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STRONG CONVERGENCE THEOREM FOR FINITE FAMILY OF GENERALISED ASYMPTOTICALLY NONEXPANSIVE MAPS

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Abstract. Let K be a nonexpansive retract of a uniformly convex Banach space X with retraction P and $T_{i:1\cdots,m} : K \longrightarrow X$ a finite family of uniformly continuous *generalised asymptotically nonexpansive* maps with a nonempty common fixed point set F. We provided and proved sufficient conditions for the strong convergence of a sequence of successive approximations generated by an m-step algorithm to a point of F.

1. Introduction

Let K be a nonempty subset of a normed linear space E. K is said to be (sequentially) compact if every closed bounded sequence in K has a subsequence that converges in K. K is said to be boundedly compact if every bounded subset of K is compact. Finite dimensional spaces are boundedly compact. Given a subset S of K, we shall denote by $co(S)$ and $ccl(S)$ the convex hull and the closed convex hull of S respectively. If K is boundedly compact convex and S is bounded, then $co(S)$ and hence $ccl(S)$ are compact convex subsets of K.

A map $T : K \to E$ is said to be semi-compact if for any bounded sequence $\{x_n\} \subset K$ such that

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$\|x_n - Tx_n\| \to 0$ as $n \to \infty$ there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges strongly to some $x^* \in K$ as $j \to \infty$. The map T is said to be demi-compact at $z \in E$ if for any bounded sequence $\{x_n\} \subset K$ such that $\|x_n - Tx_n\| \to z$ as $n \to \infty$ there exist a subsequence $\{x_{n_j}\}$ of$\{x_n\}$ and a point $p \in K$ such that x_{n_j} converges strongly to p as $j \to \infty$. (Observe that if T is additionally continuous, then $p - Tp = z$). A nonlinear map $T : K \to E$ is said to be completely continuous if it maps bounded sets into relatively compact sets. A mapping $T : K \longrightarrow E$ is called *nonexpansive* if and only if for all $x, y \in K$, we have that

$$\|Tx - Ty\| \leq \|x - y\|. \tag{1.1}$$

K is called a nonexpansive retract of E if there exists a nonexpansive map $P : E \longrightarrow K$ which is onto and such that $P^2 = I$. The map P is called the nonexpansive retraction of E onto K. Let K be a nonempty subset of a real normed space E. Let $P : E \longrightarrow K$ be a nonexpansive retraction of E onto K. A nonself map $T : K \longrightarrow E$ is called *asymptotically nonexpansive mapping* if and only if there exists a sequence $\{\mu_n\}_{n\geq 1} \subset [0, +\infty)$, with $\lim\limits_{n\to\infty} \mu_n = 0$ such that for all $x, y \in K$,

$$\|T(P_KT)^{n-1}x - T(P_KT)^{n-1}y\| \leq (1 + \mu_n)\|x - y\| \quad \forall\, n \in \mathbb{N} \tag{1.2}$$

where $P_K : X \longrightarrow K$ is nonexpansive retraction of E onto K. T is called *generalised asymptotically nonexpansive mapping* if and only if there exist a sequences $\{\mu_n\}_{n\geq 1}, \{\eta_n\}_{n\geq 1} \subset [0, +\infty)$, with $\lim\limits_{n\to\infty} \mu_n = 1$ and $\lim\limits_{n\to\infty} \eta_n = 0$ such that for all $x, y \in D(T)$,

$$\|T(P_KT)^{n-1}x - T(P_KT)^{n-1}y\| \leq \mu_n\|x - y\| + \eta_n \quad n \geq 1. \tag{1.3}$$

Goebel and Kirk [3] introduced the class of asymptotically nonexpansive mappings as a generalisation of nonexpansive mappings, Zegeye and Shahzad [13] introduced the class of generalised asymptotically nonexpansive mappings as a generalization of asymptotically nonexpansive maps. As further generalisation, Alber, Chidume and Zegeye [1] introduced the class of total asymptotically nonexpansive mappings, where $T : K \longrightarrow H$ is called *total asymptotically nonexpansive* if and only if there exist two sequences $\{\mu_n\}_{n\geq 1}, \{\eta_n\}_{n\geq 1} \subset [0, +\infty)$, with $\lim\limits_{n\to\infty} \mu_n = 0 = \lim\limits_{n\to\infty} \eta_n$ and nondecreasing continuous function $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\|T(P_KT)^{n-1}x - T(P_KT)^{n-1}y\| \leq \|x - y\| + \mu_n\phi(\|x - y\|) + \eta_n \quad n \geq 1. \tag{1.4}$$

Ofoedu and Nnubia [8] gave an example to show that the class of asymptotically nonexpansive mappings is a proper subset of the class of total asymptotically nonexpansive mappings. The class of total asymptotically nonexpansive type mappings includes the class of mappings which are asymptotically nonexpansive in the intermediate sense. These classes of mappings had been studied by several authors (see e.g., [3], [5], [11], [14]).

2. Preliminaries

We shall make use of the following result:

A Banach space E is said to satisfy *Opial's condition* if for each sequence $\{x_n\}_{n\geq 1} \in E$ which converges weakly to a point $z \in E$, we have that $\liminf_{n\to\infty} \|x_n - z\| < \liminf_{n\to\infty} \|x_n - y\|, \ \forall y \in E$, such that $y \neq z$. It is well known that every Hilbert space satisfies Opial's condition(see e.g., [9]).

A map T is said to satisfy *condition B* if there exists $f : [0, \infty) \to [0, \infty)$ strictly increasing, continuous, $f(0) = 0$ such that for all $x \in D(T), \|x - Tx\| \geq f(d(x, F))$ where $F = F(T) = \{x \in D(T) : x = Tx\}$ and $d(x, F) = \inf\{\|x - y\| : y \in F\}$.

Lemma 2.1. *[2] Let H be a real Hilbert space. Then for all $x, y \in H$ the following inequality holds.*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.2. *[2] For any x, y, z in a real Hilbert space H and a real number $\lambda \in [0, 1]$,*

$$\|\lambda x + (1 - \lambda)y - z\|^2 = \lambda\|x - z\|^2 + (1 - \lambda)\|y - z\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

Lemma 2.3. *[12] Let $\{x_n\}$ be sequence of nonegative real numbers satisfying the following relation:*

$$x_{n+1} \leq x_n - \alpha_n x_n + \delta_n, \ \ n \geq n_0,$$

where $\{\alpha_n\}_{n\geq 1} \subset (0, 1)$ and $\{\delta_n\}_{n\geq 1} \subset \mathbb{R}$ satisfying the following conditions:
$\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\limsup_{n\to\infty} \delta_n \leq 0$, *then* $\lim_{n\to\infty} x_n = 0$.

Lemma 2.4. *Let $\{\mu_n\}, \{\nu_n\}$ and$\{\eta_n\}$ be nonnegative sequences such that*
$\sum_{n\geq 0} \nu_n < \infty$, $\sum_{n\geq 0} \eta_n < \infty$ *and* $\mu_{n+1} \leq (1 + \nu_n)\mu_n + \eta_n$. *Then* $\lim_{n\to\infty} \mu_n$ *exists.*

The nearest point projection $P_K : H \longrightarrow K$ defined from H onto K is the function which assign to each $x \in H$ its nearest point denoted by $P_K x \ \in K$. Thus $P_K x$ is the unique point in K such that $\|x - P_K x\| \leq \|x - y\|$ for all $y \in K$ and we have the following Lemmas.

Lemma 2.5. *[12]. Let K be a closed convex nonempty subset of a real Hilbert space H. Let $x \in H$, then $z = P_K x$ if and only if*

$$\langle x - z, y - z\rangle \leq 0 \ \forall \, y \in K.$$

Lemma 2.6. *Let K be a closed convex nonempty subset of a Banach space E and let $T_{i\in I} : K \longrightarrow K$* where $i \in I = \{1, 2, ..., m\}$. *be finite family of continous nonlinear maps in K such that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and let $\{x_n\}_{n\geq 1}$ be a sequence of successive approximation satisfying*

(1) $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0 \ \ \forall i \in I$

(2) $\|x_{n+1} - x^*\| \le (1 + \tau_n)\|x_n - x^*\| + \nu_n$

where $\sum_{n\ge 0} \nu_n < \infty$ *and* $\sum_{n\ge 0} \tau_n < \infty$*. Then,* $\{x_n\}$ *converges strongly to a common fixed point of* T_i*'s if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof Now,

$$d(x_{n+1}, F) \quad \le \quad (1 + \tau_n)d(x_n, F) + \nu_n \tag{2.1}$$

hence $\liminf_{n\to\infty} d(x_n, F) = 0 \Rightarrow \lim_{n\to\infty} d(x_n, F)$. Also, $\forall\ k > 0$

$$\begin{aligned}
\|x_{n+k+1} - x^*\| &\le \Pi_{j=0}^{k}(1 + \tau_{n+j})\|x_n - x^*\| + \sum_{j=0}^{k} \nu_{n+k-j}\Pi_{r=0}^{j-1}(1 + \tau_{n+k-r}) \\
&\le \Pi_{j=0}^{k}(1 + \tau_{n+j})(\|x_n - x^*\| + \sum_{j=0}^{k} \nu_{n+k-j}) \\
&\le Q(\|x_n - x^*\| + \sum_{j=0}^{k} \nu_{n+j})
\end{aligned}$$

So that given any $\varepsilon > 0$ there exists an integer $n_0 > 0$, such that for all $n \ge n_0$, $d(x_n, F) < \frac{\varepsilon}{4(Q+1)}$ and $\nu_{n+j} < \frac{\varepsilon}{4m(Q+1)} \forall\ j = 1, 2, ..., m$. So $\exists\ x^* \in F$ such that $d(x_{n_0}, x^*) < \frac{\varepsilon}{4(Q+1)}$ that is, $\|x_{n_0} - x^*\| < \frac{\varepsilon}{4(Q+1)}$ Now,

$$\begin{aligned}
\|x_{n+k} - x_n\| \le \|x_{n+k} - x^*\| + \|x_n - x^*\| &\le 2Q(\|x_{n_0} - x^*\| + \sum_{j=0}^{k} \nu_{n_0+j}) \\
&\le 2Q(\frac{\varepsilon}{4(Q+1)} + m\frac{\varepsilon}{4m(Q+1)}) \\
&= 2Q(\frac{\varepsilon}{2(Q+1)} < \varepsilon.
\end{aligned}$$

So, $\{x_n\}$ is a Cauchy sequence in E and so it converges to some $u^* \in K$. But, $x_n - T_i x_n \to 0$ as $n \to \infty\ \forall\ i$ and T_i is continuous $\forall\ i$. Hence, $0 = \lim_{n\to\infty}(x_n - T_i x_n) = \lim_{n\to\infty} x_n - T_i(\lim_{n\to\infty} x_n) = u^* - T_i u*$. So that $u^* \in F$. i.e $u^* = x^* \in F$. Hence, $x_n \to x^*$ as $n \to \infty$. This completes the proof since the other part is obvious.

Lemma 2.7. *Let* K *be a closed convex nonempty subset of a Banach space* E *and let* $T_i : K \longrightarrow K$ where $i \in I = \{1, 2, ..., m\}$. *be finite family of continuous nonlinear maps in* K *such that* $F = \bigcap_{i=1}^{m} F(T_i) \ne \emptyset$ *and let* $\{x_n\}_{n\ge 1}$ *be a sequence of successive approximation satisfying*

(1) $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0\ \ \forall i \in I$

(2) $\|x_{n+1} - x^*\| \le (1 + \tau_n)\|x_n - x^*\| + \nu_n$

where $\sum_{n\ge 0} \nu_n < \infty$ *and* $\sum_{n\ge 0} \tau_n < \infty$ *Then,* $\{x_n\}$ *converges strongly to a common fixed point of* T_i*'s if one of the* T_i*'s satisfy condition B.*

Proof Let T_{i_0} satisfy condition B. Then $\exists f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and

$$f(d(x_n, F)) \leq \|x_n - T_{i_0} x_n\|,$$

hence

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - T_{i_0} x_n\| = 0.$$

Therefore,

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Thus by Lemma 2.6, $\{x_n\}$ converges strongly to a common fixed point of T_i's, this concludes the proof.

Lemma 2.8. *Let K be a closed convex nonempty subset of a Banach space E and let $T_i : K \longrightarrow K$where $i \in I = \{1, 2, ..., m\}$. be finite family of continuous nonlinear maps in K such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose the sequence $\{x_n\}_{n\geq 1}$ of successive approximation satisfies the following conditions*

(1) $\lim_{n\to\infty} \|x_n - x^*\|$ *exists with*$x^* \in F$,
(2) $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0 \;\; \forall i \in I$,
(3) $\{x_n\}_{n\geq 1}$ *has a convergent subsequence* $\{x_{n_j}\}_{n\geq 1}$.

Then, $\{x_n\}$ converges strongly to a point of F.

Proof Suppose that $\{x_n\}$ has a convergent subsequence $\{x_{n_j}\}$ and let $x_{n_j} \to p$ as $j \to \infty$, since $x_n - T_i x_n \to 0$ as $n \to \infty$ for all $i \in \{1, 2, ..., N\}$. It implies that $x_{n_j} - T_i x_{n_j} \to 0$ as $j \to \infty$ for all $i \in I$. Also, by continuity of T_i .$T_i x_{n_j} \to T_i p$ as $j \to \infty$ for all $i \in I$. So, $\|p - T_i p\| = \lim_{n\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0, \forall i$ which implies that $p \in F$. Now, $\lim_{n\to\infty} \|x_n - p\|$ exists from our hypothesis and $\lim_{n\to\infty} \|x_{n_j} - p\| = 0$, so $\lim_{n\to\infty} \|x_n - p\| = 0$. Thus, $\{x_n\}$ converges strongly to a point of F.

Remark 2.1. *The conditions for which $\{x_n\}$ has a convergent subsequence includes*

(1) T_i *is completely continuous* $\forall i \in \{1, ..., N\}$.
(2) T_i *is demicompact* $\forall i \in \{1, ..., N\}$.
(3) T_i *is semicompact for some* $i \in \{1, ..., N\}$.
(4) K *is compact.*
(5) K *is boundedly compact.*

Proposition 2.1. *Let K be a nonexpansive retract of a uniformly convex Banach space X with nonexpansive retraction P. Let $T : K \longrightarrow X$ uniformly continuous generalised asymptotically nonexpansive map with associated sequences $\{\mu_n\}_{n\geq 1}, \{\eta_n\}_{n\geq 1} \subset [0, +\infty)$ with $\sum_{n=0}^{\infty} (\mu_n - 1) < \infty$ $\sum_{n=0}^{\infty} \eta_n < \infty$, suppose that $F(T) \neq \emptyset$. Then $F(T)$ is closed and convex (where $F(T)$ is the fixed point set of T).*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ converging to $x^* \in K$, then $x_n = Tx_n \forall\ n \geq 0$. By continuity of $T, x^* = \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} Tx_n = T(\lim\limits_{n\to\infty} x_n) = Tx^*$. Thus, $x^* \in F(T)$ and $F(T)$ is closed. Next, we show that $F(T)$ is convex. For $t \in (0,1)$ and $x, y \in F(T)$, put $z = (1-t)x + ty$, we show that $z = Tz$.

$$
\begin{aligned}
\|z - T(P_KT)^{n-1}z\|^2 &= \|z\|^2 - 2\langle z, T(P_KT)^{n-1}z\rangle + \|T(P_KT)^{n-1}z\|^2 \\
&= \|z\|^2 - 2(1-t)\langle x, T(P_KT)^{n-1}z\rangle - 2t\langle y, T(P_KT)^{n-1}z\rangle \\
&\quad + \|T(P_KT)^{n-1}z\|^2 \\
&= \|z\|^2 + (1-t)\|x - T(P_KT)^{n-1}z\|^2 + t\|y - T(P_KT)^{n-1}z\|^2 \\
&\quad -(1-t)\|x\|^2 - t\|y\|^2 \\
&\leq \|z\|^2 + (1-t)(\mu_n\|x-z\| + \eta_n)^2 + t(\mu_n\|y-z\| + \eta_n)^2 \\
&\quad -(1-t)\|x\|^2 - t\|y\|^2 \\
&= \|z\|^2 + (1-t)\big(\mu_n^2\|x-z\|^2 + (2\mu_n\|x-z\| + \eta_n)\eta_n\big) \\
&\quad + t\big(\mu_n^2\|y-z\|^2 + (2\mu_n\|y-z\| + \eta_n)\eta_n\big) - (1-t)\|x\|^2 - t\|y\|^2 \\
&= \|z\|^2 + (1-t)\mu_n^2(\|x\|^2 - \|z\|^2 - 2\langle x, z\rangle) \\
&\quad + t\mu_n^2(\|y\|^2 - \|z\|^2 - 2\langle y, z\rangle) \\
&\quad + 2\mu_n\eta_n\big((1-t)\|x-z\| + t\|y-z\|\big) + \eta_n^2 - (1-t)\|x\|^2 - t\|y\|^2 \\
&= (1+\mu_n^2)\|z\|^2 + (\mu_n^2 - 1)\big((1-t)\|x\|^2 + t\|y\|^2\big) - 2\mu_n\big((1-t)\langle x, z\rangle \\
&\quad + t\langle y, z\rangle\big) + 2\mu_n\eta_n\big((1-t)\|x-z\| + t\|y-z\|\big) + \eta_n^2 \\
&\leq (\mu_n^2 - 1)\big((1-t)\|x\|^2 + t\|y\|^2 + \|z\|^2\big) + 2\mu_n^2\|z\|^2 \\
&\quad -2\mu_n^2\big((1-t)\langle x, z\rangle + t\langle y, z\rangle\big) \\
&\quad + 2\mu_n\eta_n\big((1-t)\|x-z\| + t\|y-z\|\big) + \eta_n^2 \\
&= (\mu_n^2 - 1)\big((1-t)\|x\|^2 + t\|y\|^2 + \|z\|^2\big) - 2\mu_n^2\big((1-t)\langle x, z\rangle \\
&\quad + t\langle y, z\rangle - (1-t)\|z\|^2 - t\|z\|^2\big) \\
&\quad + 2\mu_n\eta_n\big((1-t)\|x-z\| + t\|y-z\|\big) + \eta_n^2 \\
&= (\mu_n^2 - 1)\big((1-t)\|x\|^2 + t\|y\|^2 + \|z\|^2\big) - 2\mu_n^2\big((1-t)\langle x-z, z\rangle \\
&\quad + t\langle y-z, z\rangle\big) + 2\mu_n\eta_n\big((1-t)\|x-z\| + t\|y-z\|\big) + \eta_n^2 \\
&= (\mu_n^2 - 1)\big((1-t)\|x\|^2 + t\|y\|^2 + \|z\|^2\big) - 2\mu_n^2\langle (1-t)(x-z) + t(y-z), z\rangle \\
&\quad + 2\mu_n\eta_n\big((1-t)\|x-z\| + t\|y-z\|\big) + \eta_n^2
\end{aligned}
$$

$$= (\mu_n^2 - 1)\big((1-t)\|x\|^2 + t\|y\|^2 + \|z\|^2\big) - 2\mu_n^2 t(1-t)(\langle x - y + y - x, z\rangle$$
$$+\eta_n^2 + 2\mu_n\eta_n\big((1-t)\|x-z\| + t\|y-z\|\big)$$
$$= (\mu_n^2 - 1)\big((1-t)\|x\|^2 + t\|y\|^2 + \|z\|^2\big) + 4t(1-t)\|x-y\|\mu_n\eta_n + \eta_n^2.$$

Thus, $\lim_{n\to\infty} \|z - T(P_K T)^{n-1}z\| = 0$, which implies that $\lim_{n\to\infty} T(P_K T)^{n-1}z = z$ and hence $z = \lim_{n\to\infty} T(P_K T)^{n-1}z = TP_K(\lim_{n\to\infty} T(P_K T)^{n-2}z) = TP_K z = Tz$. Thus, $z \in F(T)$. This completes the proof.

Proposition 2.2. *Let K be a nonexpansive retract of a uniformly convex Banach space X with nonexpansive retraction P. Let $T_i : K \longrightarrow X$ ($i = 1,\ldots,m$) be a finite family of uniformly continuous generalised asymptotically nonexpansive map with associated sequences $\{\mu_{in}\}_{n\geq 1}, \{\eta_{in}\}_{n\geq 1} \subset [0, +\infty)$ with $\sum_{n=0}^{\infty}(\mu_{in} - 1) < \infty$, $\sum_{n=0}^{\infty}\eta_{in} < \infty$. Suppose that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Then F is closed and convex.*

Proof. By Lemma 2.1, we have that $F(T_i)$ is closed for each i. Now, $F = \bigcap_{i=1}^m F(T_i)$ is a finite intersection of closed sets, hence closed. Also, by Lemma 2.1, we have that $F(T_i)$ is convex for each i. Since $F = \bigcap_{i=1}^m F(T_i)$ is a finite intersection of convex set, we have that F is convex.

Proposition 2.3. *Suppose that there exist $c > 0, k > 0$ constants such that $\phi(t) \leq ct$ for all $\geq k$, then T is total asymptotically nonexpansive if and only if T is generalised asymptotically nonexpansive.*

Proof It is known that every generalised asymptotically nonexpansive map is total asymptotically nonexpansive, so it suffices to show that that every total asymptotically nonexpansive with the condition of our hypothesis is generalised asymptotically nonexpansive. Now, let T be such that

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n\phi(\|x-y\|) + \eta_n \quad n \geq 1 \tag{2.2}$$

Since ϕ is continuous, it follows that ϕ reaches its maximum (say c_0) on the interval $[0, k]$; moreover, $\phi(t) \leq ct$ whenever $t > k$. Thus,

$$\phi(t) \leq c_0 + ct \ \ \forall\, t \in [0, +\infty). \tag{2.3}$$

So, we have,

$$\|T^n x - T^n y\| \leq \|x-y\| + \mu_n(c_0 + c\|x-y\|) + \eta_n \quad n \geq 1$$
$$= (1 + \mu_n c)\|x - y\| + \mu_n c_0 + \eta_n$$
$$= (1 + \nu_n)\|x - y\| + \gamma_n$$

where $\nu_n = \mu_n c$ and $\gamma_n = \mu_n c_0 + \eta_n$. This completes the proof.

Corollary 2.1. *Let K be a nonexpansive retract of a uniformly convex Banach space X with nonexpansive retraction P. Let $T : K \longrightarrow X$ be a uniformly continuous total asymptotically nonexpansive map with associated sequences $\{\mu_n\}_{n\geq 1}, \{\eta_n\}_{n\geq 1} \subset [0, +\infty)$ with $\sum_{n=0}^{\infty}(\mu_n - 1) < \infty$ $\sum_{n=0}^{\infty}\eta_n < \infty$. Suppose that there exist $c > 0, k > 0$ constants such that $\phi(t) \leq ct \forall\ t \geq k$, and that $F(T) \neq \emptyset$ then $F(T)$ is closed and convex.*

Corollary 2.2. *Let K be a nonexpansive retract of a uniformly convex Banach space X with nonexpansive retraction P. Let $T_i : K \longrightarrow X$ (i = 1,...,m) be a finite family of uniformly continuous total asymptotically nonexpansive maps with associated sequences $\{\mu_{in}\}_{n\geq 1}, \{\eta_{in}\}_{n\geq 1} \subset [0, +\infty)$ with $\sum_{n=0}^{\infty}(\mu_{in} - 1) < \infty$ $\sum_{n=0}^{\infty}\eta_{in} < \infty$. Suppose that there exist $c > 0, k > 0$ constants such that $\phi(t) \leq ct$ for all $t \geq k$, and that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ then F is closed and convex.*

3. Main results

Proposition 3.1. *Let H be a normed linear space, let K be a closed convex nonempty subset of H and let $T_i K \longrightarrow H$ ($i \in I = \{1, ..., m\}$) be a finite family of continuous generalised asymptotically nonexpansive map with sequences $\{\mu_{in}\}_{n\geq 1}, \{\eta_{in}\}_{n\geq 1} \subset [0, +\infty)$ such that $\lim_{n\to\infty} \mu_{in} = 1$ and $\lim_{n\to\infty} \eta_n = 0$ with*

$$\sum_{n=0}^{\infty}(\mu_{in} - 1) < \infty \ \sum_{n=0}^{\infty}\eta_{in} < \infty$$

. Suppose that $F(T) \neq \emptyset$ and let $\{x_n\}_{n\geq 1}$ be a sequence generated iteratively by starting with an arbitrary $x_0 \in K$, define by

$$\begin{aligned} y_{n,i} &= P_K[(1-\alpha_n)x_n + \alpha_n T_i(PT_i)^{n-1} y_{n,i-1}], \\ y_{n,0} &= x_n; y_{n,m} = x_{n+1} = y_{n+1} \quad n \geq 0\,, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}_{n\geq 1}$, is a sequence in $(0, 1)$ such that $0 < \zeta < \beta_n < \epsilon < 1 \ \forall\ n \geq 1$. Let $x^* \in F$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exist

Proof. Let $x^* \in F$, $j \in I$ then from (3.1) we have that

$$\begin{aligned} \|y_{n,j} - x^*\| &= \|P_K[(1-\alpha_n)x_n + \alpha_n T_j(PT_j)^{n-1} y_{n,j-1}] - Px^*\| \quad (3.2) \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|T_j(PT_j)^{n-1}y_{n,j-1} - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n(\mu_{n,j})\|y_{n,j-1} - x^*\| + \eta_{in}). \\ &= (1-\alpha_n)\|x_n - x^*\| + \alpha_n\mu_{n,j}(\|y_{n,j-1} - x^*\| + \alpha_n\eta_{n,j}) \end{aligned} \tag{3.3}$$

$$\begin{aligned}\|y_{n,1}-x^*\| &\leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n\mu_{n,1}(\|y_{n,0}-x^*\|+\alpha_n\eta_{n,1})\\ &\leq \big(1+\alpha_n(\mu_{n,1-1})\big)\|x_n-x^*\|+\alpha_n\eta_{n,1}\end{aligned} \tag{3.4}$$

$$\begin{aligned}\|y_{n,2}-x^*\| &\leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n\mu_{n,2}\big((1+\alpha(\mu n,1-1))\|x_n-x^*\|+\alpha_n\eta_{n,1}\big)+\alpha_n\eta_{n,2}.\\ &\leq \big(1+\alpha_n(\mu_{n,2}-1)+\alpha_n^2\mu_(n,2)(\mu_{n,1}-1)\big)\|x_n-x^*\|+\alpha_n^2\mu_{n,2}\eta_{n,1}+\alpha_n\eta_{n,2}.\end{aligned} \tag{3.5}$$

$$\begin{aligned}\|y_{n,3}-x^*\| &\leq \big(1+\alpha_n(\mu_{n,3}-1)+\alpha_n^2\mu_(n,3)(\mu_{n,2}-1)+\alpha_n^3\mu_(n,3)\mu_{n,2}(\mu_{n,1}-1)\big)\|x_n-x^*\|\\ &+\alpha_n\eta_{n,3}+\alpha_n^2\mu_{n,3}\eta_{n,2}+\alpha_n^3\mu_{n,3}\mu_{n,2}\eta_{n,1}.\end{aligned}$$

Hence,

$$\begin{aligned}\|y_{n,j}-x^*\| &\leq \big(1+\sum_{t=1}^{j}\alpha_n^t\Pi_{s=1}^{t-1}\mu_{n,j-s+1}(\mu_{n,j-t+1}-1)\big)\|x_n-x^*\|\\ &+\sum_{t=1}^{j}\alpha_n^t\Pi_{s=1}^{t-1}\mu_{n,j-s+1}\eta_{n,j-t+1}.\end{aligned}$$

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq \big(1+\sum_{t=1}^{m}\alpha_n^t\Pi_{s=1}^{t-1}\mu_{n,m-s+1}(\mu_{n,m-t+1}-1)\big)\|x_n-x^*\|\\ &+\sum_{t=1}^{m}\alpha_n^t\Pi_{s=1}^{t-1}\mu_{n,m-s+1}\eta_{n,m-t+1}\\ &\leq \big(1+q^{m-1}b\sum_{j=1}^{m}(\mu_{n,j}-1)\big)\|x_n-x^*\|+q^{m-1}b\sum_{j=1}^{m}\eta_{n,j}.\end{aligned}$$

(since there exists n_0 such that $\mu_{n,i}\leq q$ for all $n\geq n_0, \forall\ j\in I$) So, $\lim_{n\to\infty}\|x_n-x^*\|$ exist; and hence $\{x_n\},\{y_{n,j}\}$ are bounded.

Theorem 3.1. *Let K be a nonexpansive retract of a uniformly convex Banach space X with nonexpansive retraction P. Let $T_i : K\longrightarrow E$ be a finite family of uniformly continuous generalised asymptotically nonexpansive maps with sequences $\{\mu_{in}\}_{n\geq1},\{\eta_{in}\}_{n\geq1}\subset[0,+\infty)$ such that*

$$\lim_{n\to\infty}\mu_{in}=1,\lim_{n\to\infty}\eta_{in}=0,\sum_{n=0}^{\infty}(\mu_{in}-1)<\infty\ \sum_{n=0}^{\infty}\eta_{in}<\infty$$

Suppose that $F = \bigcap_{i=1}^{N} F(T_i)$ *is not empty and let* $\{x_n\}_{n\geq 1}$ *be a sequence generated iteratively by (3.1) where* $\{\alpha_n\}_{n\geq 1}$ *is a sequence in* $(0,1)$ *satisfying the following conditions:*

$$\sum_{n=1}^{\infty} \alpha_n < \infty,\ 0 < \zeta < \alpha_n < \epsilon < 1\ \forall\ n \geq 1$$

, then $\forall j \in \{1,2,...,m\}$, $\lim_{n\to\infty} \|x_n - T_j x_n\| = 0$ *and* $\{x_n\}_{n\geq 1}$ *converges weakly to a point of* F.

Proof. Let $x^* \in F$

$$\begin{aligned} \|y_{n,j} - x^*\|^2 &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|T_j(PT_j)^{n-1}y_{n,j-1} - x^*\| \\ &\quad -\alpha_n(1-\alpha_n)g(\|x_n - T_j(PT_j)^{n-1}y_{n,j-1}\|) \\ &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n(\mu_{n,j}\|y_{n,j-1} - x^*\| + \eta_{n,j})^2 \\ &\quad -\alpha_n(1-\alpha_n)g(\|x_n - T_j(PT_j)^{n-1}y_{n,j-1}\|) \\ &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\mu_{n,j}^2\|y_{n,j-1} - x^*\|^2 \\ &\quad +\alpha_n(2\mu_{n,j}\|y_{n,j-1} - x^*\| + \eta_{n,j})\eta_{n,j} - \alpha_n(1-\alpha_n)g(\|x_n - T_j(PT_j)^{n-1}y_{n,j-1}\|) \end{aligned} \tag{3.6}$$

So

$$\begin{aligned} \|y_{n,1} - x^*\|^2 &\leq \big(1 + \alpha_n(\mu_{n,1}^2 - 1)\big)\|x_n - x^*\|^2 + \alpha_n(2\mu_{n,1}\|x_n - x^*\| + \eta_{n,1})\eta_{n,1} \\ &\quad -\alpha_n(1-\alpha_n)g(\|x_n - T_1(PT_1)^{n-1}x_n\|) \end{aligned}$$

$$\begin{aligned} \|y_{n,2} - x^*\|^2 &\leq \big(1 + \alpha_n(\mu_{n,2}^2 - 1) + \alpha_n^2\mu_{n,2}^2(\mu_{n,1}^2 - 1)\big)\|x_n - x^*\|^2 \\ &\quad +\alpha_n(2\mu_{n,2}\|y_{n,1} - x^*\| + \eta_{n,2})\eta_{n,2} + \alpha_n^2\mu_{n,2}^2(2\mu_{n,1}\|x_n - x^*\| + \eta_{n,1})\eta_{n,1} \\ &\quad -\alpha_n(1-\alpha_n)g(\|x_n - T_2(PT_2)^{n-1}y_{n,1}\|) \\ &\quad -\alpha_n^2\mu_{n,2}^2(1-\alpha_n)g(\|x_n - T_1(PT_1)^{n-1}x_n\|) \end{aligned}$$

So,

$$\begin{aligned} \|y_{n,j} - x^*\|^2 &\leq \big(1 + \sum_{t=1}^{j} \alpha_n^t \Pi_{s=1}^{t-1}\mu_{n,j-s+1}^2(\mu_{n,j-t+1}^2 - 1)\big)\|x_n - x^*\|^2 \\ &\quad + \sum_{t=1}^{j} \alpha_n^t \Pi_{s=1}^{t-1}\mu_{n,j-s+1}^2(2\mu_{n,j-t+1}\|y_{n,j-t-1} - x^*\| + \eta_{n,j-t+1})\eta_{n,j-t+1}\Pi_{s=0}^{t-1}\mu_{n,j-s}^2 \\ &\quad -(1-\alpha_n)\sum_{t=1}^{j} \alpha_n^t g(\|x_n - T_{j-t+1}(PT_{j-t+1})^{n-1}y_{n,j-t}\|)\Pi_{s=1}^{t-1}\mu_{n,j-s+1}^2 \end{aligned}$$

Hence,

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 \leq\ & \big(1+\sum_{t=1}^{m}\alpha_n^t\Pi_{s=1}^{t-1}\mu_{n,m-s+1}^2(\mu_{n,m-t+1}^2-1)\big)\|x_n-x^*\|^2 \\
& +\sum_{t=1}^{m}\alpha_n^t\Pi_{s=1}^{t-1}\mu_{n,m-s+1}^2(2\mu_{n,m-t+1}\|y_{n,m-t}-x^*\|+\eta_{n,m-t+1})\eta_{n,m-t+1} \\
& -(1-\alpha_n)\sum_{t=1}^{m}\alpha_n^t\Pi_{s=1}^{t-1}\mu_{n,m-s+1}^2 g\big(\|x_n-T_{m-t+1}(PT_{m-t+1})^{n-1}y_{n,m-t}\|\big) \\
\leq\ & \big(1+q^{2(m-1)}b\sum_{j=1}^{m}(\mu_{n,j}^2-1)\big)\|x_n-x^*\|+q^{2(m-1)}b\sum_{j=1}^{m}(2\mu_{n,j}\|y_{n,j-1}-x^*\| \\
& +\eta_{n,j})\eta_{n,j}\|-a^m(1-\alpha_n)\sum_{j=1}^{m}g(\|x_n-T_{m-j+1}(PT_{m-j+1})^{n-1}y_{n,m-j}\|) \\
\leq\ & \big(1+q^{2(m-1)}b\sum_{j=1}^{m}(\mu_{n,j}^2-1)\big)\|x_n-x^*\| \\
& +q^{2(m-1)}b\sum_{j=1}^{m}\eta_{n,j}-a^m(1-\alpha_n)\sum_{j=1}^{m}g(\|x_n-T_{m-j+1}(PT_{m-j+1})^{n-1}y_{n,m-j}\|)
\end{aligned}
$$

So,

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 \leq\ & \big(1+d_0\sum_{j=1}^{m}(\mu_{n,j}^2-1)\big)\|x_n-x^*\| \\
& +d_1\sum_{j=1}^{m}\eta_{n,j}-d_2\sum_{j=1}^{m}g(\|x_n-T_j(PT_j)^{n-1}y_{n,j-1}\|)
\end{aligned}
$$

So, $\lim_{n\to\infty} g(\|x_n-T_j(PT_j)^{n-1}y_{n,j-1}\|)=0$, thus $\lim_{n\to\infty}\|x_n-T_j(PT_j)^{n-1}y_{n,j-1}\|=0\ \forall\ j=1,...,m$. Now,

$$
\begin{aligned}
\|x_n-T_j(PT_j)^{n-1}x_n\| \leq\ & \|x_n-T_j(PT_j)^{n-1}y_{n,j-1}\|+\|T_j(PT_j)^{n-1}y_{n,j-1}-T_j(PT_j)^{n-1}x_n\| \\
\leq\ & \|x_n-T_j(PT_j)^{n-1}y_{n,j-1}\|+\mu_{n,j}\|y_{n,j-1}-x_n\|+\eta_{n,j} \\
\leq\ & \|x_n-T_j(PT_j)^{n-1}y_{n,j-1}\|+\mu_{n,j}\alpha_n\|x_n-T_{j-1}(PT_{j-1})^{n-1}y_{n,j-2}\|+\eta_{n,j}
\end{aligned}
$$

Hence, $\lim_{n\to\infty}\|x_n-T_j(PT_j)^{n-1}x_n\|=0\ \forall\ j=1,...,m.$
Further, $\|x_n-T_jx_n\|\leq\|x_n-T_j(PT_j)^{n-1}y_{n,j-1}\|+\|T_j(PT_j)^{n-1}y_{n,j-1}-T_jx_n\|$

$$
\begin{aligned}
\|(PT_j)^{n-1}y_{n,j-1}-x_n\| \leq\ & \|T_j(PT_j)^{n-2}y_{n,j-1}-x_n\| \\
\leq\ & \|T_j(PT_j)^{n-2}y_{n,j-1}-T_j(PT_j)^{n-2}y_{n-1,j-1}\| \\
& +\|T_j(PT_j)^{n-2}y_{n-1,j-1}-x_{n-1}\|+\|x_{n-1}-x_n\| \\
\leq\ & \mu_{n-1,j}\|y_{n,j-1}-y_{n-1,j-1}\|+\eta_{n-1,i} \\
& +\|x_{n-1}-T_j(PT_j)^{n-2}y_{n-1,j-1}\|+\|x_n-x_{n-1}\|
\end{aligned}
$$

$$
\begin{aligned}
\|y_{n,j} - y_{n-1,j}\| &\leq \|y_{n,j} - x_n\| + \|x_n - x_{n-1}\| + \|x_{n-1} - y_{n-1,j} \\
&\leq \alpha_n \|x_n - T_j(PT_j)^{n-1} y_{n,j-1}\| + \alpha_{n-1}\|x_{n-1} - T_m(PT_m)^{n-2} y_{n-1,m-1}\| \\
&\quad + \alpha_{n-1}\|x_{n-1} - T_j(PT_j)^{n-2} y_{n-1,j-1}\|
\end{aligned}
$$

So, $\lim_{n\to\infty} \|y_{n,j} - y_{n-1,j}\| = 0$. Also, $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$
so that $\lim_{n\to\infty} \|(PT_j)^{n-1} y_{n,j-1} - x_n\| = 0$. Hence $\lim_{n\to\infty} \|x_n - T_j x_n\| = 0 \; \forall \; j = 1, ..., m$. By reflexivity $\exists \; z \in K$ and $\{x_{nj}\} \subset \{x_n\}$ such that, $\{x_{nj}\} \to^w z$ as $j \to \infty$. Since, $x_{nj} - T_i x_{nj} \to 0$ as $j \to \infty \;\; \forall i$ then $z \in F(T_i) \;\; \forall i$ and so $z \in F = \bigcap_{i=1}^{m} F(T_i)$. Let $\omega_w(x_n)$ be subsequential limit set of the sequence $\{x_n\}$. Let $q \in \omega_w(x_n)$ arbitrary. Then $\exists \; \{x_{nr}\} \subset \{x_n\} \ni \{x_{nr}\}$ converges weakly q and $x_{nr} - T_i x_{nr} \to 0$ as $r \to \infty \; \forall i$. Thus, $\omega_w(x_n) \subseteq F$. Thus $\{x_n\}_{n\geq 1}$ converges weakly to a point of F.

Theorem 3.2. *Let $K, X, P,\; T_i's, F, \{x_n\}$ be as in Theorem 3.1 Then, $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$ (where F= F(T)).*

The Proof follows from Lemma 2.6, since from Theorem 3.1 and it's proof, the conditions of the lemma are satisfied.

Theorem 3.3. *Let $K, X, P,\; T_i's, F, \{x_n\}$ be as in Theorem 3.1 Then, $\{x_n\}$ converges strongly to a common fixed point of T_i's if one of the T_i's satify condition B.*

The Proof follows from Lemma 2.7, since from the proof of Theorem 3.1, the conditions of the lemma are satisfied.

Theorem 3.4. *Let $K, X, P, T_i's, F, \{x_n\}$ be as in Theorem 3.1 Then, $\{x_n\}$ converges strongly to a common fixed point of T_i's if $\{x_n\}_{n\geq 1}$ has a convergent subsequence $\{x_{n_j}\}_{n\geq 1}$*

The Proof follows from Lemma 2.8, since from the proof of Theorem 3.1, the conditions of the lemma are satisfied.
As a result of the proposition 2.3 we have the following results.

Theorem 3.5. *Let K be a nonexpansive retract of a uniformly convex Banach space X with nonexpansive retraction P. Let $T_i : K \longrightarrow E$ be a finite family of uniformly continuous total asymptotically nonexpansive maps with sequences $\{\mu_{in}\}_{n\geq 1}, \{\eta_{in}\}_{n\geq 1} \subset [0, +\infty)$ such that*

$$\lim_{n\to\infty} \mu_{in} = 0 = \lim_{n\to\infty} \eta_{in}, \sum_{n=0}^{\infty} (\mu_{in} - 1) < \infty \;\; \sum_{n=0}^{\infty} \eta_{in} < \infty$$

and with function $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying $\phi(t) \leq M_0 t$ for all $t > M_1$, for some constants $M_0, M_1 > 0$. Suppose that $F = \bigcap_{i=1}^{m} F(T_i)$ is not empty and let $\{x_n\}_{n\geq 1}$ be a sequence generated iteratively *by (3.1) where $\{\alpha_n\}_{n\geq 1}$ is a sequence in $(0,1)$ satisfying the following conditions:*

$$\sum_{n=1}^{\infty}\alpha_n < \infty,\ 0 < \zeta < \alpha_n < \epsilon < 1\ \forall\ n \geq 1,$$

then for all $j \in \{1,2,...,m\}$, $\lim_{n\to\infty} \|x_n - T_j x_n\| = 0$ and $\{x_n\}_{n\geq 1}$ converges weakly to a point of F.

Theorem 3.6. *Let $K, X, P, T_i's, F, \{x_n\}$ be as in Theorem 3.5 Then, $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Theorem 3.7. *Let $K, X, P,\ T_i's, F, \{x_n\}$ be as in Theorem 3.5. Then, $\{x_n\}$ converges strongly to a common fixed point of T_i's if one of the T_i's satify condition B.*

Theorem 3.8. *Let $K, X, P, T_i's, F, \{x_n\}$ be as in Theorem 3.5 Then, $\{x_n\}$ converges strongly to a common fixed point of T_i's if $\{x_n\}_{n\geq 1}$ has a convergent subsequence $\{x_{n_j}\}_{n\geq 1}$.*

Our iterative process generalise some of the existing ones, our theorems improves, generalise and extend several known results and our method of proof is of independent interest.

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