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ON THE BEHAVIORS OF ROUGH FRACTIONAL TYPE SUBLINEAR OPERATORS ON VANISHING GENERALIZED WEIGHTED MORREY SPACES

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Abstract. The aim of this paper is to get the boundedness of rough sublinear operators generated by fractional integral operators on vanishing generalized weighted Morrey spaces under generic size conditions which are satisfied by most of the operators in harmonic analysis. Also, rough fractional integral operator and a related rough fractional maximal operator which satisfy the conditions of our main result can be considered as some examples.

1. Introduction and useful informations

1.1. **Background.** The classical fractional integral (The classical fractional integral operator is also known as Riesz potential.) was introduced by Riesz in 1949 [6], defined by

$$\begin{aligned} I_\alpha f(x) &= (-\Delta)^{-\frac{\alpha}{2}} f(x) \qquad 0<\alpha<n, \\ &= \frac{1}{\gamma(\alpha)} \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy \end{aligned}$$

with

$$\gamma(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}-\frac{\alpha}{2}\right)},$$

where $\Gamma(\cdot)$ is the standard gamma function and I_α plays an important role in partial diferential equation as the inverse of power of Laplace operator. Especially, Its most significant feature is that I_α maps $L_p(\mathbb{R}^n)$

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continuously into $L_q(\mathbb{R}^n)$, with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $1 < p < \frac{n}{\alpha}$, through the well known Hardy-Littlewood-Sobolev imbedding theorem (see pp. 119-121, Theorem 1 and its proof in [7]) for I_α.

Let $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0$, $x \in \mathbb{R}^n \setminus \{0\}$ and satisfy the cancellation condition

$$\int\limits_{S^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$.

We first recall the definitions of rough fractional integral operator $T_{\Omega,\alpha}$ and a related rough fractional maximal operator $M_{\Omega,\alpha}$ as follows:

Definition 1.1. *Define*

$$I_{\Omega,\alpha}f(x) = \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy \qquad 0 < \alpha < n,$$

$$M_{\Omega,\alpha}f\left(x\right) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int\limits_{|x-y|<r} \left|\Omega(x-y)\right| \left|f(y)\right| dy \qquad 0 < \alpha < n.$$

Next, we give the definition of weighted Lebesgue spaces as follows:

Definition 1.2. (***Weighted Lebesgue space***) *Let $1 \le p \le \infty$ and given a weight $w\left(x\right) \in A_p\left(\mathbb{R}^n\right)$, we shall define weighted Lebesgue spaces as*

$$L_p(w) \equiv L_p(\mathbb{R}^n, w) = \left\{ f : \|f\|_{L_{p,w}} = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p w(x)dx \right)^{\frac{1}{p}} < \infty \right\}, \qquad 1 \le p < \infty.$$

$$L_{\infty,w} \equiv L_\infty(\mathbb{R}^n, w) = \left\{ f : \|f\|_{L_{\infty,w}} = \operatorname*{esssup}_{x\in\mathbb{R}^n} |f(x)|w(x) < \infty \right\}.$$

Here and later, A_p denotes the Muckenhoupt classes (see [2]).

Now, let us consider the Muckenhoupt-Wheeden class $A\left(p,q\right)$ in [5]. One says that $w\left(x\right) \in A\left(p,q\right)$ for $1 < p < q < \infty$ if and only if

$$[w]_{A(p,q)} := \sup_B \left(|B|^{-1} \int\limits_B w(x)^q dx \right)^{\frac{1}{q}} \left(|B|^{-1} \int\limits_B w(x)^{-p'} dx \right)^{\frac{1}{p'}} < \infty, \tag{1.1}$$

where the supremum is taken over all the balls B. Note that, by Hölder's inequality, for all balls B we have

$$[w]_{A(p,q)} \ge [w]_{A(p,q)(B)} = |B|^{\frac{1}{p}-\frac{1}{q}-1} \|w\|_{L_q(B)} \|w^{-1}\|_{L_{p'}(B)} \ge 1. \tag{1.2}$$

By (1.1), we have

$$\left(\int\limits_B w(x)^q dx \right)^{\frac{1}{q}} \left(\int\limits_B w(x)^{-p'} dx \right)^{\frac{1}{p'}} \lesssim |B|^{\frac{1}{q}+\frac{1}{p'}}. \tag{1.3}$$

On the other hand, let $\mu(x) = w(x)^{s'}$, $\tilde{p} = \frac{p}{s'}$ and $\tilde{q} = \frac{q}{s'}$. If $w(x)^{s'} \in A\left(\frac{p}{s'}, \frac{q}{s'}\right)$, then we get $\mu(x) \in A(\tilde{p}, \tilde{q})$. By (1.2) and (1.3),

$$\|\mu\|_{L_{\tilde{q}}(B)} \|\mu^{-1}\|_{L_{\tilde{p}'}(B)} \approx |B|^{1+\frac{1}{\tilde{q}}-\frac{1}{\tilde{p}}} \tag{1.4}$$

is valid.

Now, we introduce some spaces which play important roles in PDE. Except the weighted Lebesgue space $L_p(w)$, the weighted Morrey space $L_{p,\kappa}(w)$, which is a natural generalization of $L_p(w)$ is another important function space. Then, the definition of generalized weighted Morrey spaces $M_{p,\varphi}(w)$ which could be viewed as extension of $L_{p,\kappa}(w)$ has been given as follows:

For $1 \leq p < \infty$, positive measurable function $\varphi(x,r)$ on $\mathbb{R}^n \times (0,\infty)$ and nonnegative measurable function w on $\mathbb{R}^n$, $f \in M_{p,\varphi}(w) \equiv M_{p,\varphi}(\mathbb{R}^n, w)$ if $f \in L_{p,w}^{loc}(\mathbb{R}^n)$ and

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \|f\|_{L_p(B(x,r),w)} < \infty.$$

is finite. Note that for $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa}{p}}$, $0 < \kappa < 1$ and $\varphi(x,r) \equiv 1$, we have $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ and $M_{p,\varphi}(w) = L_p(w)$, respectively.

Extending the definition of vanishing generalized Morrey spaces in [3] to the case of generalized weighted Morrey spaces defined above, we introduce the following definition.

Definition 1.3. ***(Vanishing generalized weighted Morrey spaces)*** *For $1 \leq p < \infty$, $\varphi(x,r)$ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and nonnegative measurable function w on $\mathbb{R}^n$, $f \in VM_{p,\varphi}(w) \equiv VM_{p,\varphi}(\mathbb{R}^n, w)$ if $f \in L_{p,w}^{loc}(\mathbb{R}^n)$ and*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{1}{\varphi(x,r)} \|f\|_{L_p(B(x,r),w)} = 0. \tag{1.5}$$

Inherently, it is appropriate to impose on $\varphi(x,t)$ with the following circumstances:

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^n} \frac{(w(B(x,t)))^{\frac{1}{p}}}{\varphi(x,t)} = 0, \tag{1.6}$$

and

$$\inf_{t>1} \sup_{x\in\mathbb{R}^n} \frac{(w(B(x,t)))^{\frac{1}{p}}}{\varphi(x,t)} > 0. \tag{1.7}$$

From (1.6) and (1.7), we easily know that the bounded functions with compact support belong to $VM_{p,\varphi}(w)$. On the other hand, the space $VM_{p,\varphi}(w)$ is Banach space with respect to the following finite quasi-norm

$$\|f\|_{VM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \|f\|_{L_p(B(x,r),w)},$$

such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{1}{\varphi(x,r)} \|f\|_{L_p(B(x,r),w)} = 0,$$

we omit the details. Moreover, we have the following embeddings:

$$VM_{p,\varphi}\left(w\right)\subset M_{p,\varphi}\left(w\right),\qquad \|f\|_{M_{p,\varphi}(w)}\leq \|f\|_{VM_{p,\varphi}(w)}.$$

Henceforth, we denote by $\varphi\in\mathcal{B}\left(w\right)$ if $\varphi(x,r)$ is a positive measurable function on $\mathbb{R}^n\times(0,\infty)$ and positive for all $(x,r)\in\mathbb{R}^n\times(0,\infty)$ and satisfies (1.6) and (1.7).

The purpose of this paper is to consider the mapping properties for the rough fractional type sublinear operators $T_{\Omega,\alpha}$ satisfying the following condition

$$|T_{\Omega,\alpha}f(x)|\lesssim\int\limits_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}\,|f(y)|\,dy,\qquad x\notin \operatorname{supp} f \tag{1.8}$$

on vanishing generalized weighted Morrey spaces. Similar results still hold for the operators $I_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$, respectively. On the other hand, these operators have not also been studied so far on vanishing generalized weighted Morrey spaces and this paper seems to be the first in this direction.

At last, here and henceforth, $F\approx G$ means $F\gtrsim G\gtrsim F$; while $F\gtrsim G$ means $F\geq CG$ for a constant $C>0$; and p' and s' always denote the conjugate index of any $p>1$ and $s>1$, that is, $\frac{1}{p'}:=1-\frac{1}{p}$ and $\frac{1}{s'}:=1-\frac{1}{s}$ and also C stands for a positive constant that can change its value in each statement without explicit mention. Throughout the paper we assume that $x\in\mathbb{R}^n$ and $r>0$ and also let $B(x,r)$ denotes x-centred Euclidean ball with radius r, $B^C(x,r)$ denotes its complement. For any set E, χ_E denotes its characteristic function, if E is also measurable and w is a weight, $w(E):=\int\limits_E w(x)dx$.

2. Main Results

Our result can be stated as follows.

Theorem 2.1. *Suppose that* $0<\alpha<n$, $1\leq s'<p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $1<q<\infty$, $\Omega\in L_s(S^{n-1})$, $1<s\leq\infty$, $\Omega(\mu x)=\Omega(x)$ *for any* $\mu>0$, $x\in\mathbb{R}^n\backslash\{0\}$ *such that* $T_{\Omega,\alpha}$ *is rough fractional type sublinear operator satisfying (1.8). For* $p>1$, $w\left(x\right)^{s'}\in A\left(\frac{p}{s'},\frac{q}{s'}\right)$ *and* $s'<p$, *the following pointwise estimate*

$$\|T_{\Omega,\alpha}f\|_{L_q(B(x_0,r),w^q)}\lesssim\left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}\int\limits_{2r}^{\infty}\|f\|_{L_p(B(x_0,t),w^p)}\left(w^q\left(B\left(x_0,t\right)\right)\right)^{-\frac{1}{q}}\frac{dt}{t} \tag{2.1}$$

holds for any ball $B\left(x_0,r\right)$ *and for all* $f\in L_{p,w}^{loc}\left(\mathbb{R}^n\right)$. *If* $\varphi_1\in\mathcal{B}\left(w^p\right)$, $\varphi_2\in\mathcal{B}\left(w^q\right)$ *and the pair* (φ_1,φ_2) *satisfies the following conditions*

$$c_\delta:=\int\limits_{\delta}^{\infty}\sup_{x\in\mathbb{R}^n}\frac{\varphi_1\left(x,t\right)}{\left(w^q\left(B\left(x,t\right)\right)\right)^{\frac{1}{q}}}\frac{1}{t}dt<\infty \tag{2.2}$$

for every $\delta>0$, *and*

$$\int\limits_{r}^{\infty}\frac{\varphi_1\left(x,t\right)}{\left(w^q\left(B\left(x,t\right)\right)\right)^{\frac{1}{q}}}\frac{1}{t}dt\lesssim\frac{\varphi_2(x,r)}{\left(w^q\left(B\left(x,t\right)\right)\right)^{\frac{1}{q}}}, \tag{2.3}$$

then for $p>1$*,* $w(x)^{s'}\in A\left(\frac{p}{s'},\frac{q}{s'}\right)$ *and* $s'<p$*, the operator* $T_{\Omega,\alpha}$ *is bounded from* $VM_{p,\varphi_1}(w^p)$ *to* $VM_{q,\varphi_2}(w^q)$*. Moreover,*

$$\|T_{\Omega,\alpha}f\|_{VM_{q,\varphi_2}(w^q)}\lesssim\|f\|_{VM_{p,\varphi_1}(w^p)}. \tag{2.4}$$

Proof. Since inequality (2.1) is the heart of the proof of (2.4), we first prove (2.1).

For any $x_0\in\mathbb{R}^n$, we write as $f=f_1+f_2$, where $f_1(y)=f(y)\chi_{B(x_0,2r)}(y)$, $f_2(y)=f(y)\chi_{(B(x_0,2r))^C}(y)$, $r>0$ and $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0,2r)$. Then

$$\|T_{\Omega,\alpha}f\|_{L_q(w^q,B(x_0,r))}\leq\|T_{\Omega,\alpha}f_1\|_{L_q(w^q,B(x_0,r))}+\|T_{\Omega,\alpha}f_2\|_{L_q(w^q,B(x_0,r))}.$$

Let us estimate $\|T_{\Omega,\alpha}f_1\|_{L_q(w^q,B(x_0,r))}$ and $\|T_{\Omega,\alpha}f_2\|_{L_q(w^q,B(x_0,r))}$, respectively.

Since $f_1\in L_p(w^p,\mathbb{R}^n)$, by the boundedness of $T_{\Omega,\alpha}$ from $L_p(w^p,\mathbb{R}^n)$ to $L_q(w^q,\mathbb{R}^n)$ (see Theorem 3.4.2 in [4]), (1.4) and since $1\leq s'<p<q$ we get

$$\begin{aligned}
\|T_{\Omega,\alpha}f_1\|_{L_q(w^q,B(x_0,r))}&\leq\|T_{\Omega,\alpha}f_1\|_{L_q(w^q,\mathbb{R}^n)}\\
&\lesssim\|f_1\|_{L_p(w^p,\mathbb{R}^n)}\\
&=\|f\|_{L_p(w^p,B(x_0,2r))}\\
&\lesssim r^{n-\alpha s'}\|f\|_{L_p(w^p,B(x_0,2r))}\int_{2r}^{\infty}\frac{dt}{t^{n-\alpha s'+1}}\\
&\approx\|w^{s'}\|_{L_{\frac{q}{s'}}(B(x_0,r))}\|w^{-s'}\|_{L_{\left(\frac{p}{s'}\right)'}(B(x_0,r))}\int_{2r}^{\infty}\|f\|_{L_p(w^p,B(x_0,t))}\frac{dt}{t^{n-\alpha s'+1}}\\
&\lesssim\left(w^q\left(B(x_0,r)\right)\right)^{\frac{1}{q}}\int_{2r}^{\infty}\|f\|_{L_p(w^p,B(x_0,t))}\|w^{-s'}\|_{L_{\left(\frac{p}{s'}\right)'}(B(x_0,t))}\frac{dt}{t^{n-\alpha s'+1}}\\
&\lesssim\left(w^q\left(B(x_0,r)\right)\right)^{\frac{1}{q}}\int_{2r}^{\infty}\|f\|_{L_p(w^p,B(x_0,t))}\left[\|w^{s'}\|_{L_{\left(\frac{q}{s'}\right)}(B(x_0,t))}\right]^{-1}\frac{1}{t}dt\\
&\lesssim\left(w^q\left(B(x_0,r)\right)\right)^{\frac{1}{q}}\\
&\times\int_{2r}^{\infty}\|f\|_{L_p(w^p,B(x_0,t))}\left(w^q\left(B(x_0,t)\right)\right)^{-\frac{1}{q}}\frac{1}{t}dt.
\end{aligned}$$

Now, let's estimate the second part $(=\|T_{\Omega,\alpha}f_2\|_{L_q(w^q,B(x_0,r))})$. For the estimate used in $\|T_{\Omega,\alpha}f_2\|_{L_q(w^q,B(x_0,r))}$, we first have to prove the below inequality:

$$|T_{\Omega,\alpha}f_2(x)|\lesssim\int_{2r}^{\infty}\|f\|_{L_p(w^p,B(x_0,t))}\left(w^q\left(B(x_0,t)\right)\right)^{-\frac{1}{q}}\frac{1}{t}dt. \tag{2.5}$$

By [1] (see pp. 7 in the proof of Lemma2:), we get

$$|T_{\Omega,\alpha}f_2(x)|\lesssim\int_{2r}^{\infty}\|\Omega(x-\cdot)\|_{L_s(B(x_0,t))}\|f\|_{L_{s'}(B(x_0,t))}\frac{dt}{t^{n+1-\alpha}}. \tag{2.6}$$

On the other hand, by Hölder's inequality we have

$$
\begin{aligned}
\|f\|_{L_{s'}(B(x_0,t))} &= \left(\int\limits_{B(x_0,t)} |f(y)|^{s'} dy\right)^{\frac{1}{s'}} \\
&\leq \left(\int\limits_{B(x_0,t)} |f(y)|^{p} |\mu(y)|^{\widetilde{p}} dy\right)^{\frac{1}{p}} \left(\int\limits_{B(x_0,t)} |\mu(y)|^{-\widetilde{p}'} dy\right)^{\frac{1}{\widetilde{p}'s'}} \\
&\leq \left(\int\limits_{B(x_0,t)} |f(y)|^{p} |\mu(y)|^{\widetilde{p}} dy\right)^{\frac{1}{p}} \left(w^q\left(B(x_0,t)\right)\right)^{-\frac{1}{q}} |B(x_0,t)|^{\frac{1}{s'}+\frac{1}{q}-\frac{1}{p}} \\
&= \|f\|_{L_p(w^p,B(x_0,t))} \left(w^q\left(B(x_0,t)\right)\right)^{-\frac{1}{q}} |B(x_0,t)|^{\frac{1}{s'}+\frac{1}{q}-\frac{1}{p}},
\end{aligned} \tag{2.7}
$$

where in the second inequality we have used the following fact:

By (1.4), we get the following:

$$
\begin{aligned}
\left(\int\limits_{B(x_0,t)} |\mu(y)|^{-\widetilde{p}'} dy\right)^{\frac{1}{\widetilde{p}'s'}} &\approx \left[\|\mu\|_{L_{\tilde{q}}(B(x_0,t))}\right]^{-\frac{1}{s'}} \left[|B(x_0,t)|^{1+\frac{1}{\tilde{q}}-\frac{1}{\tilde{p}}}\right]^{\frac{1}{s'}} \\
&= \left[\left(\|w^{s'}\|_{L_{\tilde{q}}(B(x_0,t))}\right)^{-1} |B(x_0,t)|^{1+\frac{1}{\tilde{q}}-\frac{1}{\tilde{p}}}\right]^{\frac{1}{s'}} \\
&= \left[\left(\int\limits_{B(x_0,t)} |w(y)|^{q} dy\right)^{-\frac{s'}{q}} |B(x_0,t)|^{1+\frac{s'}{q}-\frac{s'}{p}}\right]^{\frac{1}{s'}} \\
&= \left(w^q\left(B(x_0,t)\right)\right)^{-\frac{1}{q}} |B(x_0,t)|^{\frac{1}{s'}+\frac{1}{q}-\frac{1}{p}}.
\end{aligned} \tag{2.8}
$$

At last, substituting (3.10) in [1] and (2.7) into (2.6), the proof of (2.5) is completed. Thus, by (2.5) we get

$$
\begin{aligned}
\|T_{\Omega,\alpha}f_2\|_{L_q(w^q,B(x_0,r))} &\lesssim \left(w^q\left(B(x_0,r)\right)\right)^{\frac{1}{q}} \\
&\quad\times \int\limits_{2r}^{\infty} \|f\|_{L_p(w^p,B(x_0,t))} \left(w^q\left(B(x_0,t)\right)\right)^{-\frac{1}{q}} \frac{1}{t} dt.
\end{aligned}
$$

Combining all the estimates for $\|T_{\Omega,\alpha}f_1\|_{L_q(w^q,B(x_0,r))}$ and $\|T_{\Omega,\alpha}f_2\|_{L_q(w^q,B(x_0,r))}$, we get (2.1).

Now, let's estimate the second part (2.4) of Theorem 2.1. Indeed, by the definition of vanishing generalized weighted Morrey spaces, (2.1) and (2.3), we have

$$\begin{aligned}
\|T_{\Omega,\alpha}f\|_{VM_{q,\varphi_2}(w^q)} &= \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)} \|T_{\Omega,\alpha}f\|_{L_q(w^q,B(x_0,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)} \left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}} \\
&\times \int_r^\infty \|f\|_{L_p(B(x_0,t),w^p)} \left(w^q\left(B\left(x_0,t\right)\right)\right)^{-\frac{1}{q}} \frac{dt}{t} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)} \left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}} \\
&\times \int_r^\infty \left(w^q\left(B\left(x_0,t\right)\right)\right)^{-\frac{1}{q}} \varphi_1\left(x,t\right) \left[\varphi_1\left(x,t\right)^{-1} \|f\|_{L_p(B(x_0,t),w^p)}\right] \frac{dt}{t} \\
&\lesssim \|f\|_{VM_{p,\varphi_1}(w^p)} \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)} \left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}} \\
&\times \int_r^\infty \left(w^q\left(B\left(x_0,t\right)\right)\right)^{-\frac{1}{q}} \varphi_1\left(x,t\right) \frac{dt}{t} \\
&\lesssim \|f\|_{VM_{p,\varphi_1}(w^p)} .
\end{aligned}$$

At last, we need to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{1}{\varphi_2(x,r)} \|T_{\Omega,\alpha}f\|_{L_q(w^q,B(x_0,r))} \lesssim \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{1}{\varphi_1(x,r)} \|f\|_{L_p(w^p,B(x_0,r))} = 0.$$

But, because the proof of above inequality is similar to Theorem 2 in [3], we omit the details, which completes the proof. □

Corollary 2.1. *Under the conditions of Theorem 2.1, the operators $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$ are bounded from $VM_{p,\varphi_1}(w^p)$ to $VM_{q,\varphi_2}(w^q)$.*

Corollary 2.2. *For $w \equiv 1$, under the conditions of Theorem 2.1, we get the Theorem 2 in [3].*

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