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CERTAIN SUBFAMILY OF HARMONIC FUNCTIONS RELATED TO SĂLĂGEAN q-DIFFERENTIAL OPERATOR

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ABSTRACT. The theory of q–calculus operators are applied in many areas of sciences such as complex analysis. In this paper we apply Sălăgean q–differential operator to harmonic functions and introduce sharp coefficient bounds, extreme points, distortion inequalities and convexity results.

1. INTRODUCTION

We state some notations regarding to q–calculus used in this article, see [1, 4] and [6]. For $0 < q < 1$ and positive integer n, the q–integer number is denoted by $[n]_q$ and introduced by:

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}. \tag{1.1}$$

We can easily conclude that:

$$\lim_{q \to 1^-} [n]_q = n.$$

If $f(z)$ be analytic in this open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$, then the q–difference operator of q–calculus operated on f given by:

$$\partial_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \tag{1.2}$$

where $\lim_{q \to 1^-} \partial_q f(z) = f'(z)$, for example see [2, 5] and [8].

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For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, the Sălăgean q-differential operator is defined by:

$$\begin{aligned} \mathcal{S}_q^0 f(z) &= f(z) \\ \mathcal{S}_q^1 f(z) &= z\partial_q f(z) = \frac{f(z) - f(qz)}{(1-q)} \\ &\vdots \\ \mathcal{S}_q^m f(z) &= z\partial_q\left(\mathcal{S}_q^{m-1} f(z)\right) = f(z) * \left(z + \textstyle\sum_{k=2}^{\infty} [k]_q^m z^k\right) \\ &= z + \sum_{k=2}^{\infty} [k]_q^m a_k z^k, \end{aligned} \tag{1.3}$$

where m is a positive integer and "$*$" is the familiar Hadamard product or convolution of two analytic functions.

Since

$$\lim_{q \to 1^-} \mathcal{S}_q^m(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k,$$

is the famous Sălăgean operator [9], so the operator $\mathcal{S}_q^m$ is called Sălăgean q-differential operator.

Let $\mathcal{S}_h$ denote the class of functions:

$$f = h + \overline{g} \tag{1.4}$$

which are harmonic, univalent and sense-preserving in $\mathbb{U}$ and normalized by $f(0) = f'(0) - 1 = 0$, where h and g are analytic in $\mathbb{U}$ take the form:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad (0 \leqslant b_1 < 1). \tag{1.5}$$

Also, we call h and g analytic part and co-analytic part of f respectively, see [3].

Hence $f \in \mathcal{S}_h$ is of the type:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k} \tag{1.6}$$

Now, we consider the Sălăgean q-differential operator of harmonic functions $f = h + \overline{g}$, by:

$$\mathcal{S}_q^m f(z) = \mathcal{S}_q^m h(z) + (-1)^m \overline{\mathcal{S}_q^m g(z)}, \tag{1.7}$$

where $\mathcal{S}_q^m$ is defined by (1.3) and h and g are of the type (1.5). For more details see [7].

We denote by $\mathcal{S}_h^*$ the family of functions of the type (1.4) where:

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \quad , \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1. \tag{1.8}$$

For $A \geqslant 0$, $0 \leqslant B, C \leqslant 1$, $0 \leqslant D < 1$ and $\gamma \in \mathbb{R}$ let $\mathcal{S}_{h(\gamma)}^*(A, B, C, D)$ denote the class of functions in $\mathcal{S}_h^*$ of the type (1.5) such that:

$$\operatorname{Re}\left\{(1-A)(1-B)\frac{\mathcal{S}_q^0 f(z)}{z} + (A+B)\frac{\left(\mathcal{S}_q^m f(z)\right)'}{z'} - Ce^{i\gamma}\frac{\left(\mathcal{S}_q^m f(z)\right)''}{z''} + \left(Ce^{i\gamma} - AB\right)\right\} \geqslant D, \tag{1.9}$$

where

$$\begin{aligned} z' &= \frac{\partial}{\partial\theta}(z) = iz, \\ \big(\mathcal{S}_q^m f(z)\big)' &= \frac{\partial}{\partial\theta}\big(\mathcal{S}_q^m f(re^{i\theta})\big) = iz\big(\mathcal{S}_q^m h\big)' - \overline{iz\big(\mathcal{S}_q^m g\big)'} \end{aligned} \tag{1.10}$$

$$\begin{aligned} z'' &= \frac{\partial^2}{\partial\theta^2}(z) = -z, \\ \big(\mathcal{S}_q^m f(z)\big)'' &= \frac{\partial^2}{\partial\theta^2}\big(\mathcal{S}_q^m f(re^{i\theta})\big) = -z\big(\mathcal{S}_q^m h\big)' - z^2\big(\mathcal{S}_q^m h\big)'' - z\big(\mathcal{S}_q^m g\big)' - \overline{z^2\big(\mathcal{S}_q^m g\big)''}. \end{aligned} \tag{1.11}$$

We further denote by $\mathcal{S}^*_{h(\gamma)}(A,B,C,D)$ the subclass of $\mathcal{S}_{h(\gamma)}(A,B,C,D)$ consisting of harmonic functions $f = h + \overline{g}$ so that h and g are of the form (1.8) and satisfying (1.9).

2. Main results

In our first theorem, we introduce a sufficient coefficient condition for functions in $\mathcal{S}_{h(\gamma)}(A,B,C,D)$ and then we show that this condition is also necessary for $f(z) \in \mathcal{S}^*_{h(\gamma)}(A,B,C,D)$.

Theorem 2.1. *Suppose $f = h + \overline{g}$, where h and g be given by (1.5) and:*

$$\begin{aligned} &\sum_{k=2}^{\infty} \left|(A+B)k - (1-A-B+AB) - Ck^2\right| [k]_q^m |a_k| + \\ &\sum_{k=1}^{\infty} \left|(A+B)k - (1-A-B+AB) - Ck^2\right| [k]_q^m |b_k| \leqslant 1 - D. \end{aligned} \tag{2.1}$$

Then $f(z) \in \mathcal{S}_{h(\gamma)}(A,B,C,D)$.

Proof. In view of the fact that:

$$\text{“}\operatorname{Re}\{W\} \geqslant 0 \iff |W + 1 - D| \geqslant |w - 1 - D|,\text{”}$$

and letting:

$$W = (1-A)(1-B)\frac{\mathcal{S}_q^m f(z)}{z} + (A+B)\frac{\big(\mathcal{S}_q^m f(z)\big)'}{z'} - Ce^{i\gamma}\frac{\big(\mathcal{S}_q^m f(z)\big)''}{z''} + \big(Ce^{i\theta} - AB\big),$$

it is enough to show that:

$$|W + 1 - D| - |W - 1 - D| \geqslant 0.$$

But by using (1.10) and (1.11) we have:

$$\begin{aligned} |W + 1 - D| = \Bigg|&(1-A)(1-B)\Big(1 + \sum_{k=2}^{\infty} a_k [k]_q^m z^{k-1} + \sum_{k=1}^{\infty} b_k [k]_q^m (\overline{z})^{k-1}\Big) \\ &+ (A+B)\Big(1 + \sum_{k=2}^{\infty} k a_k [k]_q^m z^{k-1} - \sum_{k=1}^{\infty} k b_k [k]_q^m (\overline{z})^{k-1}\Big) \\ &- Ce^{i\gamma}\Bigg(1 + \sum_{k=2}^{\infty} k a_k [k]_q^m z^{k-1} + \sum_{k=2}^{\infty} k(k-1) a_k [k]_q^m z^{k-1} \\ &+ \sum_{k=1}^{\infty} k b_k [k]_q^m (\overline{z})^{k-1} + \sum_{k=1}^{\infty} k(k-1) b_k [k]_q^m (\overline{z})^{k-1}\Bigg) \end{aligned}$$

$$+Ce^{i\gamma}-AB+1-D\Bigg|$$

$$\leqslant 2-D-\sum_{k=1}^{\infty}\left|1+(A+B)(k-1)+AB-Ck^2[k]_q^m\right||a_k|\left|\frac{z^k}{z}\right|$$

$$-\sum_{k=1}^{\infty}\left|1-(A+B)(k-1)+AB-Ck^2\right|[k]_q^m|b_k|\left|\frac{z^k}{z}\right|,$$

and

$$|W-1-D|\leqslant D+\sum_{k=2}^{\infty}\left|(A+B)(k-1)+AB-Ck^2\right|[k]_q^m|a_k|\left|\frac{z^k}{z}\right|$$

$$+\sum_{k=1}^{\infty}\left|1-(A+B)(k-1)+AB-Ck^2\right|[k]_q^m|b_k|\left|\frac{z^k}{z}\right|.$$

So by using (2.1), we get:

$$|W+1-D|-|W-1-D|\geqslant$$

$$2\Bigg[1-D-\sum_{k=2}^{\infty}\left|(A+B)k-(1-A-B+AB)-Ck^2\right|[k]_q^m|a_k|-$$

$$\sum_{k=1}^{\infty}\left|(A+B)k-(1-A-B+AB)-Ck^2\right|[k]_q^m|b_k|\Bigg]\geqslant 0.$$

□

Remark 2.1. *The coefficient bound* (2.1) *is sharpt for the function:*

$$F(z)=z+\sum_{k=2}^{\infty}\frac{x_k}{\left|(A+B)k-(1-A-B+AB)-Ck^2\right|[k]_q^m}z^k$$

$$+\sum_{k=1}^{\infty}\frac{\overline{y_k}}{\left|(A+B)k-(1-A-B+AB)-Ck^2\right|[k]_q^m}(\overline{z})^k,$$

where

$$\frac{1}{1-D}\left(\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|\right)=1.$$

Theorem 2.2. *Let $f=h+\overline{g}\in\mathcal{S}_h^*$. Then $f(z)\in\mathcal{S}_{h(\gamma)}^*(A,B,C,D)$ if and only if:*

$$\sum_{k=2}^{\infty}\left|(A+B)k-(1-A-B+AB)-Ck^2\right|[k]_q^m|a_k|$$
$$+\sum_{k=1}^{\infty}\left|(A+B)k-(1-A-B+AB)-Ck^2\right|[k]_q^m|b_k|\leqslant 1-D. \tag{2.2}$$

Proof. From Theorem 2.1, and since $\mathcal{S}_{h(\gamma)}^*(A,B,C,D)\subset\mathcal{S}_{h(\gamma)}(A,B,C,D)$, we conclude the "if" part. For the "only if" part, suppose that $f(z)\in\mathcal{S}_{h(\gamma)}^*(A,B,C,D)$. Thus for $z=re^{i\theta}\in\mathbb{U}$, we have:

$$\operatorname{Re}\left\{(1-A)(1-B)\frac{\mathcal{S}_q^m f(z)}{z}+(A+B)\frac{\left(\mathcal{S}_q^m f(z)\right)'}{z'}-Ce^{i\gamma}\frac{\left(\mathcal{S}_q^m f(z)\right)''}{z''}+Ce^{i\gamma}+Ce^{i\gamma}-AB\right\}$$

$$=\operatorname{Re}\left\{(1-A)(1-B)\left(1+\sum_{k=2}^{\infty}a_k[k]_q^m z^{k-1}+\sum_{k=1}^{\infty}b_k[k]_q^m(\overline{z})^{k-1}\right)\right.$$

$$+(A+B)\Big(1+\sum_{k=2}^{\infty}ka_k[k]_q^m z^{k-1}-\sum_{k=1}^{\infty}kb_k[k]_q^m(\overline{z})^{k-1}\Big)$$
$$-Ce^{i\gamma}\Big(1+\sum_{k=2}^{\infty}ka_k[k]_q^m z^{k-1}+\sum_{k=2}^{\infty}k(k-1)a_k[k]_q^m z^{k-1}$$
$$+\sum_{k=1}^{\infty}kb_k[k]_q^m(\overline{z})^{k-1}+\sum_{k=1}^{\infty}k(k-1)b_k[k]_q^m(\overline{z})^{k-1}\Big)+Ce^{i\gamma}-AB\Big\}$$
$$\geqslant 1-\sum_{k=2}^{\infty}\left|(A+B)k-(1-A-B+AB)-Ck^2\right|[k]_q^m|a_k|r^{k-1}$$
$$-\sum_{k=1}^{\infty}\left|(A+B)k-(1-A-B+AB)-Ck^2\right|[k]_q^m|b_k|r^{k-1}\geqslant D.$$

The above inequality holds for all $z=re^{i\theta}\in\mathbb{U}$. So if $z=r\to 1$, we obtain the required result (2.2). Now the proof is complete. □

3. Geometric properties of $\mathcal{S}^*_{h(\gamma)}(A,B,C,D)$

In this section, we first introduce extreme points of $\mathcal{S}^*_{h(\gamma)}(A,B,C,D)$ and then we obtain the distortion bounds for $f\in\mathcal{S}^*_{h(\gamma)}(A,B,C,D)$. Finally we show that the class $\mathcal{S}^*_{h(\gamma)}(A,B,C,D)$ is a convex set.

Theorem 3.1. *$f=h+\overline{g}\in\mathcal{S}^*_{h(\gamma)}(A,B,C,D)$ if and only if it can be expressed:*

$$f(z)=X_1z+\sum_{k=2}^{\infty}X_kh_k(z)+\sum_{k=1}^{\infty}Y_kg_k(z),\qquad (z\in\mathbb{U}),\tag{3.1}$$

where

$$h_k(z)=z-\frac{1-D}{|(A+B)k-(1-A-B+AB)-Ck^2|\,[k]_q^m}z^k,\quad (k=2,3,\ldots),\tag{3.2}$$
$$g_k(z)=\frac{1-D}{|(A+B)k-(1-A-B+AB)-Ck^2|\,[k]_q^m}(\overline{z})^k,\quad (k=1,2,\ldots),\tag{3.3}$$

$X_1\geqslant 0$, $Y_1\geqslant 0$, $X_1+\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1$, $X_k\geqslant 0$ and $Y_k\geqslant 0$ for $k=2,3,\ldots$.

Proof. If f is given by (3.1), then:

$$f(z)=z-\sum_{k=2}^{\infty}\frac{1-D}{|(A+B)k-(1-A-B+AB)-Ck^2|\,[k]_q^m}X_kz^k$$
$$+\sum_{k=1}^{\infty}\frac{1-D}{|(A+B)k-(1-A-B+AB)-Ck^2|\,[k]_q^m}Y_k(\overline{z})^k.$$

Since by (2.2), we have:

$$\sum_{k=2}^{\infty}\left|(A+B)k-(1-A-B+AB)-Ck^2\right|[k]_q^m$$
$$\times\Big(\frac{1-D}{|(A+B)k-(1-A-B+AB)-Ck^2|\,[k]_q^m}\Big)|X_k|$$
$$+\sum_{k=1}^{\infty}\left|(A+B)k-(1-A-B+AB)-Ck^2\right|[k]_q^m\times$$
$$\times\Big(\frac{1-D}{|(A+B)k-(1-A-B+AB)-Ck^2|\,[k]_q^m}\Big)|Y_k|$$

$$= (1-D)\Big(\sum_{k=2}^{\infty}|X_k| + \sum_{k=1}^{\infty}|Y_k|\Big) = (1-D)(1-X_1) \leqslant 1-D.$$

So $f(z) \in \mathcal{S}^*_{h(\gamma)}(A,B,C,D)$.

Conversely, suppose $f(z) \in \mathcal{S}^*_{h(\gamma)}(A,B,C,D)$. By putting:

$$X_1 = 1 - \Big(\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k\Big),$$

where

$$X_k = \frac{\big|(A+B)k - (1-A-B+AB) - Ck^2\big|\,[k]_q^m}{1-D}|a_k|,$$
$$Y_k = \frac{\big|(A+B)k - (1-A-B+AB) - Ck^2\big|\,[k]_q^m}{1-D}|b_k|,$$

we conclude the required representation (3.1), so the proof is complete. □

Theorem 3.2. *If $f(z) \in \mathcal{S}^*_{h(\gamma)}(A,B,C,D)$, $|z| = r < 1$, then:*

$$|f(z)| \geqslant (1-|b_1|)r - \frac{1}{[2]_q^m}\Big(\frac{1-D}{(A+B)+(1+AB)-4C} - \frac{2(A+B)-(1+AB)-C}{(A+B)+(1+AB)-4C}|b_1|\Big)r^2, \tag{3.4}$$

and

$$|f(z)| \leqslant (1-|b_1|)r + \frac{1}{[2]_q^m}\Big(\frac{1-D}{(A+B)+(1+AB)-4C} - \frac{2(A+B)-(1+AB)-C}{(A+B)+(1+AB)-4C}|b_1|\Big)r^2. \tag{3.5}$$

Proof. Suppose $f(z) \in \mathcal{S}^*_{h(\gamma)}(A,B,C,D)$, then by (2.2), we have:

$$\begin{aligned}
|f(z)| &= \Big|z - \sum_{k=2}^{\infty}|a_k|z^k + \sum_{k=1}^{\infty}|b_k|(\overline{z})^k\Big| \\
&= \Big|z + |b_1|(\overline{z}) - \sum_{k=2}^{\infty}\big(|a_k|z^k - |b_k|(\overline{z})^k\big)\Big| \\
&\geqslant r - |b_1|r - \frac{1-D}{(A+B)+(1+AB)-4C}\Bigg[\sum_{k=2}^{\infty}\Big(\frac{(A+B)+(1+AB)-4C}{1-D}|a_k| \\
&+ \frac{(A+B)+(1+AB)-4C}{1-D}|b_k|\Big)r^k\Bigg] \geqslant (1-|b_1|)r \\
&- \frac{1-D}{(A+B)+(1+AB)-4C}\Bigg[\sum_{k=2}^{\infty}\Big(\frac{(A+B)(k-1)+(1+AB)-4C}{1-D}|a_k| \\
&+ \frac{(A+B)(k-1)-(1+AB)-4C}{1-D}|b_k|\Big)r^k\Bigg] \geqslant (1-|b_1|)r \\
&- \frac{1-D}{(A+B)+(1+AB)-4C}\Big(1 - \frac{2(A+B)-(1+AB)-C}{1-D}|b_1|\Big)r^2 \\
&= (1-|b_1|)r - \frac{1}{[2]_q^m}\Big(\frac{1-D}{(A+B)+(1+AB)-4C} - \frac{2(A+B)-(1+AB)-C}{(A+B)+(1+AB)-4C}|b_1|\Big)r^2.
\end{aligned}$$

Relation (3.5) can be proved by using the similar statements. So the proof is complete. □

Theorem 3.3. *If $f_j(z)$, $j = 1, 2, \ldots$, belongs to $\mathcal{S}^*_{h(\gamma)}(A, B, C, D)$, then the function $F(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z)$ is also in $\mathcal{S}^*_{h(\gamma)}(A, B, C, D)$, where $f_j(z)$ defined by:*

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k + \sum_{k=1}^{\infty} b_{k,j} (\overline{z})^k, \quad \left(j = 1, 2, \ldots, \quad \textstyle\sum_{j=1}^{\infty} \lambda_j = 1\right).$$

In the other worlds, $\mathcal{S}^*_{h(\gamma)}(A, B, C, D)$, is a convex set.

Proof. Since $f_j(z) \in \mathcal{S}^*_{h(\gamma)}(A, B, C, D)$, so by (2.2), we get:

$$\sum_{k=2}^{\infty} \left|(A+B)k - (1 - A - B + AB) - Ck^2\right| [k]_q^m |a_k|$$

$$\sum_{k=1}^{\infty} \left|(A+B)k - (1 - A - B + AB) - Ck^2\right| [k]_q^m |b_{k,j}| \leqslant 1 - D, \quad (j = 1, 2, \ldots).$$

Also

$$F(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z) = z - \sum_{k=k}^{\infty} \left(\sum_{j=1}^{\infty} \lambda_j a_{k,j} \right) z^k + \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} \lambda_j b_{k,j} \right) (\overline{z})^k.$$

Now, according to Theorem 2.2, we have:

$$\begin{aligned}
&\sum_{k=2}^{\infty} \left|(A+B)k - (1 - A - B + AB) - Ck^2\right| [k]_q^m \left| \sum_{j=1}^{\infty} \lambda_j a_{k,j} \right| \\
&+ \sum_{k=1}^{\infty} \left|(A+B)k - (1 - A - B + AB) - Ck^2\right| [k]_q^m \left| \sum_{j=1}^{\infty} \lambda_j b_{k,j} \right| \\
&= \sum_{j=1}^{\infty} \left(\sum_{k=2}^{\infty} \left|(A+B)k - (1 - A - B + AB) - Ck^2\right| [k]_q^m |a_{k,j}| \right. \\
&\left. + \sum_{k=1}^{\infty} \left|(A+B)k - (1 - A - B + AB) - Ck^2\right| [k]_q^m |b_{k,j}| \right) \lambda_j \\
&\leqslant (1 - D) \sum_{j=1}^{\infty} \lambda_j = 1 - D.
\end{aligned}$$

Thus, $F(z) \in \mathcal{S}^*_{h(\gamma)}(A, B, C, D)$. □

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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