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SHEHU TRANSFORM AND APPLICATIONS TO CAPUTO-FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this manuscript we establish the expressions of the Shehu transform for fractional Riemann-Liouville and Caputo operators. With the help of this new integral transform we solve higher order fractional differential equations in the Caputo sense. Three illustrative examples are discussed to show our approach.

1. Introduction

One of the most effective methods to solve differential equations is to use integrals transformation. The main advantage of this method is that it transforms the differential problem to an algebraic problem. We recall that the Laplace's transformation which is widely used to solve differential and integral equations. The Sumudu transform was first defined in 1993 by Watugala who used it to solve engineering control problems [17].

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The Shehu transform was introduced recently by Shehu Maitama and Weidong Zhao [16] and it is a generalization of the Laplace and the Sumudu integral transforms. The authors have used it to solve ordinary and partial differential equations [16].

The Shehu transform is obtained over the set A by [16] :

$$A = \left\{ f(t) : \exists N, \eta_1, \eta_2 > 0, \ \ |f(t)| < N \exp\left(\frac{|t|}{\eta_i}\right), \text{ if } t \in (-1)^i \times [0, \infty) \right\}. \tag{1.1}$$

by

$$\mathbb{H}[f(t)] = V(s, u) = \int_0^\infty \exp\left(-\frac{st}{u}\right) f(t)\, dt. \tag{1.2}$$

Obviously, the Shehu transform is linear as the Laplace and Sumudu transformations.

Theorem 1.1. *[16] If the function $f^{(n)}(t)$ is the nth derivative of the function $f(t) \in A$, then its Shehu transform is defined by*

$$\mathbb{H}\left[f^{(n)}(t)\right] = \left(\frac{u}{s}\right)^{-n} V(s, u) - \sum_{k=0}^{n-1} \left(\frac{u}{s}\right)^{(k+1)-n} f^{(k)}(0), \quad n \geq 1. \tag{1.3}$$

Some properties of Shehu transform are given in [16].

For our results we need some other definitions and some properties.

Definition 1.1. *A generalization of the exponential function is given by [10]*

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \alpha \in \mathbb{C}, \ \ \mathrm{Re}(\alpha) > 0. \tag{1.4}$$

A generalization of Mittag-Leffler function $E_\alpha(z)$ is defined as follows [18]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha, \beta \in \mathbb{C}, \ \ \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0. \tag{1.5}$$

A generalization of Mittag-Leffler function $E_{\alpha,\beta}(z)$ of (1.5) is introduced by Prabhakar [14], as follows:

$$E^\gamma_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{\gamma_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \qquad \alpha, \beta, \gamma \in \mathbb{C}, \ \ \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0. \tag{1.6}$$

where γ_k denotes the familiar Pochhammer symbol.

Lemma 1.1. *[6] In the complex plane C, for any $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$ and $\omega \in \mathbb{C}$.*

$$S\left(t^{\gamma-1} E^\gamma_{\alpha,\beta}(\omega t^\alpha)\right) = u^{\beta-1}(1 - \omega u^\alpha)^{-\gamma}. \tag{1.7}$$

Corollary 1.1. *Sumudu transform of Mittag-leffer function $E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+\beta)}$, $\alpha, \beta \in \mathbb{C}$, $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ exists and given by*

$$S\left(t^{\gamma-1} E_{\alpha,\beta}\left(\omega t^\beta\right)\right) = u^{\gamma-1}\left(1 - \omega u^\beta\right)^{-1}. \tag{1.8}$$

Definition 1.2. *Let $f \in L^1(a,b)$. If $\alpha \geq 0$, then left sided Riemann–Liouville fractional integral of order α is defined by [1, 12, 13]*

$$
\begin{aligned}
I_{0^+}^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau \\
&= \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t), \quad \alpha > 0, \quad t > 0, \qquad (1.9)\\
I^0 f(t) &= f(t).
\end{aligned}
$$

Definition 1.3. *Let $f \in L_1(a,b)$, and $m-1 < \alpha \leq m$. The Caputo fractional derivative of order α $(\alpha > 0)$ is defined as [1, 5, 12]*

$$
{}^{C}D_{0^+}^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau, & m-1 < \alpha \leq m. \\ \frac{\partial^m}{\partial t^m} f(t) \ \text{if } \alpha = m. \end{cases}
$$

Remark 1.1. *[13] Under the terms of the previous definition, we have*

$$
{}^{C}D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} t^{m-\alpha-1} * f^{(m)}(t). \qquad (1.10)
$$

Lemma 1.2. *[9, Lemma 2.22 p.96] If $f(t) \in AC^n[a,b]$ or $f(t) \in C^n[a,b]$, then*

$$
\left(I_{0^+}^{\alpha}{}^{C}D_{0^+}^{\alpha}\right) f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k. \qquad (1.11)
$$

As the next theorem shows, the Shehu transform is closely connected with the Sumudu transform,

Theorem 1.2. *[3] Let $f(t) \in A$ with Sumudu transform $G(u)$. Then the Shehu transform $V(s,u)$ of $f(t)$ is given by*

$$
V(s,u) = \frac{u}{s} G\left(\frac{u}{s}\right). \qquad (1.12)
$$

Lemma 1.3. *[15, p. 140-141] If $f \in L_1(a,b)$ for any $b > a$ and of exponential order. Then $I_{0^+}^{\alpha} f(t)$ also of exponential order.*

2. Main result

In this section, we present some results on the transformation of Shehu, as a complementary result of what can be seen in [16].

Theorem 2.1. *Let $a \in \mathbb{C}^*$ and let $f(at) \in A$. If $V(s,u)$ denote the Shehu transform of f. Then*

$$
\mathbb{H}(f(at)) = \frac{1}{a} V(s, au).
$$

Proof. Using the definition of Shehu transform Eq.(1.1), we get

$$\mathbb{H}\left(f\left(at\right)\right)=\int_0^\infty \exp\left(-\frac{s}{u}t\right)f\left(at\right)dt.$$

If we set $\tau=at$ $(t=\tau/a)$, then

$$\begin{aligned}\mathbb{H}\left(f\left(at\right)\right) &= \frac{1}{a}\int_0^\infty \exp\left(-\frac{s}{au}\tau\right)f\left(\tau\right)d\tau \\ &= \frac{1}{a}V\left(s,au\right).\end{aligned}$$

□

Theorem 2.2. *Let $a\in\mathbb{C}^*$ and let $f\left(t\right)\in A$ with Shehu transform $V\left(s,u\right)$. Then*

$$\mathbb{H}\left(e^{at}f\left(t\right)\right)=V\left(s-au,u\right).$$

Proof. Using Eq.(1.2), we have

$$\begin{aligned}\mathbb{H}\left(e^{at}f\left(t\right)\right) &= \int_0^\infty \exp\left(at-\frac{s}{u}t\right)f\left(t\right)dt \\ &= \int_0^\infty \exp\left(-\frac{s-au}{u}t\right)f\left(t\right)dt.\end{aligned}$$

By setting $s'=s-au$, we get

$$\begin{aligned}\mathbb{H}\left(e^{at}f\left(t\right)\right) &= \int_0^\infty \exp\left(-\frac{s'}{u}t\right)f\left(t\right)dt \\ &= V(s',u)=V(s-au,u).\end{aligned}$$

□

Theorem 2.3. *For $x>0$, the Shehu transform of t^{x-1} is*

$$V\left(s,u\right)=\Gamma\left(x\right)\left(\frac{u}{s}\right)^x. \tag{2.1}$$

Proof. For $x>0$, the Gamma function is defined by

$$\Gamma\left(x\right)=\int_0^\infty \tau^{x-1}e^{-\tau}d\tau.$$

If we set $\tau=\frac{s}{u}t\left(t=\frac{u}{s}\tau\right)$, then we have

$$\begin{aligned}\Gamma\left(x\right) &= \int_0^\infty \left(\frac{s}{u}t\right)^{x-1}e^{-\frac{s}{u}t}\frac{s}{u}dt \\ &= \left(\frac{s}{u}\right)^x\int_0^\infty t^{x-1}e^{-\frac{s}{u}t}dt \\ &= \left(\frac{s}{u}\right)^x\mathbb{H}\left(t^{x-1}\right).\end{aligned}$$

Then, $\mathbb{H}\left(t^{x-1}\right)=\Gamma\left(x\right)\left(\frac{u}{s}\right)^x$. □

Lemma 2.1. *In the complex plane C, for any* $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$, $\operatorname{Re}(\gamma) > 0$ *and* $\omega \in \mathbb{C}$. *Shehu transform of* $E^{\gamma}_{\alpha,\beta}(\omega t^{\alpha})$ *is given by*

$$\mathbb{H}\left(t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega t^{\alpha})\right) = \left(\frac{u}{s}\right)^{\beta}\left(1-\omega\left(\frac{u}{s}\right)^{\alpha}\right)^{-\gamma}. \tag{2.2}$$

Proof. Using Eqs.(1.7), (1.12), we get

$$\begin{aligned}\mathbb{H}\left(t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega t^{\alpha})\right) &= \left(\frac{u}{s}\right)\left(\frac{u}{s}\right)^{\beta-1}\left(1-\omega\left(\frac{u}{s}\right)^{\alpha}\right)^{-\gamma} \\ &= \left(\frac{u}{s}\right)^{\beta}\left(1-\omega\left(\frac{u}{s}\right)^{\alpha}\right)^{-\gamma}.\end{aligned}$$

□

Corollary 2.1. *Shehu transform of Mittag-Leffler function* $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(\alpha k+\beta)}$, $\alpha, \beta \in \mathbb{C}, \operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$ *exists and given by*

$$\mathbb{H}\left(t^{\beta-1}E_{\alpha,\beta}(\omega t^{\alpha})\right) = \left(\frac{u}{s}\right)^{\beta}\left(1-\omega\left(\frac{u}{s}\right)^{\alpha}\right)^{-1}. \tag{2.3}$$

Proof. Using Eq.(2.2) and since $E_{\alpha,\beta}(z) = E^{1}_{\alpha,\beta}(z)$, we get the desired result. □

The next Theorem shows the Shehu transform convolution theorem.

Theorem 2.4. *Let* $f(t)$ *and* $g(t)$ *be in* A, *having Shehu transforms* $V(s,u)$ *and* $W(s,u)$, *respectively. Then the Shehu transform of the convolution of* f *and* g

$$(f*g)(t) = \int_0^{\infty} f(t)\, g(t-\tau)\, d\tau, \tag{2.4}$$

is given by

$$\mathbb{H}((f*g)(t)) = V(s,u)\, W(s,u). \tag{2.5}$$

Proof. First, recall that the Sumudu transform of $f*g$ is given by [2]

$$S((f*g)(t)) = uF(u)G(u). \tag{2.6}$$

where $F(u)$ and $G(u)$, are the Sumudu transforms of $f(t)$ and $g(t)$ respectively. Now, since, by the relation (1.12),

$$\begin{aligned}H[v(t)*w(t)] &= \frac{u}{s}S[v(t)*w(t)] \\ &= \left(\frac{u}{s}\right)^{2}F\left(\frac{u}{s}\right)G\left(\frac{u}{s}\right) \\ &= \left(\frac{u}{s}\right)F\left(\frac{u}{s}\right)\times\left(\frac{u}{s}\right)G\left(\frac{u}{s}\right) \\ &= V(s,u)\, W(s,u).\end{aligned}$$

□

Theorem 2.5. *Let f satisfy the conditions of Lemma 1.3. Then the Shehu transform of $I_t^{\alpha} f(t)$ exists and given by*

$$\mathbb{H}\left(I_t^{\alpha} f(t)\right) = \left(\frac{u}{s}\right)^{\alpha} V(s,u). \tag{2.7}$$

Proof. Since by equation Eq.(1.9) above, $I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t)$, then by Theorems 2.3 and Theorem 2.4, we have,

$$\begin{aligned} \mathbb{H}\left(I_t^{\alpha} f(t)\right) &= \frac{1}{\Gamma(\alpha)} \mathbb{H}\left(t^{\alpha-1}\right) \mathbb{H}(f(t)) \\ &= \frac{1}{\Gamma(\alpha)} \Gamma(\alpha) \left(\frac{u}{s}\right)^{\alpha} V(s,u) \\ &= \left(\frac{u}{s}\right)^{\alpha} V(s,u). \end{aligned}$$

□

Theorem 2.6. *If $f \in AC^n(a,b)$ for any $b > a$ and of exponential order. Then*

$$\mathbb{H}\left({}^C D_0^{\alpha} f(t)\right) = \left(\frac{s}{u}\right)^{\alpha} V(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{\alpha-(k+1)} f^{(k)}(0). \tag{2.8}$$

Proof. Since $\left(I_{0+}^{\alpha C}\mathrm{D}_{0+}^{\alpha}\right) f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k$, by Lemma 1.2, we have

$$\mathbb{H}\left(\left(I_{0+}^{\alpha C}\mathrm{D}_{0+}^{\alpha}\right) f(t)\right) = \mathbb{H}\left(f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k\right),$$

thus

$$\left(\frac{u}{s}\right)^{\alpha} \mathbb{H}\left({}^C\mathrm{D}_{0+}^{\alpha} f(t)\right) = V(s,u) - \sum_{k=0}^{n-1} \left(\frac{u}{s}\right)^{k+1} f^{(k)}(0),$$

finally, we get

$$\mathbb{H}\left({}^C\mathrm{D}_{0+}^{\alpha} f(t)\right) = \left(\frac{u}{s}\right)^{-\alpha} V(s,u) - \sum_{k=0}^{n-1} \left(\frac{u}{s}\right)^{k+1-\alpha} f^{(k)}(0).$$

By other method, we can use Eq. (1.12), and that [8]

$$\mathbb{S}\left({}^C\mathrm{D}_0^{\alpha} f(t)\right) = u^{-\alpha}\left(G(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0)\right).$$

In fact,

$$\begin{aligned} \mathbb{H}\left({}^C\mathrm{D}_{0+}^{\alpha} f(t)\right) &= \left(\frac{u}{s}\right)\left(\frac{u}{s}\right)^{-\alpha}\left(G\left(\frac{u}{s}\right) - \sum_{k=0}^{n-1}\left(\frac{u}{s}\right)^k f^{(k)}(0)\right) \\ &= \left(\frac{u}{s}\right)^{1-\alpha}\left(\frac{s}{u} V(s,u) - \sum_{k=0}^{n-1}\left(\frac{u}{s}\right)^k f^{(k)}(0)\right) \\ &= \left(\frac{u}{s}\right)^{-\alpha}\left(V(s,u) - \sum_{k=0}^{n-1}\left(\frac{u}{s}\right)^{k+1} f^{(k)}(0)\right). \end{aligned}$$

By anothor method, we have by Remark 1.1, ${}^{C}\mathrm{D}_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} t^{n-\alpha-1} * f^{(n)}(t)$, $n-1 < \alpha \leq n$, then by using Theorems 2.3 and Theorem 2.4, we obtain

$$\begin{aligned} \mathbb{H}\left({}^{C}\mathrm{D}_{0+}^{\alpha} f(t)\right) &= \mathbb{H}\left(\frac{1}{\Gamma(n-\alpha)} t^{n-\alpha-1} * f^{(n)}(t)\right) \\ &= \frac{1}{\Gamma(n-\alpha)} \mathbb{H}\left(t^{n-\alpha-1}\right) \mathbb{H}\left(f^{(n)}(t)\right) \\ &= \frac{1}{\Gamma(n-\alpha)} \Gamma(n-\alpha) \left(\frac{u}{s}\right)^{n-\alpha} \left[\left(\frac{s}{u}\right)^{n} V(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{n-(k+1)} f^{(k)}(0)\right] \\ &= \left(\frac{u}{s}\right)^{-\alpha} \left[V(s,u) - \sum_{k=0}^{n-1} \left(\frac{u}{s}\right)^{k+1} f^{(k)}(0)\right]. \end{aligned}$$

□

3. Applications

We take into consideration a general linear ordinary differential equation with fractional order as follows:

$$ {}^{C}\mathrm{D}_{0+}^{\alpha} y(t) = \sum_{i=1}^{n} b_i y^{(i)}(t) + g(t), \qquad n-1 < \alpha \leq n \tag{3.1}$$

subject to the initial condition

$$ y^{(i)}(0) = a_i, \ i = 0, ..., n-1, \tag{3.2}$$

where $a_i, b_j \in \mathbb{R}$, $g(t) \in A$.

When we get Shehu transform of (3.1) taking into consideration (1.3) and (2.8), we obtain Shehu transform of (3.1) as follows

$$\mathbb{H}\left({}^{C}\mathrm{D}_{0+}^{\alpha} y(t)\right) = \mathbb{H}\left(\sum_{i=1}^{n} b_i y^{(i)}(t) + g(t)\right),$$

By the linearity of Shehu transform, we have

$$\begin{aligned} \mathbb{H}\left({}^{C}\mathrm{D}_{0+}^{\alpha} y(t)\right) &= \sum_{i=0}^{n} b_i \mathbb{H}\left(y^{(i)}(t)\right) + \mathbb{H}(g(t)), \\ &= b_0 y(t) + \sum_{i=1}^{n} b_i \mathbb{H}\left(y^{(i)}(t)\right) + \mathbb{H}(g(t)) \end{aligned}$$

Using Eqs.(1.3), (2.8), we obtain

$$\begin{aligned} \left(\frac{u}{s}\right)^{-\alpha} V(s,u) - \sum_{k=0}^{n-1} \left(\frac{u}{s}\right)^{k+1-\alpha} y^{(k)}(0) &= b_0 V(s,u) + \sum_{i=1}^{n} b_i \left[\left(\frac{u}{s}\right)^{-i} V(s,u)\right. \\ &\left. - \sum_{k=0}^{i-1} \left(\frac{u}{s}\right)^{k+1-i} y^{(k)}(0)\right] + \mathbb{H}(g(t)) \end{aligned}$$

$$\left(\frac{u}{s}\right)^{-\alpha}V(s,u)-\sum_{i=0}^{n}b_i\left(\frac{u}{s}\right)^{-i}V(s,u) = \sum_{k=0}^{n-1}a_k\left(\frac{u}{s}\right)^{k+1-\alpha} - \sum_{i=1}^{n}b_i\sum_{k=0}^{i-1}a_k\left(\frac{u}{s}\right)^{k+1-i}+\mathbb{H}(g(t))$$

$$V(s,u) = \left(\left(\frac{u}{s}\right)^{-\alpha}-\sum_{i=0}^{n}b_i\left(\frac{u}{s}\right)^{-i}\right)^{-1}\left(\sum_{k=0}^{n-1}a_k\left(\frac{u}{s}\right)^{k+1-\alpha} - \sum_{i=1}^{n}b_i\sum_{k=0}^{i-1}a_k\left(\frac{u}{s}\right)^{k+1-i}+\mathbb{H}(g(t))\right). \tag{3.3}$$

Operating the inverse Shehu transform on both sides of Eq. (3.3), we get the solution of Eq. (3.1) as follows:

$$y(t) = \mathbb{H}^{-1}\left[\left(\left(\frac{u}{s}\right)^{-\alpha}-\sum_{i=0}^{n}b_i\left(\frac{u}{s}\right)^{-i}\right)^{-1}\left(\sum_{k=0}^{n-1}a_k\left(\frac{u}{s}\right)^{k+1-\alpha} - \sum_{i=1}^{n}b_i\sum_{k=0}^{i-1}a_k\left(\frac{u}{s}\right)^{k+1-i}+\mathbb{H}(g(t))\right)\right]. \tag{3.4}$$

Example 3.1. *When $n=1, b_0=-1$ and $b_1=g(t)=0$, we obtain [11]*

$$^{C}D^{\alpha}y(t)+y(t)=0, \quad 0<\alpha\leq 1,\ t>0, \tag{3.5}$$

with initial condition

$$y(0)=1. \tag{3.6}$$

Substituting n, b_0, b_1 and g in (3.4), we get :

$$y(t)=\mathbb{H}^{-1}\left[\left(\left(\frac{u}{s}\right)^{-\alpha}-\sum_{i=0}^{1}b_i\left(\frac{u}{s}\right)^{-i}\right)^{-1}\left(\frac{u}{s}\right)^{1-\alpha}\right].$$

$$y(t)=\mathbb{H}^{-1}\left[\left(\frac{u}{s}\right)\left(1-(-1)\left(\frac{u}{s}\right)^{\alpha}\right)^{-1}\right]$$

Thus, by Eq.(2.3), we have

$$V(s,u)=H\left(E_{\alpha}\left(-t^{\alpha}\right)\right). \tag{3.7}$$

When we get the inverse Sumudu transform of (3.7) , we find exact solution of Eq.(3.5) as follows:

$$y(t)=E_{\alpha}\left(-t^{\alpha}\right).$$

Example 3.2. *Below we give the following particular example, which where debate in the literature and here important application in several world problems.*

Consider the Bagley-Torvik equation [7]

$$D^2y(t)+^{C}D^{3/2}y(t)+y(t)=t+1, \tag{3.8}$$

with the initial conditions

$$y(0) = y'(0) = 1. \tag{3.9}$$

In this case, we have $n = 2, b_0 = b_2 = -1, b_1 = 0$ *and* $g(t) = t + 1$. *Applying Eq.(3.4), we get*

$$\begin{aligned} y(t) &= \mathbb{H}^{-1}\left[\left(\left(\frac{u}{s}\right)^{-3/2} - \sum_{i=0}^{2} b_i \left(\frac{u}{s}\right)^{-i}\right)^{-1} \left(\sum_{k=0}^{1} a_k \left(\frac{u}{s}\right)^{k+1-3/2}\right.\right. \\ &\quad \left.\left. - \sum_{i=1}^{2} b_i \sum_{k=0}^{i-1} a_k \left(\frac{u}{s}\right)^{k+1-i} + \frac{u}{s} + \left(\frac{u}{s}\right)^2\right)\right]. \end{aligned} \tag{3.10}$$

Then,

$$\begin{aligned} y(t) &= \mathbb{H}^{-1}\left(\frac{\left(\frac{u}{s}\right)^{-1/2} + \left(\frac{u}{s}\right)^{1/2} + \left(\frac{u}{s}\right)^{-1} + 1 + \frac{u}{s} + \left(\frac{u}{s}\right)^2}{\left(\frac{u}{s}\right)^{-3/2} + \left(\frac{u}{s}\right)^{-2} + 1}\right) \\ &= \mathbb{H}^{-1}\left(\frac{\left(\frac{u}{s}\right)^{-1} + \left(\frac{u}{s}\right)^{-1/2} + \frac{u}{s}}{\left(\frac{u}{s}\right)^{-2} + \left(\frac{u}{s}\right)^{-3/2} + 1} + \frac{1 + \left(\frac{u}{s}\right)^{1/2} + \left(\frac{u}{s}\right)^2}{\left(\frac{u}{s}\right)^{-2} + \left(\frac{u}{s}\right)^{-3/2} + 1}\right) \\ &= \mathbb{H}^{-1}\left(\frac{u}{s} + \left(\frac{u}{s}\right)^2\right). \end{aligned} \tag{3.11}$$

Taking the inverse Shehu transform of Eq. (3.11), yields

$$y(t) = t + 1,$$

which is the exact solution.

Example 3.3. *Consider the following homogeneous fractional ordinary differential equation : [4]*

$${}^{C}D^{\frac{1}{2}} y(t) + y(t) = t^2 + \frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)} t^{\frac{3}{2}}, \quad t > 0, \tag{3.12}$$

with initial condition

$$y(0) = 0. \tag{3.13}$$

In order to find exact solution of (3.12), we apply Eq.(3.4), for $n = 1$, $b_0 = -1, b_1 = 0$ *and* $g(t) = t^2 + \frac{\Gamma(3)}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}}$, *we obtain*

$$\begin{aligned} y(t) &= \mathbb{H}^{-1}\left(\left(\left(\frac{u}{s}\right)^{-\frac{1}{2}} + 1\right)^{-1}\left(2\left(\frac{u}{s}\right)^3 + \Gamma(3)\left(\frac{u}{s}\right)^{\frac{5}{2}}\right)\right) \\ &= \mathbb{H}^{-1}\left(\frac{2\left(\frac{u}{s}\right)^3 + \Gamma(3)\left(\frac{u}{s}\right)^{\frac{5}{2}}}{\left(\frac{u}{s}\right)^{-\frac{1}{2}} + 1}\right) \\ &= \mathbb{H}^{-1}\left(2\left(\frac{u}{s}\right)^3\right). \end{aligned}$$

When we take the inverse Shehu transform of $2\left(\frac{u}{s}\right)^3$, *we get the analytical solution of Eq.(3.12)*

$$y(t) = t^2.$$

4. Conclusions

In the field of fractional calculus finding a new integral transform for solving the ordinary od partial fractional differential equations it is always useful. In this manuscript, the newly suggested Shehu integral transform was applied to solve hgher order fractional differential equations with Caputo derivative. We show the efficiency and and high accuracy of the suggested integral transform.

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