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ON PROPERTIES OF CERTAIN ANALYTIC MULTIPLIER TRANSFORM OF COMPLEX ORDER

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Abstract. The focus of this paper is to investigate the subclasses $S^*C(\gamma,\mu,\alpha,\lambda;b)$, $TS^*C(\gamma,\mu,\alpha,\lambda;b) = T \cap S^*C(\gamma,\mu,\alpha,\lambda;b)$ and obtain the coefficient bounds as well as establishing its relationship with certain existing results in the literature.

1. Introduction

Let A be the class of normalized analytic functions f in the open unit disc $U = \{z \in C : |z| < 1\}$ with $f(0) = f'(0) = 0$ and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in C, \tag{1.1}$$

and S the class of all functions in A that are univalent in U. Also, the subclass of functions in A that are of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \tag{1.2}$$

is denoted by T and the subclasses $S^*(\alpha)$, $\quad C(\gamma)$ are given respectively by

$$S^*(\alpha) = \left\{ f \in S : \text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma \ z \in U \right\} \tag{1.3}$$

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$$C(\alpha) = \left\{ f \in S : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma \ z \in U, \geq \gamma < 1 \right\}. \tag{1.4}$$

Moreover, the class $TS^*(\gamma)$ denoted by $T \cap S^*(\gamma)$ which is the subclass of function $f \in T$ such that f is starlike of order γ and respectively, $TC(\gamma)$ is the class of function $f \in T$ such that f is convex of order γ. An interesting unification of the classes $S^*(\alpha)$ and $C(\gamma)$ denoted by $S^*C(\gamma, \beta)$ which satisfies the condition

$$\operatorname{Re}\left\{ \frac{zf'(z) + \beta z^2 f'(z)}{\beta z f'(z) + (1-\beta) f(z)} \right\} > \gamma \quad 0 \geq \gamma < 1, z \in U. \tag{1.5}$$

has been extensively studied by different researchers, for example, see [6] and [1,2,3]. The special cases for $\beta = 0, 1$ are given by $S^*(\gamma)$ and $C((\gamma))$ respectively.

Furthermore, the class $TS^*C(\gamma, \beta)$ which is the subclass of function $f \in T$ such that f belongs the class $S^*C(\gamma, \beta)$, was studied by Altintas et al. and other researchers. For details see [3, 5, 6].

Using the unification in (5), Nizami Mustafa [6] introduced and investigated the class $S^*C(\gamma, \beta; \tau)$ and $TS^*C(\gamma, \beta; \tau)$, $0 \leq \alpha < 1; \beta \in [0, 1]; \tau \in C$ which he defined as follows

A function $f \in S$ given by (1.1) is said to belong to the class $S^*C(\gamma, \beta; \tau)$ if the following condition is satisfied

$$\operatorname{Re}\left\{ 1 + \frac{1}{\tau}\left[\frac{zf'(z) + \beta z^2 f'(z)}{\beta z f'(z) + (1-\beta) f(z)} - 1 \right] \right\} > \gamma \quad 0 \geq \gamma < 1; \beta \in [0, 1]; \tau \in C - \{0\}, z \in U. \tag{1.6}$$

Meanwhile, the author in [4] defined a linear transformation $D^m_{\alpha,\lambda} f$ by

$$D^m_{\alpha,\lambda} f(z) = z + \sum_{n=2}^{\infty} \alpha \left(\frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^m a_n z^n, \quad 0 \leq \lambda \leq 1; \alpha \geq 1; m \in \mathbb{N} \cup 0 \tag{1.7}$$

Motivated by the work of Mustafa in [6], we study the effect of the application of the linear operator $D^m_{\alpha,\lambda} f$ on the unification of the classes of the functions $S^*C(\gamma, \beta; \tau)$.

Now, we define the class $S^*C(\gamma, \alpha, \lambda; b)$ to be class of functions $f \in S$ which satisfies the condition

$$\operatorname{Re}\left\{ 1 + \frac{1}{b}\left[\frac{z(D^m_{\alpha,\lambda} f)'(z) + \mu z^2 (D^m_{\alpha,\lambda} f)''(z)}{\mu z (D^m_{\alpha,\lambda} f)'(z) + (1-\mu)(D^m_{\alpha,\lambda} f)(z)} - 1 \right] \right\} > \gamma, 0 \geq \gamma < 1, z \in U; 0 \leq \lambda, \mu \leq 1; \alpha \geq 1; m \in \mathbb{N} \cup 0 \tag{1.8}$$

Also, we denote by D_T the subclass of the class of functions in (7) which is of the form

$$D^m_{\alpha,\lambda} f(z) = z - \sum_{n=2}^{\infty} \alpha \left(\frac{1 + \lambda(n + \alpha - 2)}{1 + \lambda(\alpha - 1)} \right)^m a_n z^n, \quad 0 \leq \lambda, \mu \leq 1; \alpha \geq 1; m \in \mathbb{N} \cup 0 \tag{1.9}$$

and denote by $TS^*C(\gamma, \mu, \alpha, \lambda; b) = T \cap S^*C(\gamma, \mu, \alpha, \lambda; b)$ which is the class of functions f in (1.9) such that f belong to the class $S^*C(\gamma, \mu, \alpha, \lambda; b) = T \cap S^*C(\gamma, \mu, \alpha, \lambda; b)$.

In this paper, we investigate the subclasses $S^*C(\gamma, \mu, \alpha, \lambda; b)$ and

$TS^*C(\gamma, \mu, \alpha, \lambda; b) = T \cap S^*C(\gamma, \mu, \alpha, \lambda; b)$

2. Coeffiecient bounds for the classes $S^*C_\alpha^\lambda(\gamma,\mu;b)$ and $TS^*C_\alpha^\lambda(\gamma,\mu;b)$

Theorem 2.1. *Let f be as defined in (1.1). Then the function $D^m_{\alpha,\lambda}f$ belongs to the class $S^*C(\gamma,\mu,\alpha,\lambda;b)$, $0\geq\gamma<1, z\in U; 0\leq\lambda,\mu\leq 1;\alpha\geq 1; m\in\mathbb{N}\cup 0$*

if

$$\sum_{n=2}^{\infty}\left[\alpha\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m[1+\mu(n-1)][n+|b|(1-\gamma)-1]\right]|a_n|\leq|b|(1-\gamma)$$

The result is sharp for the function

$$D^m_{\alpha,\lambda}f(z)=z+\frac{|b|(1-\gamma)(1+\lambda(\alpha-1))^m}{\alpha[1+\mu(n-1)][n+|b|(1-\gamma)](1+\lambda(n+\alpha-2))^m}z^n\quad n\geq 2$$

Proof. By (1.8), f belong to the class $S^*C(\gamma,\mu,\alpha,\lambda;b)$ if

$$Re\left\{1+\frac{1}{b}\left[\frac{z(D^m_{\alpha,\lambda}f)'(z)+\mu z^2(D^m_{\alpha,\lambda}f)''(z)}{\mu z(D^m_{\alpha,\lambda}f)'(z)+(1-\mu)(D^m_{\alpha,\lambda}f)(z)}-1\right]\right\}>\gamma$$

It suffices to show that:

$$\left|\frac{1}{b}\left[\frac{z(D^m_{\alpha,\lambda}f)'(z)+\mu z^2(D^m_{\alpha,\lambda}f)''(z)}{\mu z(D^m_{\alpha,\lambda}f)'(z)+(1-\mu)(D^m_{\alpha,\lambda}f)(z)}-1\right]\right|<1-\gamma \tag{2.1}$$

Simple computation in (2.1), using (1.7), we have:

$$\left|\frac{1}{b}\left[\frac{z(D^m_{\alpha,\lambda}f)'(z)+\mu z^2(D^m_{\alpha,\lambda}f)''(z)}{\mu z(D^m_{\alpha,\lambda}f)'(z)+(1-\mu)(D^m_{\alpha,\lambda}f)(z)}-1\right]\right|$$

$$=\left|\frac{1}{b}\left[\frac{z+\sum_{n=2}^{\infty}n\alpha\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m a_nz^n+\mu\sum_{n=2}^{\infty}n(n-1)\alpha\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m a_nz^n}{\mu z+\sum_{n=2}^{\infty}\mu n\alpha\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m a_nz^n+(1-\mu)\left(z+\sum_{n=2}^{\infty}\alpha\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m a_nz^n\right)}-1\right]\right|$$

$$=\left|\frac{1}{b}\left[\frac{z+\sum_{n=2}^{\infty}n\alpha[1+\mu(n-1)]\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m a_nz^n}{z+\sum_{n=2}^{\infty}\alpha(1+\mu(n-1))\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m a_nz^n}-1\right]\right|$$

$$\leq\frac{1}{b}\left[\frac{\sum_{n=2}^{\infty}\alpha(n-1)[1+\mu(n-1)]\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m|a_n|}{1-\sum_{n=2}^{\infty}\alpha(1+\mu(n-1))\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m|a_n|}\right]$$

which is bounded by $1-\gamma$ if

$\sum_{n=2}^{\infty}\alpha(n-1)[1+\mu(n-1)]\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m|a_n|\leq|b|(1-\gamma)1-\sum_{n=2}^{\infty}\alpha(1+\mu(n-1))\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m|a_n|$

which is equivalent to

$\sum_{n=2}^{\infty}\left[\alpha(n-1)[1+\mu(n-1)]\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m+\alpha|b|(1-\gamma)(1+\mu(n-1))\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m\right]|a_n|\leq|b|(1-\gamma)$

Which implies that

$$\sum_{n=2}^{\infty}\left[\alpha\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m[1+\mu(n-1)][n+|b|(1-\gamma)-1]\right]|a_n|\leq|b|(1-\gamma) \tag{2.2}$$

Thus, (2.1) is satisfied if (2.2) is satisfied. □

Corollary 2.1. *Let f be as defined in (1) and the function $D^m_{\alpha,\lambda}f$ belongs to the class $S^*C(\gamma,\mu,\alpha,\lambda;b)$, $0 \geq \gamma < 1, z \in U; 0 \leq \lambda,\mu \leq 1; \alpha \geq 1; m \in \mathbb{N}\cup 0$. Then*

$$|a_n| \leq \frac{|b|(1-\gamma)(1+\lambda(\alpha-1))^m}{\alpha[1+\mu(n-1)][n+|b|(1-\gamma)-1](1+\lambda(n+\alpha-2))^m}$$

Corollary 2.2. *Let f be as defined in (1.1). Then the function $D^m_{\alpha,\lambda}f$ belongs to the class $S^*C(\gamma,\mu,1,\lambda,m;b)$, $0 \geq \gamma < 1, z \in U; 0 \leq \lambda,\mu \leq 1; m \in \mathbb{N}\cup 0$ if*

$$\sum_{n=2}^{\infty}[(1+\lambda(n-1))^m\,[1+\mu(n-1)][n+|b|(1-\gamma)-1]]\,|a_n| \leq |b|(1-\gamma) \tag{2.3}$$

The result is sharp for the function

$$D^m_{\alpha,\lambda}f(z) = z + \frac{|b|(1-\gamma)}{[1+\mu(n-1)][n+|b|(1-\gamma)-1](1+\lambda(n-1))^m}z^n, \quad n \geq 2$$

Corollary 2.3. *Let f be as defined in (1.1). Then the function $D^m_{\alpha,\lambda}f$ belongs to the class $S^*C(\gamma,\mu,1,\lambda,1;b)$, $0 \geq \gamma < 1, z \in U; 0 \leq \lambda,\mu \leq 1; m \in \mathbb{N}\cup 0$ if*

$$\sum_{n=2}^{\infty}[(1+\lambda(n-1))\,[1+\mu(n-1)][n+|b|(1-\gamma)-1]]\,|a_n| \leq |b|(1-\gamma) \tag{2.4}$$

The result is sharp for the function

$$D^m_{\alpha,\lambda}f(z) = z + \frac{|b|(1-\gamma)}{[1+\mu(n-1)][n+|b|(1-\gamma)-1](1+\lambda(n-1))}z^n, \quad n \geq 2$$

Corollary 2.4. *Let f be as defined in (1.1). Then the function $D^m_{\alpha,\lambda}f$ belongs to the class $S^*C(\gamma,\mu,1,1,1;b)$, $0 \geq \gamma < 1, z \in U; 0 \leq \lambda,\mu \leq 1; m \in \mathbb{N}\cup 0$ if*

$$\sum_{n=2}^{\infty}[n[1+\mu(n-1)][n+|b|(1-\gamma)-1]]\,|a_n| \leq |b|(1-\gamma) \tag{2.5}$$

The result is sharp for the function

$$D^m_{\alpha,\lambda}f(z) = z + \frac{|b|(1-\gamma)}{n[1+\mu(n-1)][n+|b|(1-\gamma)-1]}z^n, \quad n \geq 2$$

Corollary 2.5. *Let f be as defined in (1.1). Then the function $D^m_{\alpha,\lambda}f$ belongs to the class $S^*C(\gamma,\mu,1,0,1;b)$, $0 \geq \gamma < 1, z \in U; 0 \leq \lambda,\mu \leq 1; m \in \mathbb{N}\cup 0$ if*

$$\sum_{n=2}^{\infty}[[1+\mu(n-1)][n+|b|(1-\gamma)-1]]\,|a_n| \leq |b|(1-\gamma) \tag{2.6}$$

The result is sharp for the function

$$D^m_{\alpha,\lambda}f(z) = z + \frac{|b|(1-\gamma)}{[1+\mu(n-1)][n+|b|(1-\gamma)-]}z^n, \quad n \geq 2$$

This result agrees with the Theorem 2.1 in [6].

Corollary 2.6. *Let f be as defined in (1.1). Then the function $D^m_{\alpha,\lambda}f$ belongs to the class $S^*C(\gamma,0,1,\lambda,0;1)$, $0\geq\gamma<1, z\in U; 0\leq\lambda,\mu\leq 1; m\in\mathbb{N}\cup 0$ if*

$$\sum_{n=2}^{\infty}[[1+\mu(n-1)][n-\gamma]]\,|a_n|\leq 1-\gamma \tag{2.7}$$

The result is sharp for the function

$$D^m_{\alpha,\lambda}f(z)=z+\frac{1-\gamma}{[1+\mu(n-1)][n-\gamma]}z^n,\quad n\geq 2$$

This result agrees with the Corollary 2.1 in [6].

Corollary 2.7. *Let f be as defined in (1.1). Then the function $D^m_{\alpha,\lambda}f$ belongs to the class $S^*C(\gamma,\mu,1,\lambda,0;1)$, $0\geq\gamma<1, z\in U; 0\leq\lambda,\mu\leq 1; m\in\mathbb{N}\cup 0$ if*

$$\sum_{n=2}^{\infty}(n-\gamma)|a_n|\leq 1-\gamma \tag{2.8}$$

The result is sharp for the function

$$D^m_{\alpha,\lambda}f(z)=z+\frac{1-\gamma}{n-\gamma}z^n,\quad n\geq 2$$

This result agrees with the Corollary 2.2 in [6].

Theorem 2.2. *Let $f\in D_T$. Then the function $D^m_{\alpha,\lambda}f$ belongs to the class $D_TS^*C(\gamma,\mu,\alpha,\lambda;b)$, $0\geq\gamma<1, z\in U; 0\leq\lambda,\mu\leq 1; \alpha\geq 1; m\in\mathbb{N}\cup 0$ if and only if*

$$\sum_{n=2}^{\infty}\alpha(n-1)[1+\mu(n-1)][n+b(1-\gamma)]\left(\frac{x}{y}\right)^m|a_n|\leq|b|(1-\gamma)$$

Proof. We shall prove only the necessity part of the Theorem as the sufficiency proof is similar to the proof of Theorem 1.

Let f be as defined in (1.1) and $D^m_{\alpha,\lambda}f$ belongs to the class $TS^*C(\gamma,\mu,\alpha,\lambda;b)$, $0\geq\gamma<1, z\in U; 0\leq\lambda,\mu\leq 1; \alpha\geq 1; m\in\mathbb{N}\cup 0; b\in\mathbb{R}-\{0\}$, we have

$$Re\left\{1+\frac{1}{b}\left[\frac{z(D^m_{\alpha,\lambda}f)'(z)+\mu z^2(D^m_{\alpha,\lambda}f)''(z)}{\mu z(D^m_{\alpha,\lambda}f)'(z)+(1-\mu)(D^m_{\alpha,\lambda}f)(z)}-1\right]\right\}>\gamma \tag{2.9}$$

Using (1.7) in (2.9) and by algebraic simplification, we have

$$\mathrm{Re}\left\{\frac{-\sum_{n=2}^{\infty}\alpha(n-1)[1+\mu(n-1)]\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m a_nz^n}{b\left\{z-\sum_{n=2}^{\infty}\alpha(1+\mu(n-1))\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m a_nz^n\right\}}\right\}\geq\gamma-1$$

Choosing z to be real and $z\longrightarrow 1$, we have

$$\frac{-\sum_{n=2}^{\infty}\alpha(n-1)[1+\mu(n-1)]\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m a_n}{b\left\{1-\sum_{n=2}^{\infty}\alpha(1+\mu(n-1))\left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m a_n\right\}}\geq\gamma-1 \tag{2.10}$$

$b \in \mathbb{R} - \{0\}$ implies that b could be greater or less than zero.

Let $b > 0$ in (19), we have

$$-\sum_{n=2}^{\infty} \alpha(n-1)[1+\mu(n-1)] \left(\frac{x}{y}\right)^m a_n \geq (\gamma-1)b \left\{1 - \sum_{n=2}^{\infty} \alpha(1+\mu(n-1)) \left(\frac{x}{y}\right)^m a_n\right\} \tag{2.11}$$

where $x = 1 + \lambda(n + \alpha - 2)$ and $y = 1 + \lambda(\alpha - 1)$ From (20), we have

$$\sum_{n=2}^{\infty} \alpha(n-1)[1+\mu(n-1)][n+b(1-\gamma)] \left(\frac{x}{y}\right)^m |a_n| \leq b(1-\gamma) \tag{2.12}$$

Now suppose $b < 0$, which implies that $b = -|b|$ and substituting $b = -|b|$ in (19), we have

$$\frac{\sum_{n=2}^{\infty} \alpha(n-1)[1+\mu(n-1)] \left(\frac{x}{y}\right)^m a_n}{|b| \left\{1 - \sum_{n=2}^{\infty} \alpha(1+\mu(n-1)) \left(\frac{x}{y}\right)^m a_n\right\}} \geq \tag{2.13}$$

$\sum_{n=2}^{\infty} \alpha(n-1)[1+\mu(n-1)] \left(\frac{x}{y}\right)^m |a_n| \geq (\gamma-1)|b| \left\{1 - \sum_{n=2}^{\infty} \alpha(1+\mu(n-1)) \left(\frac{x}{y}\right)^m a_n\right\}$ which implies

$$\sum_{n=2}^{\infty} \alpha(n-1)[1+\mu(n-1)][n+b(1-\gamma)] \left(\frac{1+\lambda(n+\alpha-2)}{1+\lambda(\alpha-1)}\right)^m |a_n| \geq -b(1-\gamma) \tag{2.14}$$

From (21) and (23), the proof of the necessity is completed. □

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