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BLOW-UP, EXPONENTIAL GROUTH OF SOLUTION FOR A NONLINEAR PARABOLIC EQUATION WITH $p(x)-$ LAPLACIAN

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Abstract. In this paper, we consider the following equation

$$u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + \omega\,|u|^{m(x)-2}u_t = b\,|u|^{r(x)-2}u.$$

We prove a finite time blowup result for the solutions in the case $\omega = 0$ and exponential growth in the case $\omega > 0$, with the negative initial energy in the both case.

1. Introduction

We consider the following boundary problem:

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + \omega\,|u|^{m(x)-2}u_t = b\,|u|^{r(x)-2}u & \text{in } \Omega\times(0,T), \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0, \\ u(x,0) = u_0(x) \quad \text{in } \Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with smooth boundary $\partial\Omega$ and $b > 0$, $\omega \geq 0$ are constants, $p(.)$, $m(x)$ and $r(.)$ are given measurable functions on Ω satisfying

$$2 \leq m_1 \leq m(x) \leq m_2 < p_1 \leq p(x) \leq p_2 < r_1 \leq r(x) \leq r_2 \leq p_*(x). \tag{1.2}$$

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$$\begin{aligned} p_1 &: = ess\inf_{x\in\Omega}(p(x)), \quad p_2 := ess\sup_{x\in\Omega}(p(x)), \\ r_1 &: = ess\inf_{x\in\Omega}(r(x)), \quad r_2 := ess\sup_{x\in\Omega}(r(x)), \\ m_1 &: = ess\inf_{x\in\Omega}(m(x)), \quad m_2 := ess\sup_{x\in\Omega}(m(x)), \end{aligned}$$

and

$$p_*(x) = \begin{cases} \frac{np(x)}{ess\sup\limits_{x\in\Omega}(n-p(x))} & \text{if } p_2 < n \\ +\infty \ \text{ if } p_2 \geq n \end{cases}.$$

We also assume that $p(.)$, $m(.)$ and $r(.)$ satisfy the log-Hölder continuity condition:

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|}, \text{ for a.e. } x, \ y \in \Omega, \text{ with } |x-y| < \delta, \tag{1.3}$$

$A > 0$, $0 < \delta < 1$.

Equation (1.1) can be viewed as a generalization of the evolutional p-Laplacian equation

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \omega |u|^{m-2} u_t = b|u|^{r-2} u,$$

with the constant exponent of nonlinearity p, m, $r \in (2, \ \infty)$, which appears in various physical contexts. In particular, this equation arises from the mathematical description of the reaction-diffusion/ diffusion, heat transfer, population dynamics processus, and so on (*see* [11]) and references therein). Recently in [1], in the case $\omega = 0$, Agaki proved an existence and blow up result for the initial datum $u_0 \in L^r()$. Ôtani [17] studied the existence and the asymptotic behavior of solutions of (1.1) and overcome the difficulties caused by the use of nonmonotone perturbation theory. The quasilinear case, with $p \neq 2$, requires a strong restriction on the growth of the forcing term $|u|^{r-2}u$, which is caused by the loss of the elliptic estimate for the $p-$Laplacian operator defined by $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ (*see* [2]).

Alaoui et al [12] considered the following nonlinear heat equation

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = |u|^{r(x)-2} u + f, & \text{in } \Omega\times(0,T), \\ u(x,t) = 0, \ \ x \in \ \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) \ \text{ in } \ \Omega. \end{cases} \tag{1.4}$$

Where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Under suitable conditions on r and p and for $f = 0$, they showed that any solution with nontrivial initial datum blows up in finite time. In the absence of the diffusion term in equation (1.1) when $p(x) = p$ and $r(x) = r$ proved the existence and plow up results have been established by many authors (See $[1-3, \ 9, \ 14, 17]$).

We should also point out that Polat [18] established a blow-up result for the solution with vanishing initial energy of the following initial boundary value problem

$$u_t - u_{xx} + |u|^{m-2} u_t = |u|^{p-2} u. \tag{1.5}$$

Where m and p are real constants.

In recent years, mush attention has been paid to the study of mathematical models of electro-theological fluids. This models inclode hyperbolic, parapolic or elliptic equations which are nonlinear with respect to the gradient of the thought solution with variable exponents of nonlinearity, (see [4, 5, 10, 15]).

Our objective in this paper is to study: In the section 3, the blow up of the solutions of the problem (1.1) in the case $\omega = 0$, in the section 4, exponential growth of solution when $\omega > 0$.

2. Preliminaries

We present in this section some Lemmas about the Lebesque and sobolev space with variables conponents (See [6 − 8, 12, 13]). Let $p : \Omega \to [1, +\infty]$ be a measurable function, where Ω is adomain of $\mathbb{R}^n$.

We define the Lebesque space with a variale exponent $p(.)$ by

$$L^{p(.)}(\Omega) := \left\{ v : \Omega \to \mathbb{R} : \text{ measurable in } \Omega,\ A_{p(.)}(\lambda v) < +\infty, \text{ for some } \lambda > 0 \right\},$$

where $A_{p(.)}(v) = \int_\Omega |v(x)|^{p(x)}\, dx$.

The set $L^{p(.)}(\Omega)$ equipped with the norm (Luxemburg's norm)

$$\|v\|_{p(.)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{v(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\},$$

$L^{p(.)}(\Omega)$ is a Banach space [13].

We next, define the variable-exponent Sobolev space $W^{1,p(.)}(\Omega)$ as follows:

$$W^{1,p(.)}(\Omega) := \left\{ v \in L^{p(.)}(\Omega) \text{ such that } \nabla v \text{ exists and } |\nabla v| \in L^{p(.)}(\Omega) \right\}.$$

This is a Banach space with respect to the norm $\|v\|_{W^{1,p(.)}(\Omega)} = \|v\|_{p(.)} + \|\nabla v\|_{p(.)}$.

Furthmore, we set $W^{1,p(.)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in the space $W_0^{1,p(.)}(\Omega)$. Let us note that the space $W^{1,p(.)}(\Omega)$ has a differenet definition in the case of variable exponents.

However, under condition (1.3), both definitions are equivalent $[13]$. The space $W^{-1,p'(.)}(\Omega)$, dual of $W_0^{1,p(.)}(\Omega)$, is defined in the same way as the classical Sobolev spaces, where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$.

Lemma 2.1. (*Poincaré's inequality*) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and suppose that $p(.)$ satisfies* (1.3), *then*

$$\|v\|_{p(.)} \le c \|\nabla v\|_{p(.)}, \quad \text{for all } v \in W_0^{1,p(.)}(\Omega).$$

Where $c>0$ is a constant which depends on p_1, p_2, and Ω only. In particular, $\|\nabla v\|_{p(.)}$ define an equivalent norm on $W_0^{1,p(.)}(\Omega)$.

Lemma 2.2. *If $p(.)\in C(\overline{\Omega})$ and $q:\Omega\to[1,\ +\infty)$ is a measurable function such that*

$$ess\inf_{x\in\Omega}(p_*(x)-q(x))>0 \text{ with } p_*(x)=\begin{cases}\frac{np(x)}{ess\sup\limits_{x\in\Omega}(n-p(x))} & \text{if } p_2<n\\ +\infty & \text{if } p_2\geq n.\end{cases}$$

Then the embedding $W_0^{1,p(.)}(\Omega)\hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact.

Lemma 2.3. *(Hölder's Inequality) Suppose that $p,\ q,\ s\geq 1$ are measurable functions defined on Ω such that*

$$\frac{1}{s(y)}=\frac{1}{p(y)}+\frac{1}{q(y)},\quad \text{for a.e.}\quad y\in\Omega.$$

If $u\in L^{p(.)}(\Omega)$ and $v\in L^{q(.)}(\Omega)$, then $uv\in L^{s(.)}(\Omega)$, with

$$\|uv\|_{s(.)}\leq 2\|u\|_{p(.)}\|v\|_{q(.)}.$$

Lemma 2.4. *If p a measurable function on Ω satisfying (1.2), then we have*

$$\min\left\{\|u\|_{p(.)}^{p_1},\ \|u\|_{p(.)}^{p_2}\right\}\leq A_{p(.)}(u)\leq\max\left\{\|u\|_{p(.)}^{p_1},\ \|u\|_{p(.)}^{p_2}\right\},$$

for any $u\in L^{p(.)}(\Omega)$.

3. Blow up

In this section, we prove that the solution of equation (1.1) blow up in finite time when $\omega=0$. we recall that (1.1), becomes

$$\begin{cases} u_t-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)=b|u|^{r(x)-2}u & \text{in } \Omega\times(0,T),\\ u(x,t)=0,\ \ x\in\partial\Omega,\ \ t\geq 0,\\ u(x,0)=u_0(x) & \text{in } \Omega.\end{cases} \tag{3.1}$$

We start with a local existence result for the problem (1.1), which is a direct result of the existence theorem by Agaki and Ôtani [2].

Proposition 3.1. *For all $u_0\in W_0^{1,p(.)}(\Omega)$, there exists a number $T_0\in(0,T]$ such that the problem (1.1) has a solution u on $[0,T_0]$ satisfying:*

$$u\in C_w([0,T_0];\ W_0^{1,p(.)}(\Omega))\cap C([0,T_0],\ L^{r(.)}(\Omega))\cap W^{1,2}(0,T_0;\ L^2(\Omega)).$$

We define the energy functional associaeted of the problem (1.1)

$$E(t)=\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx-b\int_\Omega\frac{1}{r(x)}|u|^{r(x)}dx. \tag{3.2}$$

Theorem 3.1. *Let the assumptions of proposition* 1, *be satisfied and assume that*

$$E(0) < 0. \tag{3.3}$$

Then the solution of the problem (3.1), *blow up in finite time.*

Now, we let

$$H(t) := -E(t), \tag{3.4}$$

and

$$L(t) = \frac{1}{2}\int_{\Omega} u^2 dx. \tag{3.5}$$

To prove our result, we first establesh some Lemmas.

Lemma 3.1. *Assume that* (1.2) *and* (1.3), *hold and* $E(0) < 0$. *Then*

$$A_{p(.)}(\nabla u) < \frac{bp_2}{r_1} A_{r(.)}(u), \tag{3.6}$$

and

$$\frac{r_1}{b} H(0) < A_{r(.)}(u). \tag{3.7}$$

Proof. We multiply the first equation of (3.1) by u_t and integratying over the domain Ω, we get

$$\frac{d}{dt}\left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - b\int_{\Omega} \frac{1}{r(x)} |u|^{r(x)} dx\right) = -\|u_t\|_2^2,$$

then

$$E^{'}(t) = -\|u_t\|_2^2 \leq 0. \tag{3.8}$$

Integrating (3.8) over $(0,\ t)$, we obtain

$$E(t) \leq E(0) < 0. \tag{3.9}$$

By (3.2) and (3.9), we have

$$\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx < b\int_{\Omega} \frac{1}{r(x)} |u|^{r(x)} dx,$$

so that

$$\int_{\Omega} \frac{1}{p_2} |\nabla u|^{p(x)} dx < \int_{\Omega} \frac{b}{r_1} |u|^{r(x)} dx.$$

On the other hand, we have

$$\begin{aligned} H(t) &= -\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + b\int_{\Omega} \frac{1}{r(x)} |u|^{r(x)} dx \\ &\leq b\int_{\Omega} \frac{1}{r(x)} |u|^{r(x)} dx. \end{aligned} \tag{3.10}$$

Then, by (3.10) , (3.4) and (3.9) , we obtain

$$0 < H(0) < H(t) < \frac{b}{r_1} A_{r(.)}(u).$$

□

Lemma 3.2. [16] *Assume that* (1.2), (1.3) *hold and* $E(0) < 0$. *Then the solution of* (3.1) , *satisfies for some* $c > 0$,

$$A_{r(.)}(u) \geq c \|u\|_{r_1}^{r_1}. \tag{3.11}$$

Proof of theorem 1. We have

$$\begin{aligned} L^{'}(t) &= \int_{\Omega} u u_t dx \\ &= \int_{\Omega} u \left(\operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u \right) + b |u|^{r(x)-2} u \right) dx \\ &= -A_{p(.)}(\nabla u) + b A_{r(.)}(u). \end{aligned} \tag{3.12}$$

Combining of (3.12) , (3.11) and (3.6) , leads to

$$L^{'}(t) \geq cb \left(1 - \frac{p_2}{r_1}\right) \|u\|_{r_1}^{r_1}. \tag{3.13}$$

Now, we estimate $L^{\frac{r_1}{2}}(t)$, by the embedding of $L^{r_1}(\Omega) \hookrightarrow L^2(\Omega)$, we get

$$L^{\frac{r_1}{2}}(t) \leq \left(\frac{1}{2} \|u\|_{r_1}^2\right)^{\frac{r_1}{2}} \leq c \|u\|_{r_1}^{r_1}. \tag{3.14}$$

By combining (3.14) and (3.13) , we obtain

$$L^{'}(t) \geq \xi L^{\frac{r_1}{2}}(t). \tag{3.15}$$

A direct integration of (3.15) , then yields

$$L^{\frac{r_1}{2}-1}(t) \geq \frac{1}{L^{1-\frac{r_1}{2}}(0) - \xi t}.$$

Therefore, L blow up in a time $t^* \leq \frac{1}{L^{\frac{r_1}{2}-1}(0)}$. □

4. Exponential growth

In this section, we prove that the solution of equation (1.1) exponential growth when $\omega > 0$.

Lemma 4.1. *Suppose that* (1.2) *holds and* $E(0) < 0$. *Then,*

$$\int_{\Omega} |u|^{m(x)} dx \leq c \left(\|u\|_{r_1}^{r_1} + H(t) \right). \tag{4.1}$$

Proof.

$$\int_{\Omega} |u|^{m(x)}\, dx = \int_{\Omega_-} |u|^{m(x)}\, dx + \int_{\Omega_+} |u|^{m(x)}\, dx,$$

where

$$\Omega_+ = \{x \in \Omega \ / \ |u(x,\ t)| \geq 1\} \text{ and } \Omega_- = \{x \in \Omega \ / \ |u(x,\ t)| < 1\}.$$

So, we get

$$\begin{aligned} \int_{\Omega} |u|^{m(x)}\, dx &\leq c\left[\left(\int_{\Omega_-} |u|^{r_1}\, dx\right)^{\frac{m_1}{r_1}} + \left(\int_{\Omega+} |u|^{r_1}\, dx\right)^{\frac{m_2}{r_1}}\right] \\ &\leq c\left(\|u\|_{r_1}^{m_1} + \|u\|_{r_1}^{m_2}\right). \end{aligned}$$

Exploiting the algebric inequality

$$z^v \leq (z+1) \leq \left(1 + \frac{1}{a}\right)(z+a),\ \forall z > 0,\ 0 < v \leq 1,\ a \geq 0,$$

we have

$$\begin{aligned} \|u\|_{r_1}^{m_1} &\leq c\left(\|u\|_{r_1}^{r_1}\right)^{\frac{m_1}{r_1}} \leq c\left(1 + \frac{1}{H(0)}\right)\left(\|u\|_{r_1}^{r_1} + H(0)\right) \\ &\leq c\left(\|u\|_{r_1}^{r_1} + H(t)\right). \end{aligned}$$

Similarly,

$$\begin{aligned} \|u\|_{r_1}^{m_2} &\leq c\left(\|u\|_{r_1}^{r_1}\right)^{\frac{m_2}{r_1}} \leq c\left(1 + \frac{1}{H(0)}\right)\left(\|u\|_{r_1}^{r_1} + H(0)\right) \\ &\leq c\left(\|u\|_{r_1}^{r_1} + H(t)\right). \end{aligned}$$

This gives

$$\int_{\Omega} |u|^{m(x)}\, dx \leq c\left(\|u\|_{r_1}^{r_1} + H(t)\right).$$

□

Theorem 4.1. *Let the assumptions of proposition* 1, *be satisfied and assume that* (3.3) *holds. Then the solution of the problem* (1.1), *grows exponentially.*

Proof. By the same procedure of the proof the Lemma 5, we get

$$E'(t) = -\|u_t\|_2^2 - \omega \int_{\Omega} |u|^{m(x)-2}\, u_t^2 \leq 0, \tag{4.2}$$

then, we have

$$H'(t) = \|u_t\|_2^2 + \omega \int_{\Omega} |u|^{m(x)-2}\, u_t^2 dx \geq 0. \tag{4.3}$$

We define

$$G(t) = H(t) + \epsilon L(t). \tag{4.4}$$

for ϵ small to be chosen later.

The time derivative of (4.4), we obtain

$$G^{'}(t) = H^{'}(t) + \epsilon \int_{\Omega} u u_t dx.$$

By using (1.1), we get

$$G^{'}(t) = H^{'}(t) - \epsilon A_{p(.)}(\nabla u) + \epsilon b A_{r(.)}(u) - \epsilon \omega \int_{\Omega} |u|^{m(x)-2} u_t u dx. \tag{4.5}$$

To estimate the last term in the right hand side of (4.5), by using the following Young's Inequality

$$XY \leq \delta X^2 + \delta^{-1} Y^2, \qquad X,\ Y \geq 0,\ \delta > 0.$$

$$\begin{aligned} \int_{\Omega} |u|^{m(x)-2} u_t u dx &= \int_{\Omega} |u|^{\frac{m(x)-2}{2}} u_t |u|^{\frac{m(x)-2}{2}} u dx \\ &\leq \delta \int_{\Omega} |u|^{m(x)-2} u_t^2 dx + \delta^{-1} \int_{\Omega} |u|^{m(x)} dx. \end{aligned}$$

We conclude

$$\begin{aligned} G^{'}(t) \geq\ & (1 - \epsilon\delta) \int_{\Omega} |u|^{m(x)-2} u_t^2 dx + \|u_t\|_2^2 - \epsilon A_{p(.)}(\nabla u) \\ & + \epsilon b A_{r(.)}(u) - \epsilon \omega \delta^{-1} \int_{\Omega} |u|^{m(x)} dx. \end{aligned} \tag{4.6}$$

Then

$$\begin{aligned} G^{'}(t) \geq\ & (1 - \epsilon\delta) \int_{\Omega} |u|^{m(x)-2} u_t^2 dx + \|u_t\|_2^2 - \epsilon \omega \delta^{-1} \int_{\Omega} |u|^{m(x)} dx \\ & + \epsilon (1-\mu) r_1 H(t) + \epsilon b \mu A_{r(.)}(u) + \epsilon \left((1-\mu) \frac{r_1}{p_2} - 1 \right) A_{p(.)}(\nabla u), \end{aligned}$$

where μ is a constant such that $0 < \mu \leq 1 - \frac{p_2}{r_1}$.

Also, by using (3.6), we obtain

$$\begin{aligned} G^{'}(t) \geq\ & (1 - \epsilon\delta) \int_{\Omega} |u|^{m(x)-2} u_t^2 dx + \|u_t\|_2^2 - \epsilon \omega \delta^{-1} \int_{\Omega} |u|^{m(x)} dx \\ & + \epsilon (1-\mu) r_1 H(t) + \epsilon \left(b\mu + 1 - \mu - \frac{p_2}{r_1} \right) A_{r(.)}(u). \end{aligned} \tag{4.7}$$

Then, by *Lemma* 7 and (3.11), (4.7) becomes

$$\begin{aligned} G^{'}(t) &\geq (1-\epsilon\delta)\int_{\Omega}|u|^{m(x)-2}u_t^2dx+\|u_t\|_2^2-\epsilon c\,\omega\delta^{-1}\left(\|u\|_{r_1}^{r_1}+H(t)\right) \\ &\quad+\epsilon(1-\mu)r_1H(t)+\epsilon c\left(b\mu+1-\mu-\frac{p_2}{r_1}\right)\|u\|_{r_1}^{r_1}. \end{aligned} \tag{4.8}$$

So that

$$\begin{aligned} G^{'}(t) &\geq (1-\epsilon\delta)\int_{\Omega}|u|^{m(x)-2}u_t^2dx+\|u_t\|_2^2+\epsilon\left((1-\mu)r_1-c\,\omega\delta^{-1}\right)H(t) \\ &\quad+\epsilon\left(c\left(b\mu+1-\mu-\frac{p_2}{r_1}\right)-c\,\omega\delta^{-1}\right)\|u\|_{p_1}^{p_1}. \end{aligned} \tag{4.9}$$

So, we chosen δ large sufficient and ϵ small enough for that we can find $\lambda_1,\ \lambda_2>0$, such that

$$G^{'}(t)\geq\lambda_1H(t)+\lambda_2\|u\|_{r_1}^{r_1}\geq K_1\left(H(t)+\|u\|_{r_1}^{r_1}\right), \tag{4.10}$$

and

$$G(0)=H(0)+\epsilon L(0)>0.$$

Similarly in (4.7), we have

$$\|u\|_2^2\leq c\left(H(t)+\|u\|_{r_1}^{r_1}\right). \tag{4.11}$$

On the other hand, by (4.11), we get

$$G(t)\leq K_2\left(H(t)+\|u\|_{r_1}^{r_1}\right). \tag{4.12}$$

Combining with (4.12) and (4.10), we arrive at

$$G^{'}(t)\geq\eta G(t). \tag{4.13}$$

Finally, a simple integration of (4.13) gives

$$G(t)\geq G(0)e^{\eta t},\qquad\forall t\geq 0. \tag{4.14}$$

Thus completes the proof. □

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