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GENERALIZATION OF BATEMAN POLYNOMIALS

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Abstract. In this paper, generalize the Bateman polynomials in terms of generalized hypergeometric function of the type ${}_pF_p$. Establish different forms of extended polynomials such as series expansion, generating functions and recurrence relations.

1. Introduction

Bateman polynomials are the family of F_n orthogonal polynomials. Many of researchers generalized the classical results on the Bateman polynomials. A large dedicated literature, numbers of relevant properties, extensions, generalizations and applications of Bateman polynomials are available in [1], [2], [4], [7], [10] and [11]. The Bateman polynomials $f_n(x)$ generated by

$$\sum_{n=0}^{\infty} f_n(x)t^n = (1-t)^{-1}\psi\left(\frac{-4xt}{(1-t)^2}\right), \tag{1.1}$$

have the following classical properties.

$$f_n(x) = \sum_{k=0}^{\infty} \frac{(-n)_k(1+n)_k\gamma_k x^k}{(\frac{1}{2})_k(1)_k}, \tag{1.2}$$

$$f_n(x) = {}_2F_2(-n, 1+n; 1, 1; x), \tag{1.3}$$

$$xf_n^{'}(x) - nf_n(x) = -nf_{n-1}(x) - xf_{n-1}^{'}(x), \ n \geq 1, \tag{1.4}$$

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$$xf_n^{'}(x) - nf_n(x) = -\sum_{k=0}^{n-1} f_k(x) - 2x\sum_{k=0}^{n-1} f_k^{'}(x),\ n \geq 1, \tag{1.5}$$

$$xf_n^{'}(x) - nf_n(x) = \sum_{k=0}^{n-1}(-1)^{n-k}(1+2k)f_k(x),\ n \geq 1. \tag{1.6}$$

2. Main results

In this section we determine generalized properties of classical Bateman polynomials, series expansion, generating function and recurrence relations. For this let $\psi(u)$ have a formal power- series expansion

$$\psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n, \tag{2.1}$$

Define a polynomials $f_n^{(\alpha)}(x)$ by

$$\sum_{n=0}^{\infty} f_n^{(\alpha)}(x)^n = (1-t)^{-1-\alpha}\psi\left(\frac{-p^p xt}{q^q(1-t)^p}\right). \tag{2.2}$$

where $p \geq 2$, $q = p-1$ and α is any non-negative real parameter.

Theorem 2.1. *If n is non-negative integer then,*

$$f_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\sum_{k=0}^{n} \frac{(-n)_k(\frac{1+\alpha+n}{q})_k(\frac{2+\alpha+n}{q})_k....(\frac{q+\alpha+n}{q})_k x^k\gamma_k}{(\frac{1+\alpha}{p})_k(\frac{2+\alpha}{p})_k....(\frac{p+\alpha}{p})_k}. \tag{2.3}$$

Proof: From (2.1) and (2.2)

$$\sum_{n=0}^{\infty} f_n^{(\alpha)}(x)t^n = \sum_{k=0}^{\infty} \frac{(-1)^k(p)^{pk}x^k t^k\gamma_k}{(q)^{qk}(1-t)^{1+\alpha+pk}},$$

By using (1), pp 58 of [1]

$$\sum_{n=0}^{\infty} f_n^{(\alpha)}(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k(p)^{pk}(1+\alpha)_{n+pk}\gamma_k x^k t^{n+k}}{(q)^{qk}n!(1+\alpha)_{pk}},$$

By using Lemma 11, pp 57 of [1]

$$\sum_{n=0}^{\infty} f_n^{(\alpha)}(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k(p)^{pk}(x)^k(1+\alpha)_{n+qk}\gamma_k t^n}{(q)^{qk}(n-k)!(1+\alpha)_{pk}},$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(1+\alpha)_n}{n!} \frac{(-1)^k n! (p)^{pk} (x)^k (1+\alpha+n)_{qk} \gamma_k t^n}{(q)^{qk} (n-k)! (1+\alpha)_{pk}},$$

equating the coefficients of t^n, we obtain (2.3).

Theorem 2.2. *If $n \geq 1$, then*

$$f_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} {}_pF_p \left(-n, \frac{1+\alpha+n}{q}, \frac{2+\alpha+n}{q} ... \frac{q+\alpha+n}{q}; \underbrace{1, 1...1}_{p-times}; x \right). \tag{2.4}$$

Proof: If we choose

$$\gamma_k = \frac{(\frac{1+\alpha}{p})_k (\frac{2+\alpha}{p})_k .... (\frac{p+\alpha}{p})_k}{(k!)^{p+1}}.$$

in (2.3) then our yield is (2.4).

Theorem 2.3. *If n is non-negative integer then,*

$$x f_n^{'(\alpha)}(x) - n f_n^{(\alpha)}(x) = -(\alpha+n) f_{n-1}^{(\alpha)}(x) - qx f_{n-1}^{'(\alpha)}(x). \tag{2.5}$$

Proof:

In order to derive (2.5), consider

$$F = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) t^n = (1-t)^{-1-\alpha} \psi(v).$$

$$where, \ v = \frac{-p^p xt}{q^q (1-t)^p}.$$

Differentiate with respect to x

$$F_x = \sum_{n=0}^{\infty} f_n^{'(\alpha)}(x) t^n = (1-t)^{-1-\alpha} \psi^{'}(v) \frac{-p^p t}{q^q (1-t)^p},$$

Differentiate with respect to t

$$F_t = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) n t^{n-1} = (1+\alpha)(1-t)^{-2-\alpha}\psi(v) - (1-t)^{-1-\alpha}\psi^{'}(v)\frac{\partial v}{\partial t},$$

$$where,\ \frac{\partial v}{\partial t} = \frac{-p^p x(1+qt)}{q^q(1-t)^{p+1}}.$$

$$F_t = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) n t^{n-1} = (1+\alpha)(1-t)^{-2-\alpha}\psi(v) - x\frac{p^p(1-t)^{-2-\alpha-p}(1+qt)}{q^q}\psi^{'}(v),$$

Therefore F satisfies the partial differential equation

$$x(1+qt)F_x - t(1-t)F_t + (1+\alpha)tF = 0.$$

$$x(1+qt)\sum_{n=0}^{\infty} f_n^{'(\alpha)}(x)t^n - t(1-t\sum_{n=0}^{\infty} f_n^{(\alpha)}(x)nt^{n-1}) + (1+\alpha)t\sum_{n=0}^{\infty} f_n^{(\alpha)}(x)t^n = 0,$$

$$\sum_{n=0}^{\infty}[xf_n^{'(\alpha)}(x) - nf_n^{(\alpha)}(x)]t^n = -\sum_{n=0}^{\infty}(1+\alpha+n)f_n^{(\alpha)}(x)t^{n+1} - qx\sum_{n=0}^{\infty} f_n^{'(\alpha)}(x)t^{n+1},$$

$$= -\sum_{n=1}^{\infty}(\alpha+n)f_{n-1}^{(\alpha)}(x)t^n - qx\sum_{n=1}^{\infty} f_{n-1}^{'(\alpha)}(x)t^n,$$

which leads to (2.5).

Theorem 2.4. *If n is non-negative integer then,*

$$xf_n^{'(\alpha)}(x) - nf_n^{(\alpha)}(x) = -(1+\alpha)\sum_{k=0}^{n-1} f_k^{(\alpha)}(x) - px\sum_{k=0}^{n-1} f_k^{'(\alpha)}(x). \tag{2.6}$$

Proof:

F also satisfies the partial differential equation

$$xF_x - xtF_x + pxtF_x - tF_t + t^2F_t + (1+\alpha)tF = 0.$$

$$xF_x - tF_t = -\frac{(1+\alpha)t}{1-t}F - \frac{pxt}{1-t}F_x.$$

$$x\sum_{n=0}^{\infty} f_n^{'(\alpha)}(x)t^n - t\sum_{n=0}^{\infty} f_n^{(\alpha)}(x)nt^{n-1} = -(1+\alpha)\sum_{n=0}^{\infty} t^{n+1}\sum_{k=0}^{\infty} f_k^{(\alpha)}(x)t^k - px\sum_{n=0}^{\infty} t^{n+1}\sum_{k=0}^{\infty} f_k^{'(\alpha)}(x)t^k,$$

$$\sum_{n=0}^{\infty}[xf_n^{'(\alpha)}(x) - nf_n^{(\alpha)}(x)]t^n = -(1+\alpha)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f_k^{(\alpha)}(x)t^{n+k+1} - px\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f_k^{'(\alpha)}(x)t^{n+k+1}$$

$$= -\sum_{n=1}^{\infty}[(1+\alpha)\sum_{k=0}^{n-1} f_k^{(\alpha)}(x) - px\sum_{k=0}^{n-1} f_k^{'(\alpha)}(x)]t^n,$$

which leads to (2.6).

Theorem 2.5. *If n is non-negative integer then,*

$$xf_n^{'(\alpha)}(x) - nf_n^{(\alpha)}(x) = \sum_{k=0}^{n-1}(-q)^{n-k}(1+\alpha+pk)f_k^{(\alpha)}(x). \tag{2.7}$$

Proof:

F satisfies the partial differential equation

$$xF_x + qxtF_x - tF_t - qt^2F_t + pt^2F_t + (1+\alpha)tF = 0.$$

$$xF_x - tF_t = -\frac{(1+\alpha)t}{1+qt}F - \frac{pt^2}{1+qt}F_t,$$

$$\sum_{n=0}^{\infty}[xf_n^{'(\alpha)}(x) - nf_n^{(\alpha)}(x)]t^n = -(1+\alpha)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-q)^n f_k^{(\alpha)}(x)t^{n+k+1} - p\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-q)^n f_k^{(\alpha)}(x)kt^{n+k+1},$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (-q)^{n-k} (1+\alpha+pk) f_k^{(\alpha)}(x) t^n,$$

which gives (2.7).

For $\alpha = 0$ and $p = 2$ the equations (2.2) to (2.7) reduces to (1.1) to (1.6).

Theorem 2.6. *If $n \geq 1$, then the polynomials $f_n^{(\alpha)}(x)$ also satisfying the following property*

$$\sum_{n=0}^{\infty} f_n^{(\alpha)}(x) t^n = (1-t)^{-1-\alpha} {}_pF_p \left(\frac{1+\alpha+n}{p}, \frac{2+\alpha+n}{p} \ldots \frac{q+\alpha+n}{p}; \underbrace{1, 1 \ldots 1}_{p-times}; \frac{-p^p x t}{q^q} \right). \tag{2.8}$$

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