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DIRECT PRODUCT OF FINITE FUZZY NORMAL SUBRINGS OVER NON-ASSOCIATIVE RINGS

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Abstract. In this paper, we define the concept of direct product of finite fuzzy normal subrings over non-associative and non-commutative rings (LA-ring) and investigate the some fundamental properties of direct product of fuzzy normal subrings.

1. Introduction

A generalization of commutative semigroups has been established by Kazim et al [10]. In ternary commutative law: $abc = cba$, they introduced the braces on the left side of this law and explored a new pseudo associative law $(ab)c = (cb)a$. This law $(ab)c = (cb)a$ is called the left invertive law. A groupoid S is left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law: $(ab)c = (cb)a$.

A groupoid S is medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$), in [5] (resp. [1]). In [10], an LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial. Every LA-semigroup with left identity is paramedial in [19] and also satisfies $a(bc) = b(ac)$ and $(ab)(cd) = (dc)(ba)$.

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S. Kamran [6], extended the concept of LA-semigroup to the left almost group (LA-group). An LA-semigroup S is left almost group, if there exists left identity $e \in S$ such that $ea = a$ for all $a \in S$ and for every $a \in S$, there exists $b \in S$ such that $ba = e$.

Rehman et al [23], discussed the left almost ring (LA-ring) of finitely nonzero functions which is a generalization of commutative semigroup ring. By a left almost ring, we mean a non-empty set R with at least two elements such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining for all $a, b \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. In fact an LA-ring is a non-associative and non-commutative ring.

A non-empty subset A of an LA-ring R is an LA-subring of R if $a - b$ and $ab \in A$ for all $a, b \in A$. A is called a left (resp. right) ideal of R, if $(A, +)$ is an LA-group and $RA \subseteq A$ (resp. $AR \subseteq A$). A is called an ideal of R if it is both a left ideal and a right ideal of R.

First time, the concept of fuzzy set introduced by Zadeh in his classical paper [26]. This concept has provided a useful mathematical tool for describing the behavior of systems that are too complex to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields such as artificial intelligence, computer science, management science, expert systems, finite state machines, Languages, robotics, coding theory and others.

Liu [14], introduced the concept of fuzzy subrings and fuzzy ideals of a ring. Many authors have explored the theory of fuzzy rings (for example [2–4, 13, 15, 16, 25]). Gupta et al [4], gave the idea of intrinsic product of fuzzy subsets of a ring. Kuroki [13], characterized regular (intra-regular, both regular and intra-regular) rings in terms of fuzzy left (right, quasi, bi-) ideals.

Sherwood [24], introduced the concept of product of fuzzy subgroups. After this, further study on this concept continued by Osman [17, 18] and Ray [20]. Zaid [27], gave the idea of normal fuzzy subgroups.

Kausar et al [21] initiated the idea of intuitionistic fuzzy normal subrings over a non-associative ring and also characterized the non-associative rings by their intuitionistic fuzzy bi-ideals in [7]. Recently Kausar [9] explored the direct product of finite intuitionistic anti fuzzy normal subrings over non-associative rings. In this paper we explore the concept of [9, 21] in finite fuzzy normal subrings over non-associative and non-commutative rings. Recently Kausar et al [11] studied the fuzzy ideals in LA-rings and also Kausar et al [12], investigated a study on intuitionistic fuzzy ideals with thresholds $(\alpha, \beta]$ in LA-rings.

In this paper, we give the concept of direct product of fuzzy normal LA-subrings. In the first section, we investigate the some basic properties of fuzzy normal LA-subrings of an LA-ring R. In the second section, we provide the some elementary properties of direct product of fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2$. In the third section, we define the direct product of fuzzy subsets $\mu_1, \mu_2, ..., \mu_n$ of LA-rings $R_1, R_2, ..., R_n$, respectively and examine the some fundamental properties of direct product of fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Specifically we will show that:

(1) Let A and B be two LA-subrings of an LA-ring R. Then $A \cap B$ is an LA-subring of R if and only if the characteristic function χ_Z of $Z = A \cap B$ is a fuzzy normal LA-subring of R.

(2) Let $X = A \times B$ and $Y = C \times D$ be two LA-subrings of an LA-ring $R_1 \times R_2$. Then $X \cap Y$ is an LA-subring of $R_1 \times R_2$ if and only if the characteristic function χ_Z of $Z = X \cap Y$ is a fuzzy normal LA-subring of $R_1 \times R_2$.

(3) Let $A = A_1 \times A_2 \times ... \times A_n$ and $B = B_1 \times B_2 \times ... \times B_n$ be two LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Then $A \cap B$ is an LA-subring of $R_1 \times R_2 \times ... \times R_n$ if and only if the characteristic function χ_Z of $Z = A \cap B$ is a fuzzy normal LA-subring of $R_1 \times R_2 \times ... \times R_n$.

2. Fuzzy Normal LA-subrings

In this section, we investigate the some basic properties of fuzzy normal LA-subrings of an LA-ring R.

By a fuzzy subset μ of an LA-ring R, we mean a function $\mu : R \to [0, 1]$ and the complement of μ is denoted by μ', is a fuzzy subset of R defined by $\mu'(x) = 1 - \mu(x)$ for all $x \in R$.

A fuzzy subset μ of an LA-ring R is said to be a fuzzy LA-subring of R if $\mu(x-y) \geq min\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in R$.

A fuzzy LA-subring of an LA-ring R is said to be a fuzzy normal LA-subring of R if $\mu(xy) = \mu(yx)$ for all $x, y \in R$.

Let A be a non-empty subset of an LA-ring R. The characteristic function of A is denoted by χ_A and defined by

$$\chi_A \colon R \to [0,1] \mid x \to \chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

Lemma 2.1. *Let A be a non-empty subset of an LA-ring R. Then A is an LA-subring of R if and only if the characteristic function χ_A of A is a fuzzy normal LA-subring of R.*

Proof. Let A be an LA-subring of R and $a, b \in R$. If $a, b \in A$, then by definition of characteristic function $\chi_A(a) = 1 = \chi_A(b)$. Since $a - b, ab \in A$, A being an LA-subring of R. This implies that

$$\begin{aligned} \chi_A(a-b) &= 1 = 1 \wedge 1 = \chi_A(a) \wedge \chi_A(b) \\ \text{and } \chi_A(ab) &= 1 = 1 \wedge 1 = \chi_A(a) \wedge \chi_A(b). \end{aligned}$$

Thus $\chi_A(a-b) \geq min\{\chi_A(a), \chi_A(b)\}$ and $\chi_A(ab) \geq min\{\chi_A(a), \chi_A(b)\}$. Since ab and $ba \in A$, so $\chi_A(ab) = 1 = \chi_A(ba)$, i.e., $\chi_A(ab) = \chi_A(ba)$. Similarly we have

$$\begin{aligned} \chi_A(a-b) &\geq min\{\chi_A(a), \chi_A(b)\}, \\ \chi_A(ab) &\geq min\{\chi_A(a), \chi_A(b)\}, \\ \chi_A(ab) &= \chi_A(ba), \end{aligned}$$

when $a, b \notin A$. Hence the characteristic function χ_A of A is a fuzzy normal LA-subring of R.

Conversely, suppose that the characteristic function χ_A of A is a fuzzy normal LA-subring of R. Let $a, b \in A$, then by definition $\chi_A(a) = 1 = \chi_A(b)$. By the supposition

$$\begin{aligned} \chi_A(a-b) &\geq \chi_A(a) \wedge \chi_A(b) = 1 \wedge 1 = 1 \\ \text{and } \chi_A(ab) &\geq \chi_A(a) \wedge \chi_A(b) = 1 \wedge 1 = 1. \end{aligned}$$

Thus $\chi_A(a-b) = 1 = \chi_A(ab)$, i.e., $a-b, ab \in A$. Hence A is an LA-subring of R. □

Lemma 2.2. *If A and B are two LA-subrings of an LA-ring R, then their intersection $A \cap B$ is also an LA-subring of R.*

Proof. Straight forward. □

Theorem 2.1. *Let A and B be two LA-subrings of an LA-ring R. Then $A \cap B$ is an LA-subring of R if and only if the characteristic function χ_Z of $Z = A \cap B$ is a fuzzy normal LA-subring of R.*

Proof. Let $Z = A \cap B$ be an LA-subring of R and $a, b \in R$. If $a, b \in Z = A \cap B$, then by definition of characteristic function $\chi_Z(a) = 1 = \chi_Z(b)$. Since $a-b, ab \in A, B$, A and B being LA-subrings of R. This implies that

$$\begin{aligned} \chi_Z(a-b) &= 1 = 1 \wedge 1 = \chi_Z(a) \wedge \chi_Z(b) \\ \text{and } \chi_Z(ab) &= 1 = 1 \wedge 1 = \chi_Z(a) \wedge \chi_Z(b). \end{aligned}$$

Thus $\chi_Z(a-b) \geq min\{\chi_Z(a), \chi_Z(b)\}$ and $\chi_Z(ab) \geq min\{\chi_Z(a), \chi_Z(b)\}$. As ab and $ba \in Z$, so $\chi_Z(ab) = 1 = \chi_Z(ba)$, i.e., $\chi_Z(ab) = \chi_Z(ba)$. Similarly we have

$$\begin{aligned} \chi_Z(a-b) &\geq min\{\chi_Z(a), \chi_Z(b)\}, \\ \chi_Z(ab) &\geq min\{\chi_Z(a), \chi_Z(b)\}, \\ \chi_Z(ab) &= \chi_Z(ba), \end{aligned}$$

when $a, b \notin Z$. Hence the characteristic function χ_Z of Z is a fuzzy normal LA-subring of R.

Conversely, assume that the characteristic function χ_Z of $Z = A \cap B$ is a fuzzy normal LA-subring of R. Let $a, b \in Z = A \cap B$, then by definition of characteristic function $\chi_Z(a) = 1 = \chi_Z(b)$. By our assumption

$$\begin{aligned} \chi_Z(a-b) &\geq \chi_Z(a) \wedge \chi_Z(b) = 1 \wedge 1 = 1 \\ \text{and } \chi_Z(ab) &\geq \chi_Z(a) \wedge \chi_Z(b) = 1 \wedge 1 = 1. \end{aligned}$$

Thus $\chi_Z(a-b) = 1 = \chi_Z(ab)$, i.e., $a - b$ and $ab \in Z$. Hence Z is an LA-subring of R. □

Corollary 2.1. *Let $\{A_i\}_{i \in I}$ be a family of LA-subrings of an LA-ring R. Then $A = \cap A_i$ is an LA-subring of R if and only if the characteristic function χ_A of $A = \cap A_i$ is a fuzzy normal LA-subring of R.*

Lemma 2.3. *If μ and γ are two fuzzy normal LA-subrings of an LA-ring R, then their intersection $\mu \cap \gamma$ is also a fuzzy normal LA-subring of R.*

Proof. Let μ and γ be two fuzzy normal LA-subrings of an LA-ring R. We have to show that $\beta = \mu \cap \gamma$ is also a fuzzy normal LA-subring of R. Now

$$\begin{aligned} \beta(z_1 - z_2) &= (\mu \cap \gamma)(z_1 - z_2) \\ &= min\{\mu(z_1 - z_2), \gamma(z_1 - z_2)\} \\ &\geq \{\{\mu(z_1) \wedge \mu(z_2)\} \wedge \{\gamma(z_1) \wedge \gamma(z_2)\}\} \\ &= \{\mu(z_1) \wedge \{\mu(z_2) \wedge \gamma(z_1)\} \wedge \gamma(z_2)\} \\ &= \{\mu(z_1) \wedge \{\gamma(z_1) \wedge \mu(z_2)\} \wedge \gamma(z_2)\} \\ &= \{\{\mu(z_1) \wedge \gamma(z_1)\} \wedge \{\mu(z_2) \wedge \gamma(z_2)\}\} \\ &= min\{(\mu \cap \gamma)(z_1), (\mu \cap \gamma)(z_2)\} \\ &= min\{\beta(z_1), \beta(z_2)\}. \\ &\Rightarrow \beta(z_1 - z_2) \geq min\{\beta(z_1), \beta(z_2)\}. \end{aligned}$$

Similarly, we have $\beta(z_1 \circ z_2) \geq min\{\beta(z_1), \beta(z_2)\}$. Thus β is a fuzzy LA-subring of an LA-ring R. Now

$$\begin{aligned} \beta(z_1 \circ z_2) &= (\mu \cap \gamma)(z_1 \circ z_2) \\ &= min\{\mu(z_1 \circ z_2), \gamma(z_1 \circ z_2)\} \\ &= min\{\mu(z_2 \circ z_1), \gamma(z_2 \circ z_1)\} \\ &= (\mu \cap \gamma)(z_2 \circ z_1) \\ &= \beta(z_2 \circ z_1). \end{aligned}$$

Hence $\beta = \mu \cap \gamma$ is a fuzzy normal LA-subring of R. □

Corollary 2.2. *If $\{\mu_i\}_{i\in I}$ is a family of fuzzy normal LA-subrings of an LA-ring R, then $\mu = \cap\mu_i$ is also a fuzzy normal LA-subring of R.*

3. Direct Product of Fuzzy Normal LA-subrings

In this section, we define the direct product of fuzzy subsets μ_1, μ_2 of LA-rings R_1, R_2, respectively and investigate the some elementary properties of direct product of fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2$.

Let μ_1, μ_2 be fuzzy subsets of LA-rings R_1, R_2, respectively. The direct product of fuzzy subsets μ_1, μ_2 is denoted by $\mu_1 \times \mu_2$ and defined as $(\mu_1 \times \mu_2)(x_1, x_2) = min\{\mu_1(x_1), \mu_2(x_2)\}$.

A fuzzy subset $\mu_1 \times \mu_2$ of an LA-ring $R_1 \times R_2$ is said to be a fuzzy LA-subring of $R_1 \times R_2$ if

(1) $(\mu_1 \times \mu_2)(x - y) \geq min\{(\mu_1 \times \mu_2)(x), (\mu_1 \times \mu_2)(y)\}$,

(2) $(\mu_1 \times \mu_2)(xy) \geq min\{(\mu_1 \times \mu_2)(x), (\mu_1 \times \mu_2)(y)\}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in R_1 \times R_2$.

A fuzzy LA-subring of an LA-ring $R_1 \times R_2$ is said to be a fuzzy normal LA-subring of $R_1 \times R_2$ if $(\mu_1 \times \mu_2)(xy) = (\mu_1 \times \mu_2)(yx)$ for all $x = (x_1, x_2), y = (y_1, y_2) \in R_1 \times R_2$.

Let $A \times B$ be a non-empty subset of an LA-ring $R_1 \times R_2$. The characteristic function of $A \times B$ is denoted by $\chi_{A\times B}$ and defined as

$$\chi_{A\times B}\colon R_1 \times R_2 \to [0,1] \mid x = (x_1, x_2) \to \chi_{A\times B}(x) = \begin{cases} 1 \text{ if } x \in A \times B \\ 0 \text{ if } x \notin A \times B \end{cases}$$

Lemma 3.1. *[21, Lemma 4.2] If A and B are LA-subrings of LA-rings R_1 and R_2, respectively, then $A \times B$ is an LA-subring of an LA-ring $R_1 \times R_2$ under the same operations defined as in $R_1 \times R_2$.*

Proposition 3.1. *Let A and B be LA-subrings of LA-rings R_1 and R_2, respectively. Then $A \times B$ is an LA-subring of an LA-ring $R_1 \times R_2$ if and only if the characteristic function χ_Z of $Z = A \times B$ is a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.*

Proof. Let $Z = A\times B$ be an LA-subring of $R_1\times R_2$ and $a = (a_1, a_2), b = (b_1, b_2) \in R_1\times R_2$. If $a, b \in Z = A\times B$, then by definition of characteristic function $\chi_Z(a) = 1 = \chi_Z(b)$. Since $a - b$ and $ab \in Z$, Z being an LA-subring of an LA-ring $R_1 \times R_2$. This implies that

$$\begin{aligned} \chi_Z(a-b) &= 1 = 1 \wedge 1 = \chi_Z(a) \wedge \chi_Z(b) \\ \text{and } \chi_Z(ab) &= 1 = 1 \wedge 1 = \chi_Z(a) \wedge \chi_Z(b). \end{aligned}$$

Thus $\chi_Z(a-b) \geq min\{\chi_Z(a), \chi_Z(b)\}$ and $\chi_Z(ab) \geq min\{\chi_Z(a), \chi_Z(b)\}$. Since ab and $ba \in Z$, so $\mu_{\chi_Z}(ab) = 1 = \mu_{\chi_Z}(ba)$, i.e., $\chi_Z(ab) = \chi_Z(ba)$. Similarly we have

$$\begin{aligned} \chi_Z(a-b) &\geq min\{\chi_Z(a), \chi_Z(b)\}, \\ \chi_Z(ab) &\geq min\{\chi_Z(a), \chi_Z(b)\}, \\ \chi_Z(ab) &= \chi_Z(ba), \end{aligned}$$

when $a, b \notin Z$. Hence the characteristic function χ_Z of $Z = A \times B$ is a fuzzy normal LA-subring of $R_1 \times R_2$.

Conversely, suppose that the characteristic function χ_Z of $Z = A \times B$ is a fuzzy normal LA-subring of $R_1 \times R_2$. We have to show that $Z = A \times B$ is an LA-subring of $R_1 \times R_2$. Let $a, b \in Z$, where $a = (a_1, a_2)$ and $b = (b_1, b_2)$, $a_1, b_1 \in A$, $a_2, b_2 \in B$. By definition, we have $\chi_Z(a) = 1 = \chi_Z(b)$. By our supposition

$$\begin{aligned} \chi_Z(a-b) &\geq \chi_Z(a) \wedge \chi_Z(b) = 1 \wedge 1 = 1 \\ \text{and } \chi_Z(ab) &\geq \chi_Z(a) \wedge \chi_Z(b) = 1 \wedge 1 = 1. \end{aligned}$$

Thus $\chi_Z(a-b) = 1 = \chi_Z(ab)$, i.e., $a - b$ and $ab \in Z$. Hence $Z = A \times B$ is an LA-subring of $R_1 \times R_2$. □

Lemma 3.2. *If $X = A \times B$ and $Y = C \times D$ are two LA-subrings of an LA-ring $R_1 \times R_2$, then their intersection $X \cap Y$ is also an LA-subring of $R_1 \times R_2$.*

Proof. Straight forward. □

Theorem 3.1. *Let $X = A \times B$ and $Y = C \times D$ be two LA-subrings of an LA-ring $R_1 \times R_2$. Then $X \cap Y$ is an LA-subring of $R_1 \times R_2$ if and only if the characteristic function χ_Z of $Z = X \cap Y$ is a fuzzy normal LA-subring of $R_1 \times R_2$.*

Proof. Let $Z = X \cap Y$ be an LA-subring of an LA-ring $R_1 \times R_2$ and $a = (a_1, a_2), b = (b_1, b_2) \in R_1 \times R_2$. If $a, b \in Z = X \cap Y$, then by definition of characteristic function $\chi_Z(a) = 1 = \chi_Z(b)$. Since $a - b$ and $ab \in Z$, Z being an LA-subring of $R_1 \times R_2$. This implies that

$$\begin{aligned} \chi_Z(a-b) &= 1 = 1 \wedge 1 = \chi_Z(a) \wedge \chi_Z(b) \\ \text{and } \chi_Z(ab) &= 1 = 1 \wedge 1 = \chi_Z(a) \wedge \chi_Z(b). \end{aligned}$$

Thus $\chi_Z(a-b) \geq min\{\chi_Z(a), \chi_Z(b)\}$ and $\chi_Z(ab) \geq min\{\chi_Z(a), \chi_Z(b)\}$. Since ab and $ba \in Z$, then by definition $\chi_Z(ab) = 1 = \chi_Z(ba)$, i.e., $\chi_Z(ab) = \chi_Z(ba)$. Similarly we have

$$\begin{aligned} \chi_Z(a-b) &\geq min\{\chi_Z(a), \chi_Z(b)\}, \\ \chi_Z(ab) &\geq min\{\chi_Z(a), \chi_Z(b)\}, \\ \chi_Z(ab) &= \chi_Z(ba), \end{aligned}$$

when $a, b \notin Z$. Hence the characteristic function χ_Z of Z is a fuzzy normal LA-subring of $R_1 \times R_2$.

Conversely, assume that the characteristic function χ_Z of $Z = X \cap Y$ is a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. Let $a, b \in Z = X \cap Y$, then by definition $\chi_Z(a) = 1 = \chi_Z(b)$. By our assumption

$$\begin{aligned} \chi_Z(a-b) &\geq \chi_Z(a) \wedge \chi_Z(b) = 1 \wedge 1 = 1 \\ \text{and } \chi_Z(ab) &\geq \chi_Z(a) \wedge \chi_Z(b) = 1 \wedge 1 = 1. \end{aligned}$$

Thus $\chi_Z(a-b) = 1 = \chi_Z(ab)$, i.e., $a-b$ and $ab \in Z$. Hence Z is an LA-subring of an LA-ring $R_1 \times R_2$. $\square$

Corollary 3.1. *Let $\{C_i\}_{i \in I} = \{A_i \times B_i\}_{i \in I}$ be a family of LA-subrings of an LA-ring $R_1 \times R_2$. Then $C = \cap C_i$ is an LA-subring of $R_1 \times R_2$ if and only if the characteristic function χ_C of $C = \cap C_i$ is a fuzzy normal LA-subring of $R_1 \times R_2$.*

Lemma 3.3. *If μ and γ are fuzzy normal LA-subrings of LA-rings R_1 and R_2, respectively, then $\mu \times \gamma$ is a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$.*

Proof. Let μ and γ be fuzzy normal LA-subrings of LA-ring R_1 and R_2, respectively. We have to show that $\beta = \mu \times \gamma$ is a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. Now

$$\begin{aligned} \beta((a,b)-(c,d)) &= (\mu \times \gamma)(a-c, b-d) \\ &= min\{\mu(a-c), \gamma(b-d)\} \\ &= \mu(a-c) \wedge \gamma(b-d) \\ &\geq \{\mu(a) \wedge \mu(c)\} \wedge \{\gamma(b) \wedge \gamma(d)\} \\ &= \mu(a) \wedge \{\mu(c) \wedge \gamma(b)\} \wedge \gamma(d) \\ &= \mu(a) \wedge \{\gamma(b) \wedge \mu(c)\} \wedge \gamma(d) \\ &= \{\mu(a) \wedge \gamma(b)\} \wedge \{\mu(c) \wedge \gamma(d)\} \\ &= min\{(\mu \times \gamma)(a,b), (\mu \times \gamma)(c,d)\} \\ &= min\{\beta(a,b), \beta(c,d)\}. \\ &\Rightarrow \beta((a,b)-(c,d)) \geq min\{\beta(a,b), \beta(c,d)\}. \end{aligned}$$

Similarly, we have $\beta((a,b)\circ(c,d)) \geq min\{\beta(a,b), \beta(c,d)\}$. Thus $\mu\times\gamma$ is a fuzzy LA-subring of $R_1\times R_2$. Now

$$\begin{aligned}\beta((a,b)\circ(c,d)) &= (\mu\times\gamma)(ac,bd)\\ &= min\{\mu(ac),\gamma(bd)\}\\ &= min\{\mu(ca),\gamma(db)\}\\ &= (\mu\times\gamma)(ca,db)\\ &= \beta((c,d)\circ(a,b)).\end{aligned}$$

Hence $\mu\times\gamma$ is a fuzzy normal LA-subring of $R_1\times R_2$. □

Proposition 3.2. *If $\mu=\mu_1\times\mu_2$ and $\gamma=\gamma_1\times\gamma_2$ are two fuzzy normal LA-subrings of an LA-ring $R_1\times R_2$, then their intersection $\beta=\mu\cap\gamma$ is also a fuzzy normal LA-subring of $R_1\times R_2$.*

Proof. Let $\mu=\mu_1\times\mu_2$ and $\gamma=\gamma_1\times\gamma_2$ be two fuzzy normal LA-subrings of an LA-ring $R_1\times R_2$. We have to show that $\beta=\mu\cap\gamma$ is also a fuzzy normal LA-subring of $R_1\times R_2$. Now

$$\begin{aligned}\beta((z_1,z_2)-(z_3,z_4)) &= (\mu\cap\gamma)((z_1,z_2)-(z_3,z_4))\\ &= min\{\mu((z_1,z_2)-(z_3,z_4)),\gamma((z_1,z_2)-(z_3,z_4))\}\\ &\geq \{\{\mu(z_1,z_2)\wedge\mu(z_3,z_4)\}\wedge\{\gamma(z_1,z_2)\wedge\gamma(z_3,z_4)\}\}\\ &= \{\mu(z_1,z_2)\wedge\{\mu(z_3,z_4)\wedge\gamma(z_1,z_2)\}\wedge\gamma(z_3,z_4)\}\\ &= \{\mu(z_1,z_2)\wedge\{\gamma(z_1,z_2)\wedge\mu(z_3,z_4)\}\wedge\gamma(z_3,z_4)\}\\ &= \{\{\mu(z_1,z_2)\wedge\gamma(z_1,z_2)\}\wedge\{\mu(z_3,z_4)\wedge\gamma(z_3,z_4)\}\}\\ &= min\{(\mu\cap\gamma)(z_1,z_2),(\mu\cap\gamma)(z_3,z_4)\}\\ &= min\{\beta(z_1,z_2),\beta(z_3,z_4)\}.\\ &\Rightarrow \beta((z_1,z_2)-(z_3,z_4))\geq min\{\beta(z_1,z_2),\beta(z_3,z_4)\}.\end{aligned}$$

Similarly, we have $\beta((z_1,z_2)\circ(z_3,z_4)) \geq min\{\beta(z_1,z_2),\beta(z_3,z_4)\}$. Thus $\beta=\mu\cap\gamma$ is a fuzzy LA-subring of an LA-ring $R_1\times R_2$. Now

$$\begin{aligned}\beta((z_1,z_2)\circ(z_3,z_4)) &= (\mu\cap\gamma)((z_1,z_2)\circ(z_3,z_4))\\ &= min\{\mu((z_1,z_2)\circ(z_3,z_4)),\gamma((z_1,z_2)\circ(z_3,z_4))\}\\ &= min\{\mu((z_3,z_4)\circ(z_1,z_2)),\gamma((z_3,z_4)\circ(z_1,z_2))\}\\ &= (\mu\cap\gamma)((z_3,z_4)\circ(z_1,z_2))\\ &= \beta((z_3,z_4)\circ(z_1,z_2)).\end{aligned}$$

Hence $\beta = \mu \cap \gamma$ is a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$. □

Corollary 3.2. *If $\{\beta_i\}_{i\in I} = \{\mu_i \times \gamma_i\}_{i\in I}$ is a family of fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2$, then $\beta = \cap\beta_i$ is also a fuzzy normal LA-subring of $R_1 \times R_2$.*

Theorem 3.2. *If $\mu = \mu_1 \times \mu_2$ and $\gamma = \gamma_1 \times \gamma_2$ are fuzzy normal LA-subrings of LA-rings $R' = R_1 \times R_2$ and $R'' = R_3 \times R_4$, respectively, then $\beta = \mu \times \gamma$ is a fuzzy normal LA-subring of an LA-ring $R' \times R'' = (R_1 \times R_2) \times (R_3 \times R_4)$.*

Proof. Let $\mu = \mu_1 \times \mu_2$ and $\gamma = \gamma_1 \times \gamma_2$ be fuzzy normal LA-subrings of LA-rings $R' = R_1 \times R_2$ and $R'' = R_3 \times R_4$, respectively. We have to show that $\beta = \mu \times \gamma$ is a fuzzy normal LA-subring of an LA-ring $R' \times R''$. Now

$$\begin{aligned}
& \beta(((z_1, z_2), (z_3, z_4)) - ((z_5, z_6), (z_7, z_8))) \\
= \; & \mu \times \gamma(((z_1, z_2), (z_3, z_4)) - ((z_5, z_6), (z_7, z_8))) \\
= \; & \mu \times \gamma(((z_1, z_2) - (z_5, z_6)), ((z_3, z_4) - (z_7, z_8))) \\
= \; & min\{\mu((z_1, z_2) - (z_5, z_6)), \gamma((z_3, z_4) - (z_7, z_8))\} \\
\geq \; & min\{(\mu(z_1, z_2) \wedge \mu(z_5, z_6)), (\gamma(z_3, z_4) \wedge \gamma(z_7, z_8))\} \\
= \; & ((\mu(z_1, z_2) \wedge \mu(z_5, z_6)) \wedge (\gamma(z_3, z_4) \wedge \gamma(z_7, z_8))) \\
= \; & ((\mu(z_1, z_2) \wedge \gamma(z_3, z_4)) \wedge (\mu(z_5, z_6) \wedge \gamma(z_7, z_8))) \\
= \; & min\{(\mu(z_1, z_2) \wedge \gamma(z_3, z_4)), (\mu(z_5, z_6) \wedge \gamma(z_7, z_8))\} \\
= \; & min\{\mu \times \gamma((z_1, z_2), (z_3, z_4)), \mu \times \gamma((z_5, z_6), (z_7, z_8))\} \\
= \; & min\{\beta((z_1, z_2), (z_3, z_4)), \beta((z_5, z_6), (z_7, z_8))\}, .
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \beta(((z_1, z_2), (z_3, z_4)) \circ ((z_5, z_6), (z_7, z_8))) \\
\geq \; & min\{\beta((z_1, z_2), (z_3, z_4)), \beta((z_5, z_6), (z_7, z_8))\}.
\end{aligned}$$

Thus $\beta = \mu \cap \gamma$ is a fuzzy LA-subring of an LA-ring $R\prime \times R''$. Now

$$\begin{aligned} & \beta(((z_1, z_2), (z_3, z_4)) \circ ((z_5, z_6), (z_7, z_8))) \\ = \; & \mu \times \gamma(((z_1, z_2), (z_3, z_4)) \circ ((z_5, z_6), (z_7, z_8))) \\ = \; & \mu \times \gamma(((z_1, z_2) \circ (z_5, z_6)), ((z_3, z_4) \circ (z_7, z_8))) \\ = \; & min\{\mu((z_1, z_2) \circ (z_5, z_6)), \gamma((z_3, z_4) \circ (z_7, z_8))\} \\ = \; & min\{\mu((z_5, z_6) \circ (z_1, z_2)), \gamma((z_7, z_8) \circ (z_3, z_4))\} \\ = \; & \mu \times \gamma(((z_5, z_6) \circ (z_1, z_2)), ((z_7, z_8) \circ (z_3, z_4))) \\ = \; & \mu \times \gamma(((z_5, z_6), (z_7, z_8)) \circ ((z_1, z_2), (z_3, z_4))) \\ = \; & \beta(((z_5, z_6), (z_7, z_8)) \circ ((z_1, z_2), (z_3, z_4))). \end{aligned}$$

Hence $\beta = \mu \cap \gamma$ is a fuzzy normal LA-subring of an LA-ring $R\prime \times R''$. □

Lemma 3.4. *Let μ and γ be fuzzy subsets of LA-rings R_1 and R_2 with left identities e_1 and e_2, respectively. If $\mu \times \gamma$ is a fuzzy LA-subring of an LA-ring $R_1 \times R_2$, then at least one of the following two statements must hold.*

(1) $\mu(x) \leq \gamma(e_2)$, *for all* $x \in R_1$.

(2) $\mu(x) \leq \gamma(e_1)$, *for all* $x \in R_2$.

Proof. Let $\mu \times \gamma$ be a fuzzy LA-subring of $R_1 \times R_2$. By contraposition, suppose that none of the statements (1) and (2) holds. Then we can find a and b in R_1 and R_2, respectively such that

$$\mu(a) \geq \gamma(e_2) \text{ and } \mu(b) \geq \gamma(e_1).$$

Thus, we have

$$\begin{aligned} (\mu \times \gamma)(a, b) &= min\{\mu(a), \gamma(b)\} \\ &\geq min\{\mu(e_1), \gamma(e_2)\} \\ &= (\mu \times \gamma)(e_1, e_2). \end{aligned}$$

So $\mu \times \gamma$ is not a fuzzy LA-subring of $R_1 \times R_2$. Hence either $\mu(x) \leq \gamma(e_2)$, for all $x \in R_1$ or $\mu(x) \leq \gamma(e_1)$ for all $x \in R_2$. □

Lemma 3.5. *Let μ and γ be fuzzy subsets of LA-rings R_1 and R_2 with left identities e_1 and e_2, respectively and $\mu \times \gamma$ is a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2$, then the following conditions are true.*

(1) *If $\mu(x) \leq \gamma(e_2)$, for all $x \in R_1$, then μ is a fuzzy normal LA-subring of R_1.*

(2) *If $\mu(x) \leq \gamma(e_1)$, for all $x \in R_2$, then γ is a fuzzy normal LA-subring of R_2.*

Proof. (1) Let $\mu(x) \leq \gamma(e_2)$ for all $x \in R_1$, and $y \in R_1$. We have to show that μ is a fuzzy normal LA-subring of R_1. Now

$$\begin{aligned}
\mu(x-y) &= \mu(x+(-y)) \\
&= min\{\mu(x+(-y)), \gamma(e_2+(-e_2))\} \\
&= (\mu \times \gamma)(x+(-y), e_2+(-e_2)) \\
&= (\mu \times \gamma)((x, e_2)+(-y, -e_2)) \\
&= (\mu \times \gamma)((x, e_2)-(y, e_2)) \\
&\geq (\mu \times \gamma)(x, e_2) \wedge (\mu \times \gamma)(y, e_2) \\
&= min\{min\{\mu(x), \gamma(e_2)\}, min\{\mu(y), \gamma(e_2)\}\} \\
&= \mu(x) \wedge \mu(y).
\end{aligned}$$

and

$$\begin{aligned}
\mu(xy) &= min\{\mu(xy), \gamma(e_2e_2)\} \\
&= (\mu \times \gamma)(xy, e_2e_2) \\
&= (\mu \times \gamma)((x, e_2) \circ (y, e_2)) \\
&\geq (\mu \times \gamma)(x, e_2) \wedge (\mu \times \gamma)(y, e_2) \\
&= min\{min\{\mu(x), \gamma(e_2)\}, min\{\mu(y), \gamma(e_2)\}\} \\
&= \mu(x) \wedge \mu(y).
\end{aligned}$$

Thus μ is a fuzzy LA-subring of R_1. Now

$$\begin{aligned}
\mu(xy) &= min\{\mu(xy), \gamma(e_2e_2)\} \\
&= (\mu \times \gamma)(xy, e_2e_2) \\
&= (\mu \times \gamma)((x, e_2) \circ (y, e_2)) \\
&= (\mu \times \gamma)((y, e_2) \circ (x, e_2)) \\
&= (\mu \times \gamma)(yx, e_2e_2) \\
&= min\{\mu(yx), \gamma(e_2e_2)\} \\
&= \mu(yx).
\end{aligned}$$

Hence μ is a fuzzy normal LA-subring of R_1. (2) is same as (1). □

4. Direct Product of Finite Fuzzy Normal LA-subrings

In this section, we define the direct product of fuzzy subsets $\mu_1, \mu_2, ..., \mu_n$, of LA-rings $R_1, R_2, ..., R_n$, respectively and examine the some fundamental properties of direct product of fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$.

Let $\mu_1, \mu_2, ..., \mu_n$ be fuzzy subsets of LA-rings $R_1, R_2, ..., R_n$, respectively. The direct product of fuzzy subsets $\mu_1, \mu_2, ..., \mu_n$ is denoted by $\mu_1 \times \mu_2 \times ... \times \mu_n$ and defined by $(\mu_1 \times \mu_2 \times ... \times \mu_n)(x_1, x_2, ..., x_n) = min\{\mu_1(x_1), \mu_2(x_2), ..., \mu_n(x_n)\}$.

A fuzzy subset $\mu_1 \times \mu_2 \times ... \times \mu_n$ of an LA-ring $R_1 \times R_2 \times ... \times R_n$ is said to be a fuzzy LA-subring of $R_1 \times R_2 \times ... \times R_n$ if

(1) $(\mu_1 \times \mu_2 \times ... \times \mu_n)(x - y) \geq min\{(\mu_1 \times \mu_2 \times ... \times \mu_n)(x), (\mu_1 \times \mu_2 \times ... \times \mu_n)(y)\}$,

(2) $(\mu_1 \times \mu_2 \times ... \times \mu_n)(xy) \geq min\{(\mu_1 \times \mu_2 \times ... \times \mu_n)(x), (\mu_1 \times \mu_2 \times ... \times \mu_n)(y)\}$ for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in R_1 \times R_2 \times ... \times R_n$.

A fuzzy LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ is said to be a fuzzy normal LA-subring of $R_1 \times R_2 \times ... \times R_n$ if $(\mu_1 \times \mu_2 \times ... \times \mu_n)(xy) = (\mu_1 \times \mu_2 \times ... \times \mu_n)(yx)$ for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in R_1 \times R_2 \times ... \times R_n$.

Let $A_1 \times A_2 \times ... \times A_n$ be a non-empty subset of an LA-ring $R = R_1 \times R_2 \times ... \times R_n$. The characteristic function of $A = A_1 \times A_2 \times ... \times A_n$ is denoted by χ_A and defined as

$$\chi_A : R \rightarrow [0, 1] \mid x = (x_1, x_2, ..., x_n) \rightarrow \chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

Lemma 4.1. *If $A_1, A_2, ..., A_n$ are LA-subrings of LA-rings $R_1, R_2, ..., R_n$, respectively, then $A_1 \times A_2 \times ... \times A_n$ is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ under the same operations defined as in [21].*

Proof. Straight forward. □

Proposition 4.1. *Let $A_1, A_2, ..., A_n$ be LA-subrings of LA-rings $R_1, R_2, ..., R_n$, respectively. Then $A_1 \times A_2 \times ... \times A_n$ is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$ if and only if the characteristic function χ_A of $A = A_1 \times A_2 \times ... \times A_n$ is a fuzzy normal LA-subring of $R_1 \times R_2 \times ... \times R_n$.*

Proof. Let $A = A_1 \times A_2 \times ... \times A_n$ be an LA-subring of $R_1 \times R_2 \times ... \times R_n$ and $a = (a_1, a_2, ..., a_n), b = (b_1, b_2, ..., b_n) \in R_1 \times R_2 \times ... \times R_n$. If $a, b \in A = A_1 \times A_2 \times ... \times A_n$, then by definition of characteristic function $\chi_A(a) = 1 = \chi_A(b)$. Since $a - b$ and $ab \in A$, A being an LA-subring of $R_1 \times R_2 \times ... \times R_n$. This implies that

$$\begin{aligned} \chi_A(a - b) &= 1 = 1 \wedge 1 = \chi_A(a) \wedge \chi_A(b) \\ \text{and } \chi_A(ab) &= 1 = 1 \wedge 1 = \chi_A(a) \wedge \chi_A(b). \end{aligned}$$

Thus $\chi_A(a-b) \geq min\{\chi_A(a), \chi_A(b)\}$ and $\chi_A(ab) \geq min\{\chi_A(a), \chi_A(b)\}$. Since ab and $ba \in A$, then by definition $\chi_A(ab) = 1 = \chi_A(ba)$, i.e., $\mu_{\chi_A}(ab) = \mu_{\chi_A}(ba)$. Similarly we have

$$\begin{aligned} \chi_A(a-b) &\geq min\{\chi_A(a), \chi_A(b)\}, \\ \chi_A(ab) &\geq min\{\chi_A(a), \chi_A(b)\}, \\ \chi_A(ab) &= \chi_A(ba), \end{aligned}$$

when $a, b \notin A$. Hence the characteristic function χ_A of $A = A_1 \times A_2 \times ... \times A_n$ is a fuzzy normal LA-subring of $R_1 \times R_2 \times ... \times R_n$.

Conversely, suppose that the characteristic function χ_A of $A = A_1 \times A_2 \times ... \times A_n$ is a fuzzy normal LA-subring of $R_1 \times R_2 \times ... \times R_n$. We have to show that $A = A_1 \times A_2 \times ... \times A_n$ is an LA-subring of $R_1 \times R_2 \times ... \times R_n$. Let $a, b \in A$, where $a = (a_1, a, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$, then by definition $\chi_A(a) = 1 = \chi_A(b)$. By our supposition

$$\begin{aligned} \chi_A(a-b) &\geq \chi_A(a) \wedge \chi_A(b) = 1 \wedge 1 = 1 \\ \text{and } \chi_A(ab) &\geq \chi_A(a) \wedge \chi_A(b) = 1 \wedge 1 = 1. \end{aligned}$$

Thus $\chi_A(a-b) = 1 = \chi_A(ab)$, i.e., $a-b$ and $ab \in A$. Hence $A = A_1 \times A_2 \times ... \times A_n$ is an LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. □

Lemma 4.2. *If $A = A_1 \times A_2 \times ... \times A_n$ and $B = B_1 \times B_2 \times ... \times B_n$ are two LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$, then their intersection $A \cap B$ is also an LA-subring of $R_1 \times R_2 \times ... \times R_n$.*

Proof. Straight forward. □

Theorem 4.1. *Let $A = A_1 \times A_2 \times ... \times A_n$ and $B = B_1 \times B_2 \times ... \times B_n$ be two LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Then $A \cap B$ is an LA-subring of $R_1 \times R_2 \times ... \times R_n$ if and only if the characteristic function χ_Z of $Z = A \cap B$ is a fuzzy normal LA-subring of $R_1 \times R_2 \times ... \times R_n$.*

Proof. Let $Z = A \cap B$ be an LA-subring of $R_1 \times R_2 \times ... \times R_n$ and $a = (a_1, a_2, ..., a_n), b = (b_1, b_1, ..., b_n) \in R_1 \times R_2 \times ... \times R_n$. If $a, b \in Z = A \cap B$, then by definition of characteristic function $\chi_Z(a) = 1 = \chi_Z(b)$. Since $a-b$ and $ab \in Z$, Z being an LA-subring. This implies that

$$\begin{aligned} \chi_Z(a-b) &= 1 = 1 \wedge 1 = \chi_Z(a) \wedge \chi_Z(b) \\ \text{and } \chi_Z(ab) &= 1 = 1 \wedge 1 = \chi_Z(a) \wedge \chi_Z(b). \end{aligned}$$

Thus $\chi_Z(a-b) \geq min\{\chi_Z(a), \chi_Z(b)\}$ and $\chi_Z(ab) \geq min\{\chi_Z(a), \chi_Z(b)\}$. As ab and $ba \in Z$, then by definition $\chi_Z(ab) = 1 = \chi_Z(ba)$, i.e., $\chi_Z(ab) = \chi_Z(ba)$. Similarly, we have

$$\begin{aligned} \chi_Z(a-b) &\geq min\{\chi_Z(a), \chi_Z(b)\}, \\ \chi_Z(ab) &\geq min\{\chi_Z(a), \chi_Z(b)\}, \\ \chi_Z(ab) &= \chi_Z(ba), \end{aligned}$$

when $a, b \notin Z$. Hence the characteristic function χ_Z of Z is a fuzzy normal LA-subring of $R_1 \times R_2 \times ... \times R_n$.

Conversely, assume that the characteristic function χ_Z of $Z = A \cap B$ is a fuzzy normal LA-subring of $R_1 \times R_2 \times ... \times R_n$. Let $a, b \in Z = A \cap B$, then by definition $\chi_Z(a) = 1 = \chi_Z(b)$. By our assumption

$$\begin{aligned} \chi_Z(a-b) &\geq \chi_Z(a) \wedge \chi_Z(b) = 1 \wedge 1 = 1 \\ \text{and } \chi_Z(ab) &\geq \chi_Z(a) \wedge \chi_Z(b) = 1 \wedge 1 = 1. \end{aligned}$$

Thus $\chi_Z(a-b) = 1 = \chi_Z(ab)$, i.e., $a-b$ and $ab \in Z$. Hence Z is an LA-subring of $R_1 \times R_2 \times ... \times R_n$. □

Corollary 4.1. *Let $\{A_i\}_{i \in I} = \{A_{i1} \times A_{i2} \times ... \times A_{in}\}_{i \in I}$ be a family of LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$, then $A = \cap A_i$ is an LA-subring of $R_1 \times R_2 \times ... \times R_n$ if and only if the characteristic function χ_A of $A = \cap A_i$ is a fuzzy normal LA-subring of $R_1 \times R_2 \times ... \times R_n$.*

Theorem 4.2. *If $\mu = \mu_1 \times \mu_2 \times ... \times \mu_n$ and $\gamma = \gamma_1 \times \gamma_2 \times ... \times \gamma_n$ are two fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$, then their intersection $\beta = \mu \cap \gamma$ is also a fuzzy normal LA-subring of $R_1 \times R_2 \times ... \times R_n$.*

Proof. Let $\mu = \mu_1 \times \mu_2 \times ... \times \mu_n$ and $\gamma = \gamma_1 \times \gamma_2 \times ... \times \gamma_n$ be two fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$. We have to show that $\beta = \mu \cap \gamma$ is also a fuzzy normal LA-subring of $R_1 \times R_2 \times ... \times R_n$. Let $z = (z_1, z_2, ..., z_n)$ and $w = (w_1, w_2, ..., w_n) \in R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned} \beta(z-w) &= (\mu \cap \gamma)(z-w) = min\{\mu(z-w), \gamma(z-w)\} \\ &\geq \{\{\mu(z) \wedge \mu(w)\} \wedge \{\gamma(z) \wedge \gamma(w)\}\} \\ &= \{\mu(z) \wedge \{\mu(w) \wedge \gamma(z)\} \wedge \gamma(w)\} \\ &= \{\mu(z) \wedge \{\gamma(z) \wedge \mu(w)\} \wedge \gamma(w)\} \\ &= \{\{\mu(z) \wedge \gamma(z)\} \wedge \{\mu(w) \wedge \gamma(w)\}\} \\ &= min\{(\mu \cap \gamma)(z), (\mu \cap \gamma)(w)\} \\ &= min\{\beta(z), \beta(w)\}. \end{aligned}$$

Thus $\beta((z_1, z_2, ..., z_n) - (w_1, w_2, ...w_n)) \geq min\{\beta(z_1, z_2, ..., z_n), \beta(w_1, w_2, ...w_n)\}$.

Similarly, we have

$$\beta((z_1, z_2, ..., z_n) \circ (w_1, w_2, ...w_n)) \geq min\{\beta(z_1, z_2, ..., z_n), \beta(w_1, w_2, ...w_n)\}.$$

Therefore $\beta = \mu \cap \gamma$ is a fuzzy LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. Now

$$\begin{aligned} & \beta((z_1, z_2, ..., z_n) \circ (w_1, w_2, ..., w_n)) \\ = \ & (\mu \cap \gamma)(z_1w_1, z_2w_2, ..., z_nw_n) \\ = \ & min\{\mu(z_1w_1, z_2w_2, ..., z_nw_n), \gamma(z_1w_1, z_2w_2, ..., z_nw_n)\} \\ = \ & min\{\mu(w_1z_1, w_2z_2, ..., w_nz_n), \gamma(w_1z_1, w_2z_2, ..., w_nz_n)\} \\ = \ & (\mu \cap \gamma)(w_1z_1, w_2z_2, ..., w_nz_n) \\ = \ & \beta((w_1, w_2, ..., w_n) \circ (z_1, z_2, ..., z_n)). \end{aligned}$$

Hence $\beta = \mu \cap \gamma$ is a fuzzy normal LA-subring of an LA-ring $R_1 \times R_2 \times ... \times R_n$. □

Corollary 4.2. *If $\{\mu_i\}_{i\in I} = \{\mu_{i1} \times \mu_{i2} \times ... \times \mu_{in}\}_{i\in I}$ is a family of fuzzy normal LA-subrings of an LA-ring $R_1 \times R_2 \times ... \times R_n$, then $\mu = \cap\mu_i$ is also a fuzzy normal LA-subring of $R_1 \times R_2 \times ... \times R_n$.*

Proposition 4.2. *Let $\mu = \mu_1 \times \mu_2 \times ... \times \mu_n$ and $\gamma = \gamma_1 \times \gamma_2 \times ... \times \gamma_n$ be fuzzy subsets of LA-rings $R = R_1 \times R_2 \times ... \times R_n$ and $R' = R'_1 \times R'_2 \times ... \times R'_n$ with left identities $e = (e_1, e_2, ..., e_n)$ and $e' = (e_1\prime, e_2\prime, ..., e_n\prime)$, respectively. If $\mu \times \gamma$ is a fuzzy LA-subring of an LA-ring $R \times R'$. Then at least one of the following two statements must hold.*

(1) $\mu(x) \leq \gamma(e')$, *for all* $x \in R$.

(2) $\mu(x) \leq \gamma(e)$, *for all* $x \in R'$.

Proof. Let $\mu \times \gamma$ be a fuzzy LA-subring of $R \times R'$. By contraposition, suppose that none of the statements (1) and (2) holds. Then we can find a and b in R and R', respectively such that

$$\mu(a) \geq \gamma(e') \text{ and } \mu(b) \geq \gamma(e).$$

Thus, we have

$$\begin{aligned} \mu \times \gamma(a, b) &= min\{\mu(a), \gamma(b)\} \\ &\geq min\{\mu(e), \gamma(e')\} \\ &= (\mu \times \gamma)(e, e\prime). \end{aligned}$$

Therefore $\mu \times \gamma$ is not a fuzzy LA-subring of $R \times R'$. Hence either $\mu(x) \leq \gamma(e')$, for all $x \in R$ or $\mu(x) \leq \gamma(e)$ for all $x \in R'$. □

Proposition 4.3. *Let $\mu = \mu_1 \times \mu_2 \times ... \times \mu_n$ and $\gamma = \gamma_1 \times \gamma_2 \times ... \times \gamma_n$ be fuzzy subsets of LA-rings $R = R_1 \times R_2 \times ... \times R_n$ and $R' = R'_1 \times R'_2 \times ... \times R'_n$ with left identities $e = (e_1, e_2, ..., e_n)$ and $e' = (e_1\prime, e_2\prime, ..., e_n\prime)$, respectively and $\mu \times \gamma$ is a fuzzy normal LA-subring of an LA-ring $R \times R'$. Then the following conditions are true.*

(1) *If $\mu(x) \leq \gamma(e')$, for all $x \in R$, then μ is a fuzzy normal LA-subring of R.*

(2) *If $\mu(x) \leq \gamma(e)$, for all $x \in R'$, then γ is a fuzzy normal LA-subring of R'.*

Proof. (1) Let $\mu(x) \leq \gamma(e')$, for all $x \in R$, and $y \in R$. We have to show that μ is a fuzzy normal LA-subring of R. Now

$$\begin{aligned}
\mu(x-y) &= \mu(x+(-y)) \\
&= min\{\mu(x+(-y)), \gamma(e'+(-e'))\} \\
&= (\mu\times\gamma)(x+(-y), e'+(-e')) \\
&= (\mu\times\gamma)((x,e')+(-y,-e')) \\
&= (\mu\times\gamma)((x,e')-(y,e')) \\
&\geq (\mu\times\gamma)(x,e') \wedge (\mu\times\gamma)(y,e') \\
&= min\{min\{\mu(x),\gamma(e')\}, min\{\mu(y),\gamma(e')\}\} \\
&= \mu(x) \wedge \mu(y).
\end{aligned}$$

and

$$\begin{aligned}
\mu(xy) &= min\{\mu(xy), \gamma(e'e')\} \\
&= (\mu\times\gamma)(xy, e'e') \\
&= (\mu\times\gamma)((x,e')\circ(y,e')) \\
&\geq (\mu\times\gamma)(x,e') \wedge (\mu\times\gamma)(y,e') \\
&= min\{min\{\mu(x),\gamma(e')\}, min\{\mu(y),\gamma(e')\}\} \\
&= \mu(x) \wedge \mu(y).
\end{aligned}$$

Thus μ is a fuzzy LA-subring of R. Now

$$\begin{aligned}
\mu(xy) &= min\{\mu(xy), \gamma(e'e')\} \\
&= (\mu \times \gamma)(xy, e'e') \\
&= (\mu \times \gamma)((x, e') \circ (y, e')) \\
&= (\mu \times \gamma)((y, e') \circ (x, e')) \\
&= (\mu \times \gamma)(yx, e'e') \\
&= min\{\mu(yx), \gamma(e'e')\} \\
&= \mu(yx).
\end{aligned}$$

Hence μ is a fuzzy normal LA-subring of R. (2) is same as (1). □

References

[1] R. J. Cho, J. Jezek and T. Kepka, Paramedial groupoids, Czechoslovak Math. J., 49 (1999) 277-290.

[2] K. A. Dib, N. Galhum and A. A. M. Hassan, Fuzzy rings and fuzzy ideals, Fuzzy Math., 4 (1996) 245-261.

[3] V. N. Dixit, R. Kumar and N. Ajmal, Fuzzy ideals and fuzzy prime ideals of a ring, Fuzzy Set Syst., 44 (1991) 127-138.

[4] K. C. Gupta and M. K. Kantroo, The intrinsic product of fuzzy subsets of a ring, Fuzzy Set Syst., 57 (1993) 103-110.

[5] J. Jezek and T. Kepka, Medial groupoids, Rozpravy CSAV Rada Mat. a Prir. Ved., 93/2, 1983, 93 pp.

[6] M. S. Kamran, Conditions for LA-semigroups to resemble associative structures, Ph.D. Thesis, Quaid-i-Azam University, Islamabad, 1993.

[7] N. Kausar, M. Waqar, Characterizations of non-associative rings by their intuitionistic fuzzy bi-ideals, Eur. J. Pure Appl. Math. 12 (2019), 226-250.

[8] N. Kausar, Characterizations of non-associative ordered semigroups by the properties of their fuzzy ideals with thresholds $(\alpha, \beta]$, Prikl. Diskr. Mat. 43 (2019), 37-59.

[9] N. Kausar, Direct product of finite intuitionistic fuzzy normal subrings over non-associative rings, Eur. J. Pure Appl. Math., 12 (2019), 622-648.

[10] M. A. Kazim and M. Naseeruddin, On almost semigroups, Alig. Bull. Math., 2 (1972), 1-7.

[11] N. Kausar, B. Islam, M. Javaid, S, Amjad, U. Ijaz, Characterizations of non-associative rings by the properties of their fuzzy ideals, J. Taibah Univ. Sci. 13 (2019), 820-833.

[12] N. Kausar, B. Islam, S. Amjad, M. Waqar, Intuitionistics fuzzy ideals with thresholds(,] in LA-rings, Eur. J. Pure Appli. Math. 12 (2019) 906-943.

[13] N. Kuroki, Regular fuzzy duo rings, Inform. Sci., 94 (1996), 119-139.

[14] W. J. Liu, Fuzzy invariant subgroups and ideals, Fuzzy Sets Syst., 8 (1982), 133-139.

[15] T. K. Mukherjee and M. K. Sen, On fuzzy ideals of a ring 1, Fuzzy Sets Syst., 21 (1987), 99-104.

[16] T. K. Mukherjee and M. K. Sen, Prime fuzzy ideals in rings, Fuzzy Sets Syst., 32 (1989), 337-341.

[17] M. T. A. Osman, On the direct product of fuzzy subgroups, Fuzzy Sets Syst., 12 (1984), 87-91.

[18] M. T. A. Osman, On some product of fuzzy subgroups, Fuzzy Sets Syst., 24 (1987), 79-86.

[19] P. V. Protic and N. Stevanovic, AG-test and some general properties of Abel-Grassmann's groupoids, Pure Math. Appl., 6 (1995) 371-383.

[20] A. K. Ray, Product of fuzzy subgroups, Fuzzy Sets Syst., 105 (1999), 181-183.

[21] T. Shah, N. Kausar and I. Rehman, Intuitionistic fuzzy normal subrings over a non-associative ring, An. St. Univ. Ovidius Constanta, 1 (2012) 369-386.

[22] T. Shah, N. Kausar, Characterizations of non-associative ordered semigroups by their fuzzy bi-ideals, Theor. Comput. Sci. 529 (2014), 96-110.

[23] T. Shah and I. Rehman, On LA-rings of finitely non-zero functions, Int. J. Contemp. Math. Sci., 5 (2010) 209-222.

[24] H. Sherwood, Product of fuzzy subgroups, Fuzzy Sets Syst., 11 (1983) 65-77.

[25] U. M. Swamy and K. L. N. Swamy, Fuzzy prime ideals of rings, J. Math. Anal. Appl., 134 (1988) 94-103.

[26] L. A. Zadeh, Fuzzy sets, Inform. Control, 8 (1965) 338-353.

[27] S. A. Zaid, On normal fuzzy subgroups, J. Fac. Educ. Ain Shams Univ. Cairo, 13 (1988), 115-125.