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NEW RESULTS ON THE CONFORMABLE FRACTIONAL SUMUDU TRANSFORM: THEORIES AND APPLICATIONS

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Abstract. In this paper, we generalize the formula of Sumudu transform to the conformable fractional order and some interesting and important rules of this transform and conformable fractional Laplace transform are derived and discussed. Moreover, we present the general analytical solution of a singular and nonlinear conformable fractional Poisson- Boltzmann differential equation based on the conformable fractional Sumudu transform. Also, our proposed method is applied successfully for obtaining the general solutions of some linear and nonhomogeneous conformable fractional differential equations. Finally, the results show that our proposed method is an efficient and can be applied for finding the general solutions of the all cases realted to the conformable fractional differential equations.

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1. Introduction and Preliminaries

Various types of fractional derivatives were introduced such as: Riemann-Liouville, Caputo, Modified Riemann-Liouville, Hadamard, Grunwald, Letnikov and Riesz operators. The most two popular definitions of fractional derivatives of order $\alpha > 0$ are ([1], [2], [3], [4], [5], [6], [7]):

(i) Riemann-Liouville fractional derivative:

$$ {}^{R}D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t) dt. \tag{1.1} $$

(ii) Caputo fractional derivative:

$$ {}^{c}D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n}{dx^n} f(t) dt. \tag{1.2} $$

Here $\Gamma(.)$ denotes to the gamma function.

In the last few decades, fractional differentiation has been used applied scientists for solving several fractional differential equations and they proved that the fractional calculus is very useful in several fields of applications with some restrictions such as: Physics (quantum mechanics and thermodynamics), chemistry, biology, economics, engineering, signal and image processing and control theory ([2], [3], [4], [5], [6], [7], [8], [9], [10]). Very recently, the discrepancies between known definitions can be solved in simple way by presenting a new fractional definition which is called the " *Conformable Fractional Derivative* " and defined for a given function $f : [0, \infty) \to \mathbb{R}$ of fractional (ordinary) order $\alpha > 0$ by Khalil et el. [11] as follows:

$$ D^{n\alpha} f(x) = \lim_{\varepsilon \to 0} \frac{f^{[\alpha]-1}(x + \varepsilon x^{[\alpha]-\alpha}) - f^{[\alpha]-1}(x)}{\varepsilon}, \quad n-1 < \alpha \leq n, \ x > 0, \tag{1.3} $$

where $[\alpha]$ is the smallest integer number greater than or equal α and $n \in N$.
As a special case, if $0 < \alpha \leq 1$, then we have:

$$ D^{\alpha} f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}. \tag{1.4} $$

This definition is very easy for calculating derivatives and solving fractional differential equations compared with other fractional definitions such as the definitions of Liouville-Reimman and Caputo fractional derivatives. It has received a lot of attention and many applications have been remodeled using this new definition ([12], [13], [14], [15], [16], [17], [18], [19]). Moreover, it has many interesting advantages which make its easier and flexible more than the definitions of other fractional derivatives ([12], [13], [14], [15], [16], [17], [18], [19]). Some of these adavantages are: (i) It satisfies the all concepts of ordinary calculus such as: quotient, product and chain rules, Rolle's theorem and mean-value theorem. (ii) A non-differentiable function can be α- differentiable in the conformable sense. (iii) It can be easily extended to generalize many integral transforms such as: Laplace, Mellin, Natural and Sumudu transforms.

Also, the conformable fractional integral has been also defined of order $\alpha > 0$ by:

$$I^{\alpha} f(x) = \int_0^x f(t)\ t^{\alpha-1} dt. \tag{1.5}$$

In fact, if $f(x)$ is an n -differentiable function at $x > 0$ and $\alpha \in (0, 1]$, $n \in N$, then [11]:

$$(i)\ D^{\alpha} f(x) = x^{1-\alpha} \frac{d}{dx} f(x), \tag{1.6}$$

$$(ii)\ D^{\alpha} I^{\alpha} f(x) = f(x). \tag{1.7}$$

In the literature there are several works on the theory and applications of integral transforms such as the Laplace and Sumudu transforms ([20], [21], [22], [23], [24], [25], [26], [27], [28]) that are widely used in physics, electric circuit theory, astronomy, as well as engineering and sciences. Sumudu transform was introduced by Watugala [20] and defined over the following set of functions:

$$A = \left\{ f(x) : \exists\ M, \tau_1, \tau_2 > 0, |f(x)| < M e^{\frac{|x|}{\tau_j}},\ if\ x \in (-1)^j \times [0, \infty), j = 1, 2 \right\}, \tag{1.8}$$

as follows:

$$\mathcal{S}[f(x)] = \mathcal{F}(u) = \int_0^{\infty} e^{-x} f(ux) dx,\ u \in (-\tau_1, \tau_2), \tag{1.9}$$

or equivalently:

$$\mathcal{S}[f(x)] = \mathcal{F}(u) = \frac{1}{u} \int_0^{\infty} e^{-\frac{x}{u}} f(x) dx,\ u > 0. \tag{1.10}$$

This transform has many interesting advantages over other integral transforms especially the "unity" feature which could provide convergence when solvimg differential equations and also used to solve problems without resorting to a new frequency domain while for example, Laplace transform must satisfy the Dirichlet condition which is $f(x)$ must be piecewise continuous which means that it must be single valued but can have a finite number of finite isolated discontinuues for $x > 0$. Also this transform possesses many interesting advantages which make its visualization easier and some of these advantages can be found in ([20], [21], [22], [23], [24], [25], [26], [27], [28]).

There are several methods and techniques were applied for obtaining the analytical and numerical solution of such nonlinear and singular fractional differential equations such as the fractional Laplace transform method [19], generalized Kudryashov method [29], Adomian decomposition method ([30], [31]) and modified Kudryashov method [32]. Note that the fractional order differential equations are now span a half-century or more and play a crucial role in several theoretical and applied sciences such as, but certainly not limited to, theoretical biology and ecology, solid state physics, viscoelasticity, fiber optics, signal processing and electric control theory, stochastic based finance and thermodynamics ([19], 20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32]).

Finally, the theory of thermal explosions originally proposed by Frank-Kamenetzky ([33], [34]) and revisited by Barenblatt et al. [35] has been required the solution of the nonlinear Poisson-Boltzmann differential equation which is given and solved numerically by Chambré [36]. The ordinary Poisson-Boltzmann differential equation as follows ([36], [37]):

$$\frac{d^2y}{dx^2} + \frac{\beta}{x}\frac{dy}{dx} = e^y \tag{1.11}$$

This equation is also a very useful in many settings, whether it be to understand physiological interfaces, polymer science, electron interactions in a semiconductor; in the critical theory of gravitation and combustion or explosion. and to describe the distribution of the electric potential in solution in the direction normal to a charged surface and from this equation, many other equations can been derived with a number of different assumptions.

In this paper, we extend the definition of Sumudu transform to fractional order and derive a list of interesting rules and properties of this extension which including the rules of the conformable fractional Laplace transform. Also, a very nice relationship between conformable fractional Sumudu and Laplace transforms is derived and proved. Moreover, we give two important and attractive applications for conformable fractional Sumudu transform. These applications are: Firstly, we apply the conforrmable fractional Sumudu transform together with Adomain decomposition method for presenting the general analytical solution of a singular and nonlinear conformable fractional Poisson-Boltzmann differential equation. Secondly, we also apply the conformable fractional Sumudu transform to find the general solutions of some linear and nonhomogeneous conformable fractional differential equations. Finally, the results show that our proposed method is an efficient method and applied successfully to find the general solutions of the all cases (Singular, linear, nonlinear, homogeneous and nonhomogeneous) realted to the conformable fractional differential equations.

2. Conformable fractional Laplace and Sumudu transforms

In this Section, we introduce (with proofs) a list of important basic rules and properties for the conformable fractional Laplace and Sumudu transforms involving the nice relationship between of these transforms which are playing a central role in the solutions of conformable fractional differential equations.

Definition 2.1.: Let $f : [0, \infty) \to \mathbb{R}$ be a given function and $0 < \alpha \leq 1$. Then the conformable fractional Laplace transform of f is defined as:

$$\mathcal{L}_\alpha \{f(x)\} = \mathcal{F}_\alpha(s) = \int_0^\infty e^{-s\frac{x^\alpha}{\alpha}} f(x) x^{\alpha-1} dx, \tag{2.1}$$

provided the integral exists.

Theorem 2.1.: Let $f:[0,\infty)\to\mathbb{R}$ be a given function and $0<\alpha\leq 1$. Then

$$\mathcal{L}_\alpha\left\{D^\alpha f(x)\right\}=s\mathcal{F}_\alpha(s)-f(0),\ s>0. \tag{2.2}$$

Proof. By using Definition 2.1 and integration by parts, we have:

$$\begin{aligned}\mathcal{L}_\alpha\left\{D^\alpha f(x)\right\} &= \int_0^\infty e^{-s\frac{x^\alpha}{\alpha}}D^\alpha f(x)x^{\alpha-1}dx=\int_0^\infty e^{-s\frac{x^\alpha}{\alpha}}x^{1-\alpha}f'(x)x^{\alpha-1}dx\\ &= \int_0^\infty e^{-s\frac{x^\alpha}{\alpha}}f'(x)dx=\left[e^{-s\frac{x^\alpha}{\alpha}}f(x)\right]_0^\infty-\int_0^\infty f(x)\left(-\frac{s}{\alpha}\alpha\ x^{\alpha-1}\right)e^{-s\frac{x^\alpha}{\alpha}}dx\\ &= \lim_{c\to\infty}\left[e^{-s\frac{x^\alpha}{\alpha}}f(x)\right]_0^c+s\int_0^\infty f(x)\ x^{\alpha-1}e^{-s\frac{x^\alpha}{\alpha}}dx\\ &= s\mathcal{F}_\alpha(s)-f(0),\end{aligned}$$

which completes the proof of Theorem 2.1. □

Theorem 2.2 .: Let $f:[0,\infty)\to\mathbb{R}$ be a function. Then

$$\mathcal{F}_\alpha(s)=\mathcal{L}\left\{f(\alpha x)^{\frac{1}{\alpha}}\right\}(s). \tag{2.3}$$

Proof. By using Definition 2.1 and letting $v=\frac{x^\alpha}{\alpha}$, we have:

$$\mathcal{F}_\alpha(s)=\int_0^\infty e^{-s\frac{x^\alpha}{\alpha}}f(x)x^{\alpha-1}dx=\int_0^\infty e^{-sv}f(\alpha v)^{\frac{1}{\alpha}}dv=\mathcal{L}\left\{f(\alpha v)^{\frac{1}{\alpha}}\right\},$$

which completes the proof of Theorem 2.2. □

Theorem 2.3.: Let c, a, $p\in\ \mathbb{R}$ and $0<\alpha\leq 1$. Then

$$\begin{aligned}(i)\ \mathcal{L}_\alpha\left\{c\right\}(s) &= \frac{c}{s},\ s>0. && (2.4)\\ (ii)\ \mathcal{L}_\alpha\left\{x^p\right\}(s) &= \alpha^{\frac{p}{\alpha}}\frac{\Gamma(1+\frac{p}{\alpha})}{s^{1+\frac{p}{\alpha}}},\ s>0. && (2.5)\\ (iii)\ \mathcal{L}_\alpha\left\{e^{\frac{ax^\alpha}{\alpha}}\right\}(s) &= \frac{1}{s-a},\ s>a. && (2.6)\\ (iv)\ \mathcal{L}_\alpha\left\{sin(\frac{ax^\alpha}{\alpha})\right\}(s) &= \frac{a}{s^2+a^2},\ s>0. && (2.7)\\ (v)\ \mathcal{L}_\alpha\left\{cos(\frac{ax^\alpha}{\alpha})\right\}(s) &= \frac{s}{s^2+a^2},\ s>0. && (2.8)\\ (vi)\ \mathcal{L}_\alpha\left\{sinh(\frac{ax^\alpha}{\alpha})\right\}(s) &= \frac{a}{s^2-a^2},\ s>\left|a\right|. && (2.9)\\ (vii)\ \mathcal{L}_\alpha\left\{cosh(\frac{ax^\alpha}{\alpha})\right\}(s) &= \frac{s}{s^2-a^2},\ s>\left|a\right|. && (2.10)\end{aligned}$$

Proof. Follows by applying Definition 2.1. □

Theorem 2.4. : Let f and $g : [0,\infty) \to \mathbb{R}$ andl let λ, μ, $a \in \mathbb{R}$ and $0 < \alpha \leq 1$. Then

$$(i)\ \mathcal{L}_\alpha \{\lambda f(x) + \mu g(x)\} = \lambda \mathcal{F}_\alpha(s) + \mu \mathcal{G}_\alpha(s), s > 0, \tag{2.11}$$

$$(ii)\ \mathcal{L}_\alpha \left\{e^{-a\frac{x^\alpha}{\alpha}} f(x)\right\} = \mathcal{F}_\alpha(s+a),\ \ s > |a|\,, \tag{2.12}$$

$$(iii)\ \mathcal{L}_\alpha \{I^\alpha f(x)\} = \frac{\mathcal{F}_\alpha(s)}{s}, s > 0, \tag{2.13}$$

$$(iv)\ \mathcal{L}_\alpha \left\{\frac{x^{n\alpha}}{\alpha^n} f(x)\right\} = (-1)^n \frac{d^n}{ds^n} \mathcal{F}_\alpha(s), s > 0, \tag{2.14}$$

$$(v)\ \mathcal{L}_\alpha \{(f * g)(x)\} = \mathcal{F}_\alpha(s)\mathcal{G}_\alpha(s),\ \ s > 0, \tag{2.15}$$

where $f * g$ is the convolution product of f and g.

Proof. *(i)* Straightforward

(ii) By using Thereom 2.2, we get:

$$\begin{aligned}
\mathcal{L}_\alpha \left\{e^{-a\frac{x^\alpha}{\alpha}} f(x)\right\} &= \mathcal{L}\left\{e^{\frac{-a}{\alpha}(\alpha x)^{\alpha\cdot\frac{1}{\alpha}}} f(\alpha x)^{\frac{1}{\alpha}}\right\} = \mathcal{L}\left\{e^{-ax} f(\alpha x)^{\frac{1}{\alpha}}\right\} \\
&= \mathcal{L}\left\{f(\alpha x)^{\frac{1}{\alpha}}\right\}|_{s\to s+a} = \mathcal{F}_\alpha(s+a).
\end{aligned}$$

(iii) By using Theorem 2.1, we have:

$$\mathcal{L}_\alpha \{D^\alpha I^\alpha f(x)\} = s\mathcal{L}_\alpha \{I^\alpha f(x)\} - I^\alpha f(0).$$

Since $I^\alpha f(0) = 0$, then we obtain:

$$\mathcal{F}_\alpha(s) = s\mathcal{L}_\alpha \{I^\alpha f(x)\}$$

$$\mathcal{L}_\alpha \{I^\alpha f(x)\} = \frac{\mathcal{F}_\alpha(s)}{s}.$$

(vi) By using Theorem 2.2 , we obtain:

$$\mathcal{L}_\alpha \left\{\frac{x^{n\alpha}}{\alpha^n} f(x)\right\} = \mathcal{L}\left\{\frac{(\alpha x)^{n\alpha.\frac{1}{\alpha}}}{\alpha^n} f(\alpha x)^{\frac{1}{\alpha}}\right\} = \mathcal{L}\left\{x^n f(\alpha x)^{\frac{1}{\alpha}}\right\} = (-1)^n \frac{d^n}{ds^n} \mathcal{F}_\alpha(s).$$

(v) By using Theorem 2.2, we get:

$$\mathcal{L}_\alpha \{(f * g)(x)\} = \mathcal{L}\left\{(f * g)(\alpha x)^{\frac{1}{\alpha}}\right\} = \mathcal{L}\left\{f(\alpha x)^{\frac{1}{\alpha}}\right\} \mathcal{L}\left\{g(\alpha x)^{\frac{1}{\alpha}}\right\} = \mathcal{F}_\alpha(s)\ \mathcal{G}_\alpha(s).$$

Which completes the proof of Theorem 2.4. □

Definition 2.2.: Over the following set of functions:

$$A_\alpha = \left\{ f(x) : \exists\, M, \tau_1, \tau_2 > 0, |f(x)| < M e^{\frac{|x^\alpha|}{\alpha \tau_j}},\ if\ x^\alpha \in (-1)^j \times [0, \infty), j = 1, 2 \right\}, \tag{2.16}$$

then the conformable fractional Sumudu transform of f can be generalized by:

$$\mathcal{S}_\alpha[f(x)] = \mathcal{F}_\alpha(u) = \frac{1}{u} \int_0^\infty e^{\frac{-x^\alpha}{\alpha u}} f(x) d_\alpha x, \tag{2.17}$$

where $d_\alpha x = x^{\alpha - 1} dx$, $0 < \alpha \leq 1$ and provided the integral exists.

The relationship between the conformable fractional Sumudu and conformable fractional Laplace transforms is given as in the next result.

Theorem 2.5.: Let $f : [0, \infty) \to \mathbb{R}$ be a given function and $0 < \alpha \leq 1$.Then

$$\mathcal{S}_\alpha[f(x)] = \frac{\mathcal{F}_\alpha(\frac{1}{u})}{u}. \tag{2.18}$$

Proof. Applying Definition 2.2, we get:

$$\mathcal{S}_\alpha[f(x)] = \frac{1}{u} \int_0^\infty e^{-\frac{x^\alpha}{\alpha u}} f(x) d_\alpha x.$$

Letting $v = \frac{x^\alpha}{\alpha} \Rightarrow dv = x^{\alpha-1} dx$, then we have:

$$\mathcal{S}_\alpha[f(x)] = \frac{1}{u} \int_0^\infty e^{-\frac{v}{u}} f(\alpha v)^{\frac{1}{\alpha}} dv = \frac{1}{u} \mathcal{L} \left\{ f(\alpha v)^{\frac{1}{\alpha}} \right\}_{s \to \frac{1}{u}} = \frac{\mathcal{F}_\alpha(\frac{1}{u})}{u},$$

which completes the proof of Theorem 2.5. □

Theorem 2.6. : Let $f : [0, \infty) \to R$ be a given function and $0 < \alpha \leq 1$.Then

$$\mathcal{S}_\alpha[D^\alpha f(x)] = \frac{\mathcal{F}_\alpha(u)}{u} - \frac{f(0)}{u}. \tag{2.19}$$

Proof. Using Theorems 2.5 and 2.1, we get:

$$\begin{aligned} \mathcal{S}_\alpha[D^\alpha f(x)] &= \frac{\mathcal{L}_\alpha \left\{ D^\alpha f(x) \right\}_{s \to \frac{1}{u}}}{u} = \frac{[s\mathcal{F}_\alpha(s) - f(0)]_{s \to \frac{1}{u}}}{u} \\ &= \frac{\mathcal{F}_\alpha(\frac{1}{u})}{u^2} - \frac{f(0)}{u} = \frac{\mathcal{F}_\alpha(u)}{u} - \frac{f(0)}{u}, \end{aligned}$$

which completes the proof of Theorem 2.6. □

Theorem 2.7.: Let $f : [0, \infty) \to R$ be an n-differentiable function and $0 < \alpha \leq 1$.Then

$$\mathcal{S}_\alpha[D^{n\alpha} f(x)] = \frac{\mathcal{S}_\alpha[f(x)]}{u^n} - \frac{f(0)}{u^n},\ 0 < \alpha \leq 1\ and\ n \in \mathbb{N}. \tag{2.20}$$

Proof. Follows by using the induction process on n and Theorem 2.6.

□

Now we introduce the basic rules of conformable fractional Sumudu transform for some certain functions as in the next result.

Theorem 2.8.: Let $a, c \in \mathbb{R}$ and $0 < \alpha \leq 1$.Then we have:

$$(i)\ \mathcal{S}_\alpha[c] = c. \tag{2.21}$$

$$(ii)\ \mathcal{S}_\alpha\left[e^{a\frac{x^\alpha}{\alpha}}\right] = \frac{1}{1-au},\ u > \frac{1}{a}. \tag{2.22}$$

$$(iii)\ \mathcal{S}_\alpha\left[sin(a\frac{x^\alpha}{\alpha})\right] = \frac{au}{1+a^2u^2},\ u > \frac{1}{|a|}. \tag{2.23}$$

$$(iv)\ \mathcal{S}_\alpha\left[cos(a\frac{x^\alpha}{\alpha})\right] = \frac{1}{1+a^2u^2},\ u > \frac{1}{|a|}. \tag{2.24}$$

$$(v)\ \mathcal{S}_\alpha\left[sinh(a\frac{x^\alpha}{\alpha})\right] = \frac{au}{1-a^2u^2},\ u > \frac{1}{|a|}. \tag{2.25}$$

$$(vi)\ \mathcal{S}_\alpha\left[cosh(a\frac{x^\alpha}{\alpha})\right] = \frac{1}{1-a^2u^2},\ u > \frac{1}{|a|}. \tag{2.26}$$

$$(vii)\ \mathcal{S}_\alpha\left[\frac{x^{n\alpha}}{\alpha^n}\right] = \Gamma(n+1)u^n,\ u > 0. \tag{2.27}$$

Proof. By using Theorems 2.5 and 2.2, we have:

(i)

$$\mathcal{S}_\alpha[c] = \frac{\mathcal{L}_\alpha\left\{c\right\}_{s\to\frac{1}{u}}}{u} = \frac{\left\{\frac{c}{s}\right\}_{s\to\frac{1}{u}}}{u} = c.$$

(ii)

$$\mathcal{S}_\alpha\left[e^{a\frac{x^\alpha}{\alpha}}\right] = \frac{\mathcal{L}_\alpha\left\{e^{a\frac{x^\alpha}{\alpha}}\right\}_{s\to\frac{1}{u}}}{u} = \frac{\mathcal{L}\left\{e^{ax}\right\}_{s\to\frac{1}{u}}}{u} = \frac{\left\{\frac{1}{s-a}\right\}_{s\to\frac{1}{u}}}{u} = \frac{1}{1-au}.$$

(iii)

$$\mathcal{S}_\alpha\left[sin(a\frac{x^\alpha}{\alpha})\right] = \frac{\mathcal{L}_\alpha\left\{sin(a\frac{x^\alpha}{\alpha})\right\}_{s\to\frac{1}{u}}}{u} = \frac{\mathcal{L}\left\{sin(ax)\right\}_{s\to\frac{1}{u}}}{u} = \frac{\left\{\frac{a}{s^2+a^2}\right\}_{s\to\frac{1}{u}}}{u}$$
$$= \frac{au}{1+a^2u^2}.$$

Similarly we can prove (iv) and then can easy to prove (v) and (vi) based on (iii) and (iv) of Theorem 2.8.

(vii)

$$\mathcal{S}_\alpha\left[\frac{x^{n\alpha}}{\alpha^n}\right] = \frac{\mathcal{L}_\alpha\left\{\frac{x^{n\alpha}}{\alpha^n}\right\}_{s\to\frac{1}{u}}}{u} = \frac{\Gamma(n+1)u^{n+1}}{u} = \Gamma(n+1)u^n.$$

Which completes the proof of Theorem 2.8. □

Theorem 2.9.: Let f and $g : [0, \infty) \to \mathbb{R}$ given functiins and let $\lambda, \mu \in \mathbb{R}$ and $0 < \alpha \leq 1$.Then we have:

(i) Linearly property:

$$\mathcal{S}_\alpha \left\{ \lambda f(x) + \mu g(x) \right\} = \lambda \mathcal{F}_\alpha(u) + \mu \mathcal{G}_\alpha(u), \tag{2.28}$$

(ii) Shifting property:

$$\mathcal{S}_\alpha \left[e^{-a\frac{x^\alpha}{\alpha}} f(x) \right] = \frac{\mathcal{F}_\alpha(\frac{1}{u} + a)}{u}. \tag{2.29}$$

(iii) Integral property:

$$\mathcal{S}_\alpha \left[I^\alpha f(x) \right] = \mathcal{F}_\alpha(\frac{1}{u}). \tag{2.30}$$

(iv) Convolution property:

$$\mathcal{S}_\alpha \left[(f * g)(x) \right] = u \mathcal{F}_\alpha(u) \mathcal{G}_\alpha(u). \tag{2.31}$$

(v) Power product property:

$$\mathcal{S}_\alpha \left[\frac{x^{n\alpha}}{\alpha^n} f(x) \right] = \frac{1}{u} \left[(-1)^n \frac{d^n}{ds^n} \mathcal{F}_\alpha(s) \right]_{s \to \frac{1}{u}}. \tag{2.32}$$

Proof. (i) Straightforward by using Definition 2.2.

(ii) By applying Theorems 2.5 and 2.2, we have:

$$\begin{aligned} \mathcal{S}_\alpha \left[e^{-a\frac{x^\alpha}{\alpha}} f(x) \right] &= \frac{\mathcal{L}_\alpha \left\{ e^{-a\frac{x^\alpha}{\alpha}} f(x) \right\}_{s \to \frac{1}{u}}}{u} = \frac{\mathcal{L} \left\{ e^{-ax} f(\alpha x)^{\frac{1}{\alpha}} \right\}_{s \to \frac{1}{u}}}{u} \\ &= \frac{\mathcal{F}_\alpha(\frac{1}{u} + a)}{u}. \end{aligned}$$

(iii) By using Theorem 2.5 and Eq.(2.13), we have:

$$\mathcal{S}_\alpha \left[I^\alpha f(x) \right] = \frac{\mathcal{L}_\alpha \left\{ I^\alpha f(x) \right\}_{s \to \frac{1}{u}}}{u} = \mathcal{F}_\alpha(\frac{1}{u}).$$

(iv) By applying Theorem 2.5 and Eq. (2.15), we have:

$$\begin{aligned} \mathcal{S}_\alpha \left[(f * g)(x) \right] &= \frac{\mathcal{L}_\alpha \left\{ f * g \right\}_{s \to \frac{1}{u}}}{u} = \frac{\left[\mathcal{F}_\alpha(s) \mathcal{G}_\alpha(s) \right]_{s \to \frac{1}{u}}}{u} = \frac{\mathcal{F}_\alpha(\frac{1}{u}) \mathcal{G}_\alpha(\frac{1}{u})}{u} \\ &= u \mathcal{F}_\alpha(u) \mathcal{G}_\alpha(u). \end{aligned}$$

(v) By using Theorem 2.5 and Eq.(2.3), we have:

$$\begin{aligned} \mathcal{S}_\alpha \left[\frac{x^{n\alpha}}{\alpha^n} f(x) \right] &= \frac{\mathcal{L}_\alpha \left\{ \frac{x^{n\alpha}}{\alpha^n} f(x) \right\}_{s \to \frac{1}{u}}}{u} = \frac{1}{u} \mathcal{L} \left\{ x^n f(\alpha x)^{\frac{1}{\alpha}} \right\}_{s \to \frac{1}{u}} \\ &= \frac{1}{u} \left[(-1)^n \frac{d^n}{ds^n} \mathcal{F}_\alpha(s) \right]_{s \to \frac{1}{u}}. \end{aligned}$$

Which completes the proof of Theorem 2.9. $\square$

Theorem 2.10.: Let $y = f(x)$ be an n-differentiable function and $0 < \alpha \leq 1$. Then

$$\mathcal{S}_\alpha\left[\frac{x^\alpha}{\alpha}D^{n\alpha}y\right] = u\frac{d}{du}\left(u\ \mathcal{S}_\alpha[D^{n\alpha}y]\right), \quad n = 1, 2, ... \tag{2.33}$$

Proof. Let $d_\alpha x = x^{\alpha-1}dx$, then we have:

$$\begin{aligned}
\mathcal{S}_\alpha[D^{n\alpha}y] &= \frac{1}{u}\int_0^\infty e^{-\frac{x^\alpha}{\alpha u}}D^{n\alpha}y\ d_\alpha x \\
\frac{d}{du}\left(u\ \mathcal{S}_\alpha[D^{n\alpha}y]\right) &= \int_0^\infty \frac{d}{du}\ e^{-\frac{x^\alpha}{\alpha u}}D^{n\alpha}y\ d_\alpha x = \frac{1}{u^2}\int_0^\infty e^{-\frac{x^\alpha}{\alpha u}}\ \frac{x^\alpha}{\alpha}D^{n\alpha}y\ d_\alpha x \\
&= \frac{1}{u}\int_0^\infty \frac{1}{u}\ e^{-\frac{x^\alpha}{\alpha u}}\ \frac{x^\alpha}{\alpha}D^{n\alpha}y\ d_\alpha x \\
&= \frac{1}{u}\mathcal{S}_\alpha\left[\frac{x^\alpha}{\alpha}D^{n\alpha}y\right].
\end{aligned}$$

which concludes the result as in Eq. (2.33). □

3. Analytical solution of a singular and nonlinear conformable fractional Poisson-Boltzmann differential equation

In this Section, we apply the conformable fractional Sumudu transform togethor with Adomain decomposition method to present the general analytical solution of the following singular and nonlinear conformable fractional Poisson Boltzmann differential equation:

$$D^{2\alpha}y + \frac{\beta}{x^\alpha}D^\alpha y = e^y,\ \ y(0) = 0, \tag{3.1}$$

where $0 < \alpha \leq 1$ and $\beta > 0$.

To solve this problem: Multiply Eq. (3.1) by $\frac{x^\alpha}{\alpha}$ and take $S_\alpha$of both sides, then by applying Theorems 2.10 and 2.6, we get:

$$u\frac{d}{du}\left(\frac{Y_\alpha(u)}{u}\right) + \gamma\left(\frac{Y_\alpha(u)}{u}\right) = \mathcal{S}_\alpha\left(\frac{x^\alpha}{\alpha}e^y\right), \tag{3.2}$$

where $\gamma = \frac{\beta}{\alpha}$.

Since $u \neq 0$ and integrating Eq. (3.2) with respect to z, we get:

$$\frac{Y_\alpha(u)}{u} = \int_0^u \frac{1}{z}\mathcal{S}_\alpha\left[\frac{x^\alpha}{\alpha}e^y\right]dz - \gamma\int_0^u \frac{Y_\alpha(z)}{z^2}dz \tag{3.3}$$

$$Y_\alpha(u) = u\int_0^u \frac{1}{z}\mathcal{S}_\alpha\left[\frac{x^\alpha}{\alpha}e^y\right]dz - u\gamma\int_0^u \frac{Y_\alpha(z)}{z^2}dz. \tag{3.4}$$

Suppose the solution $y(x)$ and non-linear function e^y by the following infinite series:

$$y(x) = \sum_{n=0}^\infty y_n(x)\ ,\ \ e^y = \sum_{n=0}^\infty A_n(x), \tag{3.5}$$

where A_n is the Adomain polynomial of e^y.

Note that:

$$\begin{aligned} A_\circ &= e^{y_\circ}, \\ A_1 &= y_1 e^{y_\circ}, \\ A_2 &= y_2 e^{y_\circ} + \frac{1}{2!} y_1^2 e^{y_\circ}, \\ &\vdots \end{aligned} \tag{3.6}$$

Substituting Eq. (3.5) into Eq.(3.4), we have:

$$\mathcal{S}_\alpha\left[\sum_{n=0}^{\infty} y_n(x)\right] = u\int_0^u \frac{1}{z}\mathcal{S}_\alpha\left[\frac{x^\alpha}{\alpha}A_n\right]dz - u\gamma\int_0^u \frac{\mathcal{S}_\alpha[\sum_{n=0}^{\infty} y_n(x)]}{z^2}dz. \tag{3.7}$$

By taking $\mathcal{S}_\alpha^{-1}$ of both sides of Eq (3.7), we obtain:

$$\sum_{n=0}^{\infty} y_n(x) = \mathcal{S}_\alpha^{-1}\left[u\int_0^u \frac{1}{z}\mathcal{S}_\alpha[\frac{x^\alpha}{\alpha}A_n]dz - u\gamma\int_0^u \frac{\mathcal{S}_\alpha[\sum_{n=0}^{\infty} y_n(x)]}{z^2}dz\right].$$

Thus, the general solution of Eq.(3.1) is given by:

$$y(x) = \mathcal{S}_\alpha^{-1}\left[u\int_0^u \frac{1}{z}\mathcal{S}_\alpha[\frac{x^\alpha}{\alpha}A_n]dz - u\gamma\int_0^u \frac{\mathcal{S}_\alpha[\sum_{n=0}^{\infty} y_n(x)]}{z^2}dz\right], \tag{3.8}$$

where

$$\begin{aligned} y_\circ &= 0 \\ y_{n+1} &= \mathcal{S}_\alpha^{-1}\left[u\int_0^u \frac{1}{z}\mathcal{S}_\alpha[\frac{x^\alpha}{\alpha}A_n]dz - u\gamma\int_0^u \frac{\mathcal{S}_\alpha[y_n(x)]}{z^2}dz\right]. \end{aligned} \tag{3.9}$$

Now,

(i) For $n = 0$, then:

$$\begin{aligned} y_1 &= \mathcal{S}_\alpha^{-1}\left[u\int_0^u \frac{1}{z}\mathcal{S}_\alpha[\frac{x^\alpha}{\alpha}A_\circ]dz - u\gamma\int_0^u \frac{\mathcal{S}_\alpha[y_\circ]}{z^2}dz\right] = \mathcal{S}_\alpha^{-1}\left[u\int_0^u \frac{1}{z}\mathcal{S}_\alpha[\frac{x^\alpha}{\alpha}]dz\right] \\ &= \frac{1}{2!}\frac{x^{2\alpha}}{\alpha^2}. \end{aligned} \tag{3.10}$$

(ii) For $n = 1$, then:

$$\begin{aligned} y_2 &= \mathcal{S}_\alpha^{-1}\left[u\int_0^u \frac{1}{z}\mathcal{S}_\alpha[\frac{x^\alpha}{\alpha}A_1]dz - u\gamma\int_0^u \frac{\mathcal{S}_\alpha[y_1]}{z^2}dz\right] \\ &= \mathcal{S}_\alpha^{-1}\left[u\int_0^u \frac{1}{\Gamma(3)z}\mathcal{S}_\alpha[\frac{x^{3\alpha}}{\alpha^3}]dz - u\gamma\int_0^u \frac{\mathcal{S}_\alpha[\frac{x^{2\alpha}}{\alpha^2}]}{\Gamma(3)z^2}dz\right] \\ &= \mathcal{S}_\alpha^{-1}\left[\frac{\Gamma(4)}{3\Gamma(3)}u^4 - \gamma u^2\right] \\ &= \frac{1}{4!}\frac{x^{4\alpha}}{\alpha^4} - \frac{\gamma}{2!}\frac{x^{2\alpha}}{\alpha^2}. \end{aligned} \tag{3.11}$$

(iii) For $n = 2$, then

$$\begin{aligned} y_3 &= \mathcal{S}_\alpha^{-1}\left[u\int_0^u \frac{1}{z}\mathcal{S}_\alpha\left[\frac{x^\alpha}{\alpha}A_2\right]dz - u\gamma\int_0^u \frac{\mathcal{S}_\alpha[y_2]}{z^2}dz\right] \\ &= \mathcal{S}_\alpha^{-1}\left[u\int_0^u \frac{1}{z}\mathcal{S}_\alpha\left[\frac{x^\alpha}{\alpha}\ y_2e^{y_\circ} + \frac{1}{2!}\frac{x^\alpha}{\alpha}\ y_1^2e^{y_\circ}\right]dz - u\gamma\int_0^u \frac{\mathcal{S}_\alpha[y_2]}{z^2}dz\right] \\ &= \frac{\Gamma(6)}{5\Gamma(7)}\frac{x^{6\alpha}}{\alpha^6} - \frac{\Gamma(4)}{6\Gamma(5)}\gamma\frac{x^{4\alpha}}{\alpha^4} + \frac{\Gamma(6)}{40\Gamma(7)}\frac{x^{6\alpha}}{\alpha^6} - \frac{\gamma^2}{2!}\frac{x^{2\alpha}}{\alpha^2} - \frac{1}{3\Gamma(5)}\frac{x^{4\alpha}}{\alpha^4}. \end{aligned} \tag{3.12}$$

Now by above discussion, then it is easy to get the power series solution of Eq. (3.1) as follows:

$$\begin{aligned} y(x) &= y_\circ + y_1 + y_2 + y_3 + ... \\ &= \frac{1}{2!}\frac{x^{2\alpha}}{\alpha^2} + \frac{1}{4!}\frac{x^{4\alpha}}{\alpha^4} - \frac{\gamma}{2!}\frac{x^{2\alpha}}{\alpha^2} + ... \end{aligned} \tag{(3.13)}$$

4. Analytical solutions of some linear and nonhomogeneous conformable fractional differential equations

In this Section, we apply the conformable fractional Sumudu transform to present the general analytical solutions of some linear and nonhomogeneous conformable fractional differential equations as in the following problems.

Problem 4.1.: Consider the following linear and nonhomogeneous conformable fractional differential equation:

$$D^{3\alpha}y + D^\alpha y = \frac{x^\alpha}{\alpha},\ y(0) = 0. \tag{4.1}$$

By applying the conformable fractional Sumudu transform $\mathcal{S}_\alpha$ of both sides of Eq. (4.1) and using Theorem 2.7, then we get:

$$\frac{Y_\alpha(u)}{u^3} + \frac{Y_\alpha(u)}{u} = \Gamma(2)u,$$

which implies that:

$$Y_\alpha(u)\left[\frac{1+u^2}{u^3}\right] = u.$$

Thus,

$$Y_\alpha(u) = \frac{u^4}{1+u^2}. \tag{4.2}$$

By taking $\mathcal{S}_\alpha^{-1}$ of both sides of Eq. (4.2), then we obtain the general solution of Eq. (4.1) as follows:

$$y(x) = \mathcal{S}_\alpha^{-1}\left[u^2 - 1 + \frac{1}{u^2+1}\right] = \frac{x^{2\alpha}}{2\alpha^2} + cos\frac{x^\alpha}{\alpha} - 1. \tag{4.3}$$

Problem 4.2.: Consider the following linear and nonhomogeneous conformable fractional differential equation:

$$D^{2\alpha}y + y = e^{\frac{2x^{\alpha}}{\alpha}},\ y(0) = 1. \tag{4.4}$$

By applying $\mathcal{S}_{\alpha}$ of both sides of Eq. (4.4) and using Theorems 2.7 and 2.8, we obtain:

$$\frac{Y_{\alpha}(u)}{u^2} - \frac{1}{u^2} + Y_{\alpha}(u) = \frac{1}{1-2u},$$

which implies that:

$$Y_{\alpha}(u)\left[\frac{1+u^2}{u^2}\right] = \frac{1}{1-2u} + \frac{1}{u^2}. \tag{4.5}$$

By taking $\mathcal{S}_{\alpha}^{-1}$ of both sides of Eq. (4.5), then we obtain the general solution of Eq. (4.4) as follows:

$$\begin{aligned} y(x) &= \mathcal{S}_{\alpha}^{-1}\left[\frac{u^2}{(1+u^2)(1-2u)}\right] + \mathcal{S}_{\alpha}^{-1}\left[\frac{1}{1+u^2}\right] \\ &= \mathcal{S}_{\alpha}^{-1}\left[\frac{Au+B}{1+u^2}\right] + \mathcal{S}_{\alpha}^{-1}\left[\frac{c}{1-2u}\right] + \mathcal{S}_{\alpha}^{-1}\left[\frac{1}{1+u^2}\right] \\ &= \frac{-2}{5}\mathcal{S}_{\alpha}^{-1}\left[\frac{u}{1+u^2}\right] - \frac{1}{5}\mathcal{S}_{\alpha}^{-1}\left[\frac{1}{1+u^2}\right] + \frac{1}{5}\mathcal{S}_{\alpha}^{-1}\left[\frac{1}{1-2u}\right] + \mathcal{S}_{\alpha}^{-1}\left[\frac{1}{1+u^2}\right] \\ &= -\frac{2}{5}sin\frac{x^{\alpha}}{\alpha} + \frac{4}{5}cos\frac{x^{\alpha}}{\alpha} + \frac{1}{5}e^{2\frac{x^{\alpha}}{\alpha}}. \end{aligned} \tag{4.6}$$

Problem 4.3.: Consider the following linear and nonhomogeneous conformable fractional differential equation:

$$D^{2\alpha}y - y = cos(\frac{x^{\alpha}}{\alpha}),\ y(0) = 0. \tag{4.7}$$

By applying $\mathcal{S}_{\alpha}$ of both sides of Eq. (4.7) and using Theorems 2.7 and 2.8, we obtain:

$$\frac{Y_{\alpha}(u)}{u^2} - Y_{\alpha}(u) = \frac{1}{1+u^2},$$

which implies that:

$$Y_{\alpha}(u) = \frac{u^2}{(1-u^2)(1+u^2)}. \tag{4.8}$$

By taking $\mathcal{S}_{\alpha}^{-1}$ of both sides of Eq. (4.8), then we obtain the general solution of Eq. (4.7) as follows:

$$\begin{aligned} y(x) &= \mathcal{S}_{\alpha}^{-1}\left[\frac{u^2}{(1-u^2)(1+u^2)}\right] \\ &= \mathcal{S}_{\alpha}^{-1}\left[\frac{Au+B}{1-u^2}\right] + \mathcal{S}_{\alpha}^{-1}\left[\frac{Du+C}{1+u^2}\right] \\ &= \frac{1}{2}\mathcal{S}_{\alpha}^{-1}\frac{1}{1-u^2} - \frac{1}{2}\mathcal{S}_{\alpha}^{-1}\left[\frac{1}{1+u^2}\right] \\ &= \frac{1}{2}cosh\frac{x^{\alpha}}{\alpha} - \frac{1}{2}cos\frac{x^{\alpha}}{\alpha}. \end{aligned} \tag{4.9}$$

5. Conclusion

We have developed the conformable fractional Sumudu transform for presenting the general analytical solution of the singular and nonlinear conformable fractional Poisson -Boltzmann equation and also for presenting the general solutions of some linear and nonhomogeneuos conformable fractional differential equations. In our opinion, it is worth to extend some other conformable fractional transforms such as: The conformable fractional Natural and conformable fractional Mellin trabsforns and using them in many application and comparisons. How to use conformable fractional Sumudu and Laplace transforns for solving some other nonlinear and singular conformable fractional differential equations such as conformable fractional Lane-Emden and conformable fractional Van Der Pol oscillator differential equations still need further researches.

References

[1] M. Herzallah, Notes on some fractional calculus operators and their properties, J. Frac. Calc. Appl., 5 (35) (19) (2014), 1-10.

[2] M. Caputo, Linear model of dissipation whose Q is almost frequency independent, Geophys. J. Int., 13 (5) (1967), 529-539.

[3] K. S. Miller and B. Ross , An introduction to the fractional calculus and fractional differential equations, Wiley and Sons, NY, USA, 1993.

[4] I. Pondlubny, Fractional differential equations:An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Academic Press, USA, 1998.

[5] A. El-Ajou, O. Abu Arqub, Z. Al-Zhour and S. Momani, New results on fractional power series: Theories and applications, Entropy, 15 (2013), 5305-5323.

[6] A. Kilicman and Z. Al-Zhour, Kronecker operational matrices for fractional calculus and some applications, Appl. Math. Comput., 187 (2007), 250-265.

[7] V. Daftardar-Gejji, Fractional calculus theory and applications, Narosa Publishing House, 2014.

[8] J. Leszczynski, An introduction of fractional mechanics, Czestochowa University of Technology, 2011.

[9] A. Atangana, S. Demiray and H. Bulut, Modelling the non-linear wave motion within the scope of the fractional calculus, Abstr. Appl. Anal., 2014 (2014), Art. ID 481657, 7 pages.

[10] G. Adomian, Solving frontier problems of physics the decomposition method, Kluwer Academic Publishers, 1994.

[11] R. Khalil, M. Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math., 264 (2014), 65-70.

[12] T. Abdeljawad, On conformable fractional calculus, J. Comput. Appl. Math., 279 (2015), 57-66.

[13] M. Abu Hammad and R. Khalil, Conformable fractional heat differential equation, Int. J. Pure Appl. Math., 94(2) (2014), 215-221.

[14] Z. Ayati, J. Biazar and M. Ilei, General solution of Bernoulli and Riccati fractional differential equations based on conformable fractional derivative, Int. J. Appl. Math. Res., 6(2) (2017), 49-51.

[15] U. Ghosh, M. Sarkar and D. Shantanu, Solution of linear fractional non-homogeneous differential equations with Jumarie fractional derivative and evaluation of particular integral, Amer. J. Math. Anal., 3(3) (2015), 54-64.

[16] L. Wang and J. Fu, Non-Noether symmetries of Hamiltonian systems with conformable fractional derivatives, Chin. Phys. B, 25(1) (2016), 4501.

[17] H. Guebbai and M. Ghiat, New conformable fractional Derivative definition for positive and increasing functions and its generalization, Adv. Dyn. Syst. Appl., 11 (2) (2016), 105-111.

[18] A Kareem, Conformable fractional derivatives and it is applications for solving fractional differential equations, IOSR J. Math., 13 (2017), 81-87.

[19] A. H. Khader, The conformable Laplace transform of the fractional Chebyshev and Legendre polynnomials, MSc.Thesis, Zarqa University, 2017.

[20] G. K. Watugala, Sumudu transform: a new integral transform to solve differential equations and control engineering problems, Int. J. Math. Educ. Sci. Technol., 24 (1) (1993), 35-43.

[21] M. A. Asiru, Sumudu transform and solution of integral equations of convolution type, Int. J. Math. Educ. Sci., 32 (6)(2001), 906-910.

[22] S. Demiray, H. Bulut and F. Bin Muhammad, Sumudu transform method for analytical solutions of fractional type ordinary differential equations, Math. Problems Eng., 6 (2015), 131-690.

[23] V. G. Gupta, B. Shrama and A. Kilicman, A note on fractional Sumudu transform, J. Appl. Math., 2010 (2010), Art. ID 154189, 9 pages.

[24] M. A Asiru, Further properties of the Sumudu transform and its applications, Int. J. Math. Educ. Sci. Technol., 33 (3) (2002), 441–449.

[25] M. A Asiru, Classroom note: application of the Sumudu transform to discrete dynamic systems, Int. J. Math. Educ. Sci. Technol., 34 (6) (2003), 944–949

[26] A. Kılıcman, H. Eltayeb and P. R. Agarwal, On Sumudu transform and system of differential equations, Abstr. Appl. Anal., 2010 (2010), Art. ID 598702, 11 pages.

[27] Kılıcman A. and Gadain H., On the applications of Laplace and Sumudu transforms, J. Franklin Inst., 347 (5) (2010) 848-862.

[28] J. Vashi and M. G.Timol, Laplace and Sumudu transforms and their application, Int. J. Innov. Sci., Eng. Technol., 3 (8) (2016), 538-542.

[29] S. Demiray, Y. Pandir and H. Bulut, Generalized Kudryashov method for time-fractional differential equations, Abstr. Appl. Anal., 2014 (2014), Art. ID 901540, 13 pages.

[30] R. Mittal and R. Nigam, Solution of fractional integral- differential equations by A domian decomposition method, Int. J. Appl. Math. Mech., 4 (2) (2008), 87-94.

[31] G. Adomian, Differential equation with singular coefficients, J. Appl. Math. Comput., 47 (1992), 179-184.

[32] A.Tandogan, Y. Pandir and Y. Gurefe, Solutions of the nonlinear differential equations by use of modified Kudryashov method, Turk. J. Math Computer Sci., 2013 (2013), Art. ID 20130021, 7 pages.

[33] D. A. Frank-Kamenetsky, Dokl. Akad. Nauk USSR 18, 411, 1938.

[34] D. A. Frank-Kamenetsky, Diffusion and Heat Transfer in Chemical Kinetics, 2nd ed., Plenum, New York, 1969.

[35] G. I. Barenblatt, J. B. Bell and W. Y. Crutchfield, The thermal explosion revisited, Proc. Natl. Acad. Sci., 95 (1998), 13384-13386.

[36] P. L. Chambré, On the solution of the Poisson-Boltzmann equation with application to the theory of thermal explosions, J. Chem. Phys., 20 (1952), 1795-1797.

[37] L. Samaj and E. Trizac, Poisson-Boltzmann thermodynamics of counterions confined by curved hard walls, Phys. Rev., 93 (2016), 012601.