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HERMITE-HADAMARD TYPE INEQUALITIES FOR m-CONVEX AND (α, m)-CONVEX STOCHASTIC PROCESSES

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Abstract. In this paper, the concepts of m-convex and (α, m)-convex stochastic processes are introduced. Several new inequalities of Hermite-Hadamard type for differentiable m-convex and (α, m)-convex stochastic processes are established. The results obtained in this work are the generalizations of the known results.

1. Introduction

Stochastic convexity and its applications is of great importance in statistics and probability, because it provides numerical approximations for existing probabilistic quantities.

In 1980, Nikodem [10] defined convex stochastic processes and investigated their properties. In 1988, Shaked et al. [16] defined stochastic convexity and gave its applications. In 1992, Skowronski [17] introduced some new types of convex stochastic processes and obtained some further results on these processes. In 2012, Kotrys [6] extended classical Hermite-Hadamard inequality to convex stochastic processes. In recent years, there have been many studies on the above mentioned processes. For recent generalizations and improvements on convex stochastic processes, please refer to [4]- [8], [11]- [15], [19].

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2. Preliminaries

Let (Ω, κ, P) be a probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is κ-measurable. Let $I \subset \mathbb{R}$ be an interval. Then, a function $X : I \times \Omega \to \mathbb{R}$ is called a stochastic process if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

Let $P - \lim$ and $E[X(t, \cdot)]$ denote the limit in probability and the expectation value of random variable $X(t, \cdot)$, respectively. Then, a stochastic process $X : I \times \Omega \to \mathbb{R}$ is called

(1) continuous in probability in the interval I, if

$$P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot)$$

for all $t_0 \in I$.

(2) mean square continuous in the interval I, if

$$\lim_{t \to t_0} E\left[(X(t, \cdot) - X(t_0, \cdot))^2\right] = 0$$

for all $t_0 \in I$.

(3) mean-square differentiable at a point $t \in I$ if there is a random variable $X'(t, \cdot) : I \times \Omega \to \mathbb{R}$ such that

$$X'(t, \cdot) = P - \lim_{t \to t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}.$$

Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process with $E\left[(X(t, \cdot))^2\right] < \infty$ for all $t \in I$. Let $u = t_0 < t_1 < t_2 < \ldots < t_n = b$ be a partition of $[u, v]$ if the identity

$$\lim_{n \to \infty} E\left[\left(\sum X(\Theta_k)(t_k - t_{k-1}) - Y\right)^2\right] = 0$$

holds for all normal sequences of partitions of the interval $[u, v]$ and for all $\Theta_k \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$. Then, we can write

$$Y(\cdot) = \int_u^v X(t, \cdot)\, dt \quad \text{(a.e.)}.$$

The assumption of the mean-square continuity of the stochastic process X is enough for the mean-square integral to exist.

Definition 2.1. *[10] The stochastic process $X : I \times \Omega \to \mathbb{R}$ is convex if for all $\lambda \in [0, 1]$ and $u, v \in I$ the inequality*

$$X(\lambda u + (1 - \lambda) v, \cdot) \leq \lambda X(u, \cdot) + (1 - \lambda) X(v, \cdot) \quad \text{(a.e.)} \tag{2.1}$$

is satisfied. If the inequality (2.1) is assumed only for $\lambda = \frac{1}{2}$, then the stochastic process X is called Jensen-convex or $\frac{1}{2}$-convex.

In [6], Kotrys defined convex stochastic processes as following:

Theorem 2.1. *Let $X : I \times \Omega \to \mathbb{R}$ be a Jensen-convex stochastic process and mean-square continuous in the interval I. Then the following inequality holds for all $u, v \in I$, $u < v$.*

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{1}{v-u}\int_u^v X(t,\cdot)\,dt \leq \frac{X(u,\cdot)+X(v,\cdot)}{2} \quad (a.e.). \tag{2.2}$$

Definition 2.2. *[18] Let $m \in [0,1]$. The function $f : [0,c] \to \mathbb{R}$, $c > 0$, is said to be m-convex, if*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

is satisfied for every $x, y \in [0,c]$ and $t \in [0,1]$.

Definition 2.3. *[9] Let $\alpha, m \in [0,1]$. The function $f : [0,c] \to \mathbb{R}$, $c > 0$, is said to be (α, m)-convex, if*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha})f(y)$$

is satisfied for every $x, y \in [0,c]$ and $t \in [0,1]$.

For further information about m-convex and (α, m)-convex functions, please refer to [1], [2], [5], [13].

Theorem 2.2. *[3] Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b). If $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)\left(|f'(a)|+|f'(b)|\right)}{8}.$$

Theorem 2.3. *[3] Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b). Suppose $p \in \mathbb{R}$ with $p > 1$. If $|f'|^q$ is convex on $[a,b]$ for $q \in \mathbb{R}$ with $q > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

3. Main Results

In order to establish our main results we give the following definitions and lemma:

Definition 3.1. *The stochastic process $X : [a,b] \times \Omega \to R$ is said to be m-convex where $m \in [0,1]$, if*

$$X(\lambda u + m(1-\lambda)v, \cdot) \leq \lambda X(u,\cdot) + m(1-\lambda)X(v,\cdot)$$

holds for all $u, v \in [a,b]$ and $\lambda \in [0,1]$.

Definition 3.2. *The stochastic process $X : [a,b] \times \Omega \to R$ is said to be (α, m)-convex where $(\alpha, m) \in [0,1]^2$, if*

$$X(\lambda u + m(1-\lambda)v, \cdot) \leq \lambda^{\alpha} X(u,\cdot) + m(1-\lambda^{\alpha})X(v,\cdot)$$

holds for all $u, v \in [a,b]$ and $\lambda \in [0,1]$.

Lemma 3.1. *[11] Let $X : I^\circ \subseteq R \times \Omega \to R$ be a mean-square differentiable stochastic process on I° and $u, v \in I^\circ$ with $u < v$. If X' is mean-square integrable on $[u, v]$, then the following inequality holds almost everywhere:*

$$\begin{aligned} & \frac{X(u,\cdot) + X(v,\cdot)}{2} - \frac{1}{v-u} \int_u^v X(t,\cdot)\, dt \\ = & \frac{v-u}{2} \int_0^1 (1-2\lambda) X'(\lambda u + (1-\lambda) v, \cdot)\, d\lambda. \end{aligned}$$

Now we obtain results for stochastic processes whose derivatives absolute values raise to some certain power are m-convex and (α, m)-convex.

Theorem 3.1. *Suppose $b^* > 0$. Let $X : I \subset [a, b^*] \times \Omega \to R$ be a differentiable stochastic process on I° and let X' be mean-square integrable on $[u, v]$ where $u, v \in I$ with $u < v$. If $|X'|$ is m-convex stochastic process on $[u, v]$ for $m \in (0, 1]$, then the following inequality holds almost everywhere:*

$$\left| \frac{X(u,\cdot) + X(v,\cdot)}{2} - \frac{1}{v-u} \int_u^v X(t,\cdot)\, dt \right| \le \frac{v-u}{8} \left[|X'(u,\cdot)| + m \left| X'\left(\frac{v}{m}, \cdot\right) \right| \right].$$

Proof. From Lemma 3.1, we obtain

$$\begin{aligned} & \left| \frac{X(u,\cdot) + X(v,\cdot)}{2} - \frac{1}{v-u} \int_u^v X(t,\cdot)\, dt \right| \\ \le & \frac{v-u}{2} \int_0^1 |1-2\lambda| \left| X'(\lambda u + (1-\lambda) v, \cdot) \right| d\lambda. \end{aligned}$$

Since $|X'|$ is m-convex stochastic process on $[u, v]$ for all $u, v \in I$, $\lambda \in [0, 1]$ and $m \in (0, 1]$, we have

$$\begin{aligned} |X'((\lambda u + (1-\lambda) v), \cdot)| &= \left| X'\left(\lambda u + m(1-\lambda)\frac{v}{m}, \cdot\right) \right| \\ &\le \lambda |X'(u,\cdot)| + m(1-\lambda) \left| X'\left(\frac{v}{m}, \cdot\right) \right|. \end{aligned}$$

Hence we have

$$\begin{aligned} & \left| \frac{X(u,\cdot) + X(v,\cdot)}{2} - \frac{1}{v-u} \int_u^v X(t,\cdot)\, dt \right| \\ \le & \frac{v-u}{2} \left[|X'(u,\cdot)| \int_0^1 |1-2\lambda| (1-\lambda)\, d\lambda + m \left| X'\left(\frac{v}{m}, \cdot\right) \right| \int_0^1 |1-2\lambda| \lambda d\lambda \right]. \end{aligned}$$

Since

$$\int_0^1 |1-2\lambda| (1-\lambda)\, d\lambda = \int_0^1 |1-2\lambda| \lambda d\lambda = \frac{1}{4},$$

we obtain the desired result. □

Remark 3.1. *For $m = 1$, Theorem 3.1 becomes to Theorem 5 in [11].*

Theorem 3.2. *Suppose $b^* > 0$. Let $X : I \subset [a, b^*] \times \Omega \to R$ be a differentiable stochastic process on I° and let X' be mean-square integrable on $[u, v]$ where $u, v \in I$ with $u < v$. If $|X'|^q$ is m-convex stochastic process on $[u, v]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds almost everywhere:*

$$\begin{aligned} &\left| \frac{X(u,\cdot) + X(v,\cdot)}{2} - \frac{1}{v-u} \int_u^v X(t,\cdot)\, dt \right| \\ \leq\ & \frac{v-u}{2(p+1)^{\frac{1}{p}}} \left[\frac{|X'(u,\cdot)|^q + m\left|X'\left(\frac{v}{m},\cdot\right)\right|^q}{2} \right]^{\frac{1}{q}}. \end{aligned} \tag{3.1}$$

Proof. By Lemma 3.1 and using well known Hölder's inequality, we have

$$\begin{aligned} &\left| \frac{X(u,\cdot) + X(v,\cdot)}{2} - \frac{1}{v-u} \int_u^v X(t,\cdot)\, dt \right| \\ \leq\ & \frac{v-u}{2} \left(\int_0^1 |1-2\lambda|^p \, d\lambda \right)^{\frac{1}{p}} \left(\int_0^1 |X'(\lambda u + (1-\lambda) v, \cdot)|\, d\lambda \right)^{\frac{1}{q}}. \end{aligned} \tag{3.2}$$

Since $|X'|^q$ is m-convex stochastic process on $[u, v]$ for all $u, v \in I$ with $u < v$, $\lambda \in [0, 1]$ and $m \in (0, 1]$, we have

$$|X'(\lambda u + (1-\lambda) v, \cdot)|^q \leq \lambda |X'(u\cdot)|^q + m(1-\lambda) \left| X'\left(\frac{v}{m}, \cdot\right) \right|^q.$$

Thus we obtain

$$\begin{aligned} \int_0^1 |X'(\lambda u + (1-\lambda) v, \cdot)|^q \, d\lambda &\leq \int_0^1 \left[\lambda |X'(u,\cdot)|^q + m(1-\lambda) \left| X'\left(\frac{v}{m}, \cdot\right) \right|^q \right] d\lambda \\ &= \frac{1}{2} |X'(u,\cdot)|^q + \frac{m}{2} \left| X'\left(\frac{v}{m}, \cdot\right) \right|^q. \end{aligned} \tag{3.3}$$

Moreover, since

$$\int_0^1 |1-2\lambda|^p \, d\lambda = \int_0^{1/2} (1-2\lambda)^p \, d\lambda + \int_{1/2}^1 (2\lambda - 1)^p \, d\lambda = \frac{1}{p+1}, \tag{3.4}$$

utilizing inequalities (3.3) and (3.4) in (3.2), we get the inequality (3.1). □

Remark 3.2. *For $m = 1$, Theorem 3.2 becomes to Corollary 6 in [11].*

Theorem 3.3. *Suppose $b^* > 0$. Let $X : I \subset [a, b^*] \times \Omega \to R$ be a differentiable stochastic process on I° and let X' be mean-square integrable on $[u, v]$ where $u, v \in I$ with $u < v$. If $|X'|^q$ is m-convex stochastic process on $[u, v]$ for $m \in (0, 1]$, $q \geq 1$, then the following inequality holds almost everywhere :*

$$\begin{aligned} &\left| \frac{X(u,\cdot) + X(v,\cdot)}{2} - \frac{1}{v-u} \int_u^v X(t,\cdot)\, dt \right| \\ \leq\ & \frac{v-u}{4} \left[\frac{|X'(u,\cdot)|^q + m\left|X'\left(\frac{v}{m},\cdot\right)\right|^q}{2} \right]^{\frac{1}{q}}. \end{aligned} \tag{3.5}$$

Proof. For $q = 1$, the proof is the same as that of Theorem 3.1. Suppose that $q > 1$. From Lemma 3.1 and using well known power-mean inequality, we have

$$\begin{aligned} & \left| \frac{X(u,\cdot) + X(v,\cdot)}{2} - \frac{1}{v-u} \int_u^v X(t,\cdot)\, dt \right| \\ \leq & \frac{v-u}{2} \left(\int_0^1 |1-2\lambda|\, d\lambda \right)^{1-\frac{1}{q}} \left(\int_0^1 |1-2\lambda| \left| X'(\lambda u + (1-\lambda) v, \cdot) \right|^q d\lambda \right)^{\frac{1}{q}}. \end{aligned} \tag{3.6}$$

Using m-convexity of the stochastic process $|X'|^q$ on $[u, v]$ in the second integral on the right side of the inequality (3.6), we have

$$\begin{aligned} & \int_0^1 |1-2\lambda| \left| X'(\lambda u + (1-\lambda) v, \cdot) \right|^q d\lambda \\ \leq & \int_0^1 |1-2\lambda| \left[\lambda \left| X'(u,\cdot) \right|^q + m(1-\lambda) \left| X'\left(\frac{v}{m},\cdot\right) \right|^q \right] d\lambda \\ = & \left| X'(u,\cdot) \right|^q \int_0^1 \lambda |1-2\lambda|\, d\lambda + m \left| X'\left(\frac{v}{m},\cdot\right) \right|^q \int_0^1 (1-\lambda) |1-2\lambda|\, d\lambda \\ = & \frac{1}{4} \left| X'(u,\cdot) \right|^q + \frac{m}{4} \left| X'\left(\frac{v}{m},\cdot\right) \right|^q. \end{aligned}$$

A usage of the last inequality in (3.6) gives the desired result. □

Remark 3.3. *For $q = 1$, the inequality (3.5) reduces to the inequality proved in Theorem 3.1. If $q = \frac{p}{p-1}$ $(p > 1)$, then one has $4^p > p+1$ and so $\frac{1}{4} < \frac{1}{2(p+1)^{\frac{1}{p}}}$. This shows that the inequality (3.5) is better than the one given by (3.1) in Theorem 3.2.*

Now we establish our results for (α, m)-convex stochastic processes.

Theorem 3.4. *Suppose $b^* > 0$. Let $X : I \subset [a, b^*] \times \Omega \to R$ be a differentiable stochastic process on I° and let X' be mean-square integrable on $[u, v]$ where $u, v \in I$ with $u < v$. If $|X'|$ is (α, m)-convex stochastic process on $[u, v]$ for $m \in (0, 1]$, $q \geq 1$, then the following inequality holds almost everywhere:*

$$\begin{aligned} & \left| \frac{X(u,\cdot) + X(v,\cdot)}{2} - \frac{1}{v-u} \int_u^v X(t,\cdot)\, dt \right| \\ \leq & \frac{v-u}{2} \left[M_1 \left| X'(u,\cdot) \right| + m M_2 \left| X'\left(\frac{v}{m},\cdot\right) \right| \right] \end{aligned} \tag{3.7}$$

where

$$M_1 = \frac{1 + \alpha 2^\alpha}{2^\alpha (1+\alpha)(2+\alpha)}, \tag{3.8}$$

$$M_2 = \frac{1}{2} - M_1. \tag{3.9}$$

Proof. From Lemma 3.1, we have

$$\begin{aligned}
&\left|\frac{X(u,\cdot)+X(v,\cdot)}{2}-\frac{1}{v-u}\int_u^v X(t,\cdot)\,dt\right| \\
\leq\ & \frac{v-u}{2}\int_0^1 |1-2\lambda|\,|X'(\lambda u+(1-\lambda)v,\cdot)|\,d\lambda. \qquad (3.10)
\end{aligned}$$

Since $|X'|$ is (α,m)-convex stochastic process on $[u,v]$ for all $u,v\in I$ with $u<v$, $(\alpha,m)\in(0,1]^2$ and $\lambda\in[0,1]$, we have

$$\begin{aligned}
&\int_0^1 |1-2\lambda|\,|X'(\lambda u+(1-\lambda)v,\cdot)|\,d\lambda \\
\leq\ & |X'(u,\cdot)|\int_0^1 |1-2\lambda|\,\lambda^\alpha d\lambda + m\left|X'\left(\frac{v}{m},\cdot\right)\right|\int_0^1 |1-2\lambda|\,(1-\lambda^\alpha)\,d\lambda \\
=\ & M_1\,|X'(u,\cdot)| + m\left(\frac{1}{2}-M_1\right)\left|X'\left(\frac{v}{m},\cdot\right)\right| \qquad (3.11)
\end{aligned}$$

where

$$\int_0^1 |1-2\lambda|\,\lambda^\alpha d\lambda = \frac{1+\alpha 2^\alpha}{2^\alpha(1+\alpha)(2+\alpha)} = M_1,$$

and

$$\int_0^1 |1-2\lambda|\,(1-\lambda^\alpha)\,d\lambda = \frac{1}{2}-\frac{1+\alpha 2^\alpha}{2^\alpha(1+\alpha)(2+\alpha)} = \frac{1}{2}-M_1 = M_2.$$

Using the inequality (3.11) in the inequality (3.10), we get the required result. □

Remark 3.4. *For $(\alpha,m)=(1,1)$, Theorem 3.4 becomes to Theorem 5 in [11].*

Theorem 3.5. *Suppose $b^*>0$. Let $X: I\subset[a,b^*]\times\Omega\to R$ be a differentiable stochastic process on I° and let X' be mean-square integrable on $[u,v]$ where $u,v\in I$ with $u<v$. If $|X'|^q$ is (α,m)-convex stochastic process on $[u,v]$ for $(\alpha,m)\in(0,1]^2$, $q\geq 1$, then the following inequality holds almost everywhere:*

$$\begin{aligned}
&\left|\frac{X(u,\cdot)+X(v,\cdot)}{2}-\frac{1}{v-u}\int_u^v X(t,\cdot)\,dt\right| \\
\leq\ & \frac{v-u}{2(p+1)^{\frac{1}{p}}}\left[\frac{\alpha\,|X'(u,\cdot)|^q + m\left|X'\left(\frac{v}{m},\cdot\right)\right|^q}{1+\alpha}\right]^{\frac{1}{q}} \qquad (3.12)
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 3.1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{X(u,\cdot)+X(v,\cdot)}{2}-\frac{1}{v-u}\int_u^v X(t,\cdot)\,dt\right| \\
\leq\ & \frac{v-u}{2}\left(\int_0^1 |1-2\lambda|^p\,d\lambda\right)^{\frac{1}{p}}\left(\int_0^1 |X'(\lambda u+(1-\lambda)v,\cdot)|^q\,d\lambda\right)^{\frac{1}{q}}. \qquad (3.13)
\end{aligned}$$

By (α,m)-convexity of the stochastic processes $|X'|^q$ on $[u,v]$, we have for every $\lambda\in[0,1]$

$$|X'((\lambda u+(1-\lambda)v),\cdot)|^q \leq \lambda^\alpha\,|X'(u,\cdot)|^q + m(1-\lambda^\alpha)\left|X'\left(\frac{v}{m},\cdot\right)\right|^q$$

for $(\alpha, m) \in (0,1]^2$. Hence

$$\begin{aligned}
&\int_0^1 |X'(\lambda u + (1-\lambda) v, \cdot)| \\
\leq\ & |X'(u,\cdot)|^q \int_0^1 \lambda^\alpha d\lambda + m \left|X'\left(\frac{v}{m},\cdot\right)\right|^q \int_0^1 (1-\lambda^\alpha) d\lambda \\
=\ & \frac{1}{1+\alpha} |X'(u,\cdot)|^q + \frac{m\alpha}{1+\alpha} \left|X'\left(\frac{v}{m},\cdot\right)\right|^q .
\end{aligned}$$

Utilizing of the above inequality in (3.13) and the fact

$$\int_0^1 |1-2\lambda|^p d\lambda = \frac{1}{p+1}$$

completes the proof. □

Remark 3.5. *For $(\alpha, m) = (1,1)$, Theorem 3.5 becomes to Corollary 6 in [11].*

Theorem 3.6. *Suppose $b^* > 0$. Let $X : I \subset [a, b^*] \times \Omega \to R$ be a differentiable stochastic process on I° and let X' be mean-square integrable on $[u, v]$ where $u, v \in I$ with $u < v$. If $|X'|^q$ is (α, m)-convex stochastic process on $[u, v]$ for $(\alpha, m) \in (0,1]^2$, $q \geq 1$, then the following inequality holds almost everywhere:*

$$\begin{aligned}
&\left|\frac{X(u,\cdot) + X(v,\cdot)}{2} - \frac{1}{v-u}\int_u^v X(t,\cdot)\,dt\right| \\
\leq\ & \frac{v-u}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[M_1 |X'(u,\cdot)|^q + m M_2 \left|X'\left(\frac{v}{m},\cdot\right)\right|^q\right]^{\frac{1}{q}},
\end{aligned}$$

where

$$M_1 = \frac{1+\alpha 2^\alpha}{2^\alpha (1+\alpha)(2+\alpha)},$$

$$M_2 = \frac{1}{2} - M_1.$$

Proof. For $q = 1$, the proof is similar to that of Theorem 3.4. Now suppose that $q > 1$. Using Lemma 3.1 and power-mean inequality, we have

$$\begin{aligned}
&\left|\frac{X(u,\cdot) + X(v,\cdot)}{2} - \frac{1}{v-u}\int_u^v X(t,\cdot)\,dt\right| \\
\leq\ & \frac{v-u}{2}\left(\int_0^1 |1-2\lambda|\,d\lambda\right)^{1-\frac{1}{q}} \left(\int_0^1 |1-2\lambda|\,|X'(\lambda u + (1-\lambda) v, \cdot)|^q\,d\lambda\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.14}$$

Since $|X'|^q$ is (α, m)-convex stochastic process on $[u, v]$ for every $\lambda \in [0,1]$ and $(\alpha, m) \in (0,1]^2$, we have

$$\begin{aligned}
& \int_0^1 |1-2\lambda| \left|X'\left(\lambda u+(1-\lambda) v, \cdot\right)\right|^q d\lambda \\
\leq & \int_0^1 |1-2\lambda| \left[\lambda^\alpha \left|X'(u, \cdot)\right|^q + m\left(1-\lambda^\alpha\right)\left|X'\left(\frac{v}{m}, \cdot\right)\right|^q\right] d\lambda \\
= & \left|X'(u, \cdot)\right|^q \int_0^1 |1-2\lambda| \lambda^\alpha d\lambda + \left|X'\left(\frac{v}{m}, \cdot\right)\right|^q \int_0^1 |1-2\lambda|\left(q-\lambda^\alpha\right) d\lambda \\
= & M_1 \left|X'(u, \cdot)\right|^q + M_2\left|X'\left(\frac{v}{m}, \cdot\right)\right|^q. \qquad (3.15)
\end{aligned}$$

Using the inequality (3.15) in the inequality (3.14) we get the desired result. □

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