X$, we construct a sequence $\{y_n\}$ by $y_n = fx_n = gx_{n+1}$. If $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}^+$, then

(1) For $m \in 0 \cup \mathbb{N}^+$ and $p \in \mathbb{N}^+$, there exists $1 \le k(p) \le p$ such that

$$\delta(\mathcal{O}(y_m, m+p)) = d(y_m, y_{m+k(p)}),$$

where $\mathcal{O}(y_m, m+p) = \{y_m, y_{m+1}, \cdots, y_{m+p}\}, \delta(A) = \sup_{x,y \in A} d(x,y)$.

(2) $y_n \neq y_m$ *whenever* $n \neq m$.

(3) $\delta(\mathcal{O}(y_0, n)) \le \frac{s}{1-s\lambda}[d(y_0, y_1) + d(y_1, y_2)]$.

(4) $\delta(\mathcal{O}(y_0, \infty)) \le \frac{s}{1-s\lambda}[d(y_0, y_1) + d(y_1, y_2)]$, *where* $\mathcal{O}(y_0, \infty) = \{y_0, y_1, \cdots, y_n, \cdots\}$.

(5) $\{y_n\}$ *is a Cauchy sequence.*

Proof. (1) Let $m \in \{0, 1, 2, \cdots, \}$ and $p \in \mathbb{N}^+$. Using (2.1), for any $i, j \in \mathbb{N}^+$ with $m < i < j \le m+p$, we have that

$$\begin{aligned} d(y_i, y_j) &= d(fx_i, fx_j) \\ &\le \lambda \max\{d(gx_i, gx_j), d(gx_i, fx_i), d(gx_j, fx_j), d(gx_i, fx_j), d(gx_j, fx_i)\} \\ &= \lambda \max\{d(y_{i-1}, y_{j-1}), d(y_{i-1}, y_i), d(y_{j-1}, y_j), d(y_{i-1}, y_j), d(y_{j-1}, y_i)\} \\ &\le \lambda \delta(\mathcal{O}(y_m, m+p)) \\ &< \delta(\mathcal{O}(y_m, m+p)). \end{aligned}$$

This implies that

$$\max\{d(y_i, y_j) : i, j \in \mathbb{N}^+ \text{ and } m < i < j \le m+p\} < \delta(\mathcal{O}(y_m, m+p)).$$

Since $\delta(\mathcal{O}(y_m, m+p)) = \max\{d(y_i, y_j) : i, j \in \mathbb{N}^+ \text{ and } m \le i < j \le m+p\}$, there exists $k(p)$ with $1 \le k(p) \le p$ such that

$$\delta(\mathcal{O}(y_m, m+p)) = d(y_m, y_{m+k(p)}). \tag{2.2}$$

(2) Suppose that $y_n = y_{n+p}$ for some $n, p \in \mathbb{N}^+$. Then, by (2.1) we obtain that

$$\begin{aligned} \delta(\mathcal{O}(y_n, n+p)) &= d(y_n, y_{n+k(p)}) \\ &= d(y_{n+p}, y_{n+k(p)}) \\ &= d(fx_{n+p}, fx_{n+k(p)}) \\ &\le \lambda \max\{d(gx_{n+p}, gx_{n+k(p)}), d(gx_{n+p}, fx_{n+p}), d(gx_{n+k(p)}, fx_{n+k(p)}), \\ &\qquad d(gx_{n+k(p)}, fx_{n+p}), d(gx_{n+p}, fx_{n+k(p)})\} \end{aligned}$$

$$
\begin{aligned}
&= \lambda \max\{d(y_{n+p-1}, y_{n+k(p)-1}), d(y_{n+p-1}, y_{n+p}), d(y_{n+k(p)-1}, y_{n+k(p)}),\\
&\qquad d(y_{n+k(p)-1}, y_{n+p}), d(y_{n+p-1}, y_{n+k(p)})\}\\
&\leq \lambda\delta(\mathcal{O}(y_n, n+p)),
\end{aligned}
$$

which implies $\delta(\mathcal{O}(y_n, n+p)) = 0$. However, this is impossible because $\delta(\mathcal{O}(y_n, n+p)) \geq d(y_n, y_{n+1}) > 0$. Therefore, $y_n \neq y_m$ whenever $n \neq m$.

(3) Let $n \in \mathbb{N}^+$. Then, using (2.1) and (2.2), we get that

$$
\begin{aligned}
&\delta(\mathcal{O}(y_0, n))\\
=\ & d(y_0, y_{k(n)})\\
\leq\ & s[d(y_0, y_1) + d(y_1, y_2) + d(y_2, y_{k(n)})]\\
=\ & s[d(y_0, y_1) + d(y_1, y_2)] + sd(fx_2, fx_{k(n)})\\
\leq\ & s[d(y_0, y_1) + d(y_1, y_2)] + s\lambda \max\{d(gx_2, gx_{k(n)}), d(gx_2, fx_2), d(gx_{k(n)}, fx_{k(n)}),\\
& d(gx_2, fx_{k(n)}), d(gx_{k(n)}, fx_2)\}\\
=\ & s[d(y_0, y_1) + d(y_1, y_2)] + s\lambda \max\{d(y_1, y_{k(n)-1}), d(y_1, y_2), d(y_{k(n)-1}, y_{k(n)}),\\
& d(y_1, y_{k(n)}), d(y_{k(n)-1}, y_2))\}\\
\leq\ & s[d(y_0, y_1) + d(y_1, y_2)] + s\lambda\delta(\mathcal{O}(y_0, n)).
\end{aligned}
$$

This implies that

$$
\delta(\mathcal{O}(y_0, n)) \leq \frac{s}{1 - s\lambda}[d(y_0, y_1) + d(y_1, y_2)]. \tag{2.3}
$$

(4) Note that $\lim_{n\to\infty} \delta(\mathcal{O}(y_0, n)) = \delta(\mathcal{O}(y_0, \infty))$. Thus, from (2.3) we see that

$$
\delta(\mathcal{O}(y_0, \infty)) \leq \frac{s}{1 - s\lambda}[d(y_0, y_1) + d(y_1, y_2)].
$$

(5) For any $n, p \in \mathbb{N}^+$,

$$
\begin{aligned}
d(y_n, y_{n+p}) &\leq \lambda\delta(\mathcal{O}(y_{n-1}, n+p))\\
&\leq \lambda^2\delta(\mathcal{O}(y_{n-2}, n+p))\\
&\leq \cdots\\
&\leq \lambda^n\delta(\mathcal{O}(y_0, n+p))\\
&\leq \lambda^n\delta(\mathcal{O}(y_0, \infty))\\
&\leq \lambda^n \cdot \frac{s}{1 - s\lambda}[d(y_0, y_1) + d(y_1, y_2] \to 0 (n \to \infty).
\end{aligned}
$$

Therefore, $\{y_n\}$ is a Cauchy sequence in X. □

Theorem 2.1. *Let (X, d) be a RbMS $s \geq 1$ and $f, g : X \to X$ be two self maps such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. If there exists $\lambda \in [0, \frac{1}{s})$ such that*

$$d(fx, fy) \leq \lambda \ \max\{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}, \tag{2.4}$$

for all $x, y \in X$, then f and g have a point of coincidence in X. Moreover, if f and g are weakly compatible (i.e., they commute at their coincidence points), then they have a unique common fixed point.

Proof. Let x_0 be an arbitrary point of X. Choose $x_1 \in X$ such that $fx_0 = gx_1$. Now, we can construct a sequence $\{y_n\}$ defined by

$$y_n = fx_n = gx_{n+1}, \quad for\ n = 0, 1, 2, \ldots \tag{2.5}$$

If $y_k = y_{k+1}$ for some $k \in \mathbb{N}^+$, then $fx_{k+1} = y_{k+1} = y_k = gx_{k+1}$ and f and g have a point of coincidence. Suppose, further, that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}^+$. By Lemma 2.1, we can obtain $\{y_n\}$ is a Cauchy sequence in X. Suppose, e.g., that the subspace $g(X)$ is complete (the proof when $f(X)$ is complete is similar). Then $\{y_n\}$ tends to some $\omega \in g(X)$, where $\omega = gu$ for some $u \in X$. Suppose that $fu \neq gu$. Then

$$\begin{aligned} d(fu, y_n) &= d(fu, fx_n) \\ &\leq \lambda \max\{d(gu, gx_n), d(gu, fu), d(gx_n, fx_n), d(gu, fx_n), d(gx_n, fu)\} \\ &= \lambda \max\{d(gu, y_{n-1}), d(gu, fu), d(y_{n-1}, y_n), d(gu, y_n), d(y_{n-1}, fu)\}. \end{aligned}$$

Note that $d(gu, y_{n-1}) \to 0$, $d(y_{n-1}, y_n) \to 0$ and $d(gu, y_n) \to 0$ as $n \to \infty$. Then, for sufficiently large $n \in \mathbb{N}^+$,

$$\begin{aligned} & \max\{d(gu, y_{n-1}), d(gu, fu), d(y_{n-1}, y_n), d(gu, y_n), d(y_{n-1}, fu)\} \\ = \ & \max\{d(gu, fu), d(y_{n-1}, fu)\} \end{aligned}$$

and

$$d(fu, y_n) \leq \lambda \max\{d(gu, fu), d(y_{n-1}, fu)\}. \tag{2.6}$$

Denote $M(x_n, u) = \max\{d(gu, fu), d(y_{n-1}, fu)\}$ for $n \in \mathbb{N}^+$. Then we can consider the following cases.

Case 1. If there exists a subsequence $\{M(x_{n_k}, u)\}$ of $\{M(x_n, u)\}$ such that $M(x_{n_k}, u) = d(gu, fu)$, then $d(fu, y_{n_k}) \leq \lambda d(gu, fu)$. Note that $d(y_n, y_{n-1}) \to 0$, $d(y_n, gu) \to 0$ and

$$\frac{1}{s} d(fu, gu) \leq d(fu, y_{n_k}) + d(y_{n_k}, y_{n_k - 1}) + d(y_{n_k - 1}, gu). \tag{2.7}$$

Thus, taking upper limit as $k \to \infty$ in (2.7), we obtain that

$$\frac{1}{s} d(fu, gu) \leq \limsup_{k \to \infty} d(fu, y_{n_k}) \leq \lambda d(gu, fu).$$

This implies that $d(gu, fu) \leq s\lambda d(fu, gu)$, which is a contradiction with $s\lambda < 1$ and $fu \neq gu$.

Case 2. If there exists $N \in \mathbb{N}^+$ such that $M(x_n, u) = d(y_{n-1}, fu)$ for all $n > N$, then (2.6) implies that

$$\begin{aligned} d(fu, y_n) &\leq \lambda d(y_{n-1}, fu) \leq \lambda^2 d(y_{n-2}, fu) \leq \cdots \leq \lambda^{n-N} d(y_N, fu) \\ &= \lambda^n (\frac{1}{\lambda^N} d(y_N, fu)) \to 0 (n \to \infty), \end{aligned}$$

that is $d(fu, y_n) \to 0$ as $n \to \infty$. Since $d(gu, y_n) \to 0$ as $n \to \infty$, by Lemma 1.1 we have that $fu = gu$. This is a contradiction.

Thus, we prove that $fu = gu = \omega$, that is u is a point of coincidence of f and g.

If f,g are weakly compatible, then, by $fu = gu = \omega$, we obtain that $f\omega = fgu = gfu = g\omega$, and hence that ω is a point of coincidence of f and g. Let us prove that $\omega = f\omega = g\omega$. Using (2.1), we get that

$$\begin{aligned} d(\omega, f\omega) &= d(fu, f\omega) \\ &\leq \lambda \max\{d(gu, g\omega), d(gu, fu), d(g\omega, f\omega), d(gu, f\omega), d(g\omega, fu)\} \\ &= \lambda \max\{d(\omega, f\omega), 0, 0, d(\omega, f\omega), d(f\omega, \omega)\} \\ &= \lambda d(\omega, f\omega). \end{aligned}$$

Since $\lambda < 1$, we have that $d(\omega, f\omega) = 0$, which implies that $\omega = f\omega = g\omega$. Therefore, ω is a common fixed point of f and g.

Let us prove that the common fixed point of f and g is unique. Suppose that ω_1 and ω_2 are two common points of f and g, that is $\omega_1 = f\omega_1 = g\omega_1$ and $\omega_2 = f\omega_2 = g\omega_2$. Using (2.1), we get that

$$\begin{aligned} d(\omega_1, \omega_2) &= d(f\omega_1, f\omega_2) \\ &\leq \lambda \max\{d(g\omega_1, g\omega_2), d(g\omega_1, f\omega_1), d(g\omega_2, f\omega_2), d(g\omega_1, f\omega_2), d(g\omega_2, f\omega_1)\} \\ &= \lambda d(\omega_1, \omega_2). \end{aligned}$$

Since $\lambda < 1$, we have that $d(\omega_1, \omega_2) = 0$, that is $\omega_1 = \omega_2$. Thus, the common fixed point of f and g is unique. □

Taking $g = I_X$ (identity mapping of X) in Theorem 2.1 we obtain the following.

Corollary 2.1. *(Ćirić type contraction) Let (X, d) be a RbMS with coefficient $s \geq 1$ and $f : X \to X$ be a mapping. Assume that there exists $\lambda \in [0, \frac{1}{s})$*

$$d(fx, fy) \leq \lambda \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}$$

for all $x, y \in X$. Then f has a unique fixed point.

From Corollary 2.1, the following corollaries immediately follow.

Corollary 2.2. *(Chatterjea type contraction) Let (X, d) be a RbMS with coefficient $s \geq 1$ and $f : X \to X$ be a mapping. Assume that there exists $k \in [0, \frac{1}{s})$ such that*

$$d(fx, fy) \leq \frac{k}{2}(d(x, fy) + d(y, fx)),$$

for all $x, y \in X$. Then f has a unique fixed point.

Corollary 2.3. *(Reich type contraction) Let (X, d) be a RbMS with coefficient $s \geq 1$ and $f : X \to X$ be a mapping. Assume that there exist $\lambda, \mu, \delta \in [0, 1)$ with $\lambda + \mu + \delta < \frac{1}{s}$ such that*

$$d(fx, fy) \leq \lambda d(x, y) + \mu d(x, fx) + \delta d(y, fy),$$

for all $x, y \in X$. Then f has a unique fixed point.

Corollary 2.4. *(Hardy-Rogers type contraction) Let (X, d) be a RbMS with coefficient $s \geq 1$ and $f : X \to X$ be a mapping. Assume that there exist $\alpha_i \in [0, 1)(i = 1, 2, 3, 4, 5)$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < \frac{1}{s}$ such that*

$$d(fx, fy) \leq \alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, fy) + \alpha_4 d(x, fy) + \alpha_5 d(y, fx),$$

for all $x, y \in X$. Then f has a unique fixed point.

Remark 2.1. *From Corollary 2.1-Corollary 2.4, we see that Problem 1.2 has been fully answered.*

Finally, we give an example to illustrate our main result.

Example 2.1. *Let $X = A \bigcup B$, where $A = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right\}$ and $B = \{0, 2\}$. Define d: $X \times X \to [0, +\infty)$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$ and*

$$d(x, y) = \begin{cases} 0, & x = y; \\ |x - y|, & x, y \in A; \\ \frac{13}{6}, & x, y \in B; \\ \frac{3}{4}, & x \in A \setminus \{1\},\ y \in B; \\ 2, & x = 1,\ y \in B. \end{cases}$$

Let f: $X \to X$ be a map defined by

$$f(x) = \begin{cases} 1, & x \in B; \\ \frac{x}{2}, & x \in A \setminus \{\frac{1}{8}\}; \\ \frac{1}{8}, & x = \frac{1}{8}. \end{cases}$$

and g be an identity mapping on X. Then the following hold:

(a) *(X, d) is a complete rectangular b-metric space with coefficient $s = \frac{4}{3}$;*

(b) (X, d) *is neither a metric space nor a rectangular metric space;*

(c) *All conditions in Theorem 2.1 are satisfied with* $\lambda = \frac{1}{2}$*;*

(d) f *and* g *have a unique common fixed point* $x = \frac{1}{8}$.

Proof. First, let us prove (a). Clearly, conditions (1) and (2) of Definition 1.1 hold. To see (3), for all $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$, we consider the following three cases.

Case 1. If $x, y \in A$ or $x, y \in B$, we only need to consider the case of $x, y \in B$ with $u, v \in A \setminus \{1\}$. In this case, $d(u, v) \geq d(\frac{1}{4}, \frac{1}{8}) = \frac{1}{8}$. So we have

$$d(x, y) = \frac{13}{6} = \frac{4}{3}\left(\frac{3}{4} + \frac{1}{8} + \frac{3}{4}\right) \leq \frac{4}{3}[d(x, u) + d(u, v) + d(v, y)].$$

Case 2. If $x \in A \setminus \{1\}$ and $y \in B$, then $d(x, y) = \frac{3}{4}$. Let us consider the following three cases.

- If $v \in B \bigcup \{1\}$, then

$$d(x, y) = \frac{3}{4} < d(v, y) \leq d(x, u) + d(u, v) + d(v, y).$$

- If $u \in B$, then

$$d(x, y) = \frac{3}{4} = d(x, u) \leq d(x, u) + d(u, v) + d(v, y).$$

- If $u, v \in A$ and $v \neq 1$, then

$$d(x, y) = \frac{3}{4} = d(v, y) \leq d(x, u) + d(u, v) + d(v, y).$$

Case 3. If $x = 1$ and $y \in B$, then we consider the following two cases.

- If $u \in B$ or $v \in B$, then $d(x, u) = 2$ or $d(v, y) = \frac{13}{6}$. So we have

$$d(x, y) = 2 \leq d(x, u) + d(v, y) \leq d(x, u) + d(u, v) + d(v, y).$$

- If $u, v \in A$, then $v \neq 1$. It follows that $d(x, u) + d(u, v) \geq d(1, \frac{1}{2}) + d(\frac{1}{2}, \frac{1}{4}) = \frac{3}{4}$. So we have

$$d(x, y) = 2 = \frac{4}{3}\left(\frac{3}{4} + \frac{3}{4}\right) \leq \frac{4}{3}[d(x, u) + d(u, v) + d(v, y)].$$

  Additionally, in this case, we can also check that (b) holds.

Hence, from the above three cases, we prove that (X, d) is a rectangular b-metric space with coefficient $s = \frac{4}{3}$. Since X is a finite set, we know that $(g(X), d) = (X, d)$ is complete.

Now we prove (c). It is sufficient to prove that (2.4) holds with $\lambda = \frac{1}{2}$. Since $d(x, y) = d(y, x)$, we consider the following three cases.

Case 1. If $x, y \in B$. In this case, $d(fx, fy) = 0$. So (2.4) holds.

Case 2. If $x \in B$ and $y \in A$, then $fx = 1$, $d(gx, fx) = 2$ and $fy \in A$. In this case, we have

$$\begin{aligned} d(fx, fy) \leq & d(1, \frac{1}{8}) = \frac{7}{8} < \frac{1}{2} d(gx, fx) \\ \leq & \frac{1}{2} \max\{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(fx, gy)\}. \end{aligned}$$

Case 3. If $x, y \in A$, it is clear that $d(fx, fy) = \frac{1}{2} d(gx, gy)$ for all $x, y \in A \setminus \{\frac{1}{8}\}$, which follows that (2.4) holds. So we assume that $x = \frac{1}{8}$. In this case, we have

$$\begin{aligned} d(fx, fy) = & \frac{1}{2} y - \frac{1}{8} < \frac{1}{2} \left(y - \frac{1}{8} \right) \\ \leq & \frac{1}{2} \max\{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(fx, gy)\}. \end{aligned}$$

From the above three cases, we show that (c) holds. Obviously, f and g have a unique common fixed point $fx = gx = x = \frac{1}{8}$. □

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