International Journal of Analysis and Applications
Volume 18, Number 2 (2020), 277-303
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-18-2020-277

FUZZY IDEALS ON ORDERED AG-GROUPOIDS

NASREEN KAUSAR*, MESHARI ALESEMI, SALAHUDDIN

Department of Mathematics, University of Agriculture FSD-Pakistan, Department of Mathematics, Jazan University, Jazan, Kingdom of Saudi Arabia

Email addresses: kausar.nasreen57@gmail.com(Kausar); malesemi@jazanu.edu.sa(Alesemi); drsalah12@hotmail.com(Salahuddin)

* *Corresponding author:* kausar.nasreen57@gmail.com

Abstract. In this paper, we define the concept of direct product of finite fuzzy normal subrings over non-associative and non-commutative rings (LA-ring) and investigate the some fundamental properties of direct product of fuzzy normal subrings.

1. Introduction

In 1972, a generalization of commutative semigroups has been established by Kazim et al [12]. In ternary commutative law: $abc = cba$, they introduced the braces on the left side of this law and explored a new pseudo associative law, that is $(ab)c = (cb)a$. This law $(ab)c = (cb)a$ is called the left invertive law. A groupoid S is said to be a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law: $(ab)c = (cb)a$. This structure is also known as Abel-Grassmann's groupoid (abbreviated as AG-groupoid) in [22]. An AG-groupoid is a midway structure between an abelian semigroup and a groupoid. Mushtaq et al [21], investigated the concept of ideals of AG-groupoids.

In [4] (resp. [1]), a groupoid S is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). In [12], an AG-groupoid is medial, but in general an AG-groupoid needs not to be

Received 2019-07-16; accepted 2019-08-27; published 2020-03-02.

2010 *Mathematics Subject Classification.* 03F55, 08A72, 20N25.

Key words and phrases. fuzzy left (right, interior, quasi-, bi-, generalized bi-) ideals; regular (intra-regular) ordered AG-groupoids.

paramedial. Every AG-groupoid with left identity is paramedial by Protic et al [22] and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

In [13], if $(S, \cdot, \leq)$ is an ordered semigroup and $\emptyset \neq A \subseteq S$, we define a subset of S as follows: $(A] = \{s \in S : s \leq a$ for some $a \in A\}$. A non-empty subset A of S is called a subsemigroup of S if $A^2 \subseteq A$. A is called a left (resp. right) ideal of S if following hold (1) $SA \subseteq A$ (resp. $AS \subseteq A$). (2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$. Equivalent definition: A is called a left (resp. right) ideal of S if $(A] \subseteq A$ and $SA \subseteq A$ (resp. $AS \subseteq A$).

In [13,15], an ordered semigroup S is said to be a regular if for every $a \in S$, there exists an element $x \in S$ such that $a \leq axa$. In [14,15], an ordered semigroup S is said to be an intra-regular if for every $a \in S$ there exist elements $x, y \in S$ such that $a \leq xa^2y$.

We will define the concept of fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals of an ordered AG-groupoid S. We will establish a study by discussing the different properties of such ideals. We will also characterize regular (resp. intra-regular, both regular and intra-regular) ordered AG-groupoids by the properties of fuzzy left (right, quasi-, bi-, generalized bi-) ideals.

2. Fuzzy Ideals on ordered AG-groupoids

In [25] An ordered AG-groupoid S, is a partially ordered set, at the same time an AG-groupoid such that $a \leq b$, implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in S$. Two conditions are equivalent to the one condition $(ca)d \leq (cb)d$ for all $a, b, c, d \in S$.

Let S be an ordered AG-groupoid and $\emptyset \neq A \subseteq S$, we define a subset $(A] = \{s \in S : s \leq a$ for some $a \in A\}$ of S, obviously $A \subseteq (A]$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $\emptyset \neq A, B \subseteq S$, then $AB = \{ab \mid a \in A, b \in B\}$, $((A]] = (A]$, $(A](B] \subseteq (AB]$, $((A](B]] = (AB]$, if $A \subseteq B$ then $(A] \subseteq (B]$, $(A \cap B] \neq (A] \cap (B]$ in general.

For $\emptyset \neq A \subseteq S$. A is called an AG-subgroupoid of S if $A^2 \subseteq A$. A is called a left (resp. right) ideal of S if the following hold (1) $SA \subseteq A$ (resp. $AS \subseteq A$). (2) if $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$. Equivalent definition: A is called a left (resp. right) ideal of S if $(A] \subseteq A$ and $SA \subseteq A$ (resp. $AS \subseteq A$). A is called an ideal of S if A is both a left ideal and a right ideal of S. In particular, if A and B are any types of ideals of S, then $(A \cap B] = (A] \cap (B]$.

We denote by $L(a), S(a), I(a)$ the left ideal, the right ideal and the ideal of S, respectively generated by a. We have $L(a) = \{s \in S : s \leq a$ or $s \leq xa$ for some $x \in S\} = (a \cup Sa]$, $S(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup (Sa)S]$.

First time, Zadeh introduced the concept of fuzzy set in his classical paper [27] of 1965. This concept has provided a useful mathematical tool for describing the behavior of systems that are too complex to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields such as artificial intelligence, computer science, management science, expert systems, finite state machines, languages, robotics, coding theory and others.

Rosenfeld [23], was the first, who introduced the concept of fuzzy set in a group. The study of fuzzy set in semigroups was established by Kuroki [18, 19]. He studied fuzzy ideals and fuzzy interior (resp. quasi-, bi-, generalized bi-, semiprime, semiprime quasi-) ideals of semigroups. A systematic exposition of fuzzy semigroups appeared by Mordeson et al [20], where one can find the theoretical results on fuzzy semigroups and their use in fuzzy finite state machines and languages. Fuzzy sets in ordered semigroups/ordered groupoids were first explored by Kehayopulu et al [16, 17]. They also studied fuzzy ideals and fuzzy interior (resp. quasi-, bi-, generalized bi-) ideals in ordered semigroups.

By a fuzzy subset μ of an ordered AG-groupoid S, we mean a function $\mu : S \to [0,1]$, the complement of μ is denoted by μ', is a fuzzy subset of S given by $\mu'(x) = 1 - \mu(x)$ for all $x \in S$.

A fuzzy subset μ of S is called a fuzzy AG-subgroupoid of S if $\mu(xy) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in S$. μ is called a fuzzy left (resp. right) ideal of S if (1) $\mu(xy) \geq \mu(y)$ (resp. $\mu(xy) \geq \mu(x)$). (2) $x \leq y$, implies $\mu(x) \geq \mu(y)$ for all $x, y \in S$. μ is a fuzzy ideal of S if μ is both a fuzzy left ideal and a fuzzy right ideal of S. Every fuzzy ideal (whether left, right, two-sided) is a fuzzy AG-subgroupoid of S but the converse is not true in general.

A fuzzy subset μ of S is called a fuzzy interior ideal of S if (1) $\mu((xy)z) \geq \mu(y)$. (2) $x \leq y$, implies $\mu(x) \geq \mu(y)$ for all $x, y, z \in S$. A fuzzy subset μ of S is called a fuzzy quasi-ideal of S if (1) $(\mu \circ S) \cap (S \circ \mu) \subseteq \mu$. (2) $x \leq y$, implies $\mu(x) \geq \mu(y)$ for all $x, y \in S$. A fuzzy AG-subgroupoid μ of S is called a fuzzy bi-ideal of S if (1) $\mu((xa)y) \geq min\{\mu(x), \mu(y)\}$. (2) $x \leq y$, implies $\mu(x) \geq \mu(y)$ for all $x, a, y \in S$. A fuzzy subset μ of S is called a fuzzy generalized bi-ideal of S if (1) $\mu((xa)y) \geq min\{\mu(x), \mu(y)\}$. (2) $x \leq y$, implies $\mu(x) \geq \mu(y)$ for all $x, a, y \in S$. Every fuzzy bi-ideal of S is a fuzzy generalized bi-ideal of S. A fuzzy ideal μ of S is called a fuzzy idempotent of S if $\mu \circ \mu = \mu$.

We denote by $F(S)$, the set of all fuzzy subsets of S. We define an order relation "$\subseteq$" on $F(S)$ such that $\mu \subseteq \gamma$ if and only if $\mu(x) \leq \gamma(x)$ for all $x \in S$. Then $(F(S), \circ, \subseteq)$ is an ordered AG-groupoid.

By the symbols $\mu \wedge \gamma$ and $\mu \vee \gamma$, we mean the following fuzzy subsets:

$$(\mu \wedge \gamma)(x) = min\{\mu(x), \gamma(x)\} \text{ and } (\mu \vee \gamma)(x) = max\{\mu(x), \gamma(x)\}.$$

For $\emptyset \neq A \subseteq S$, the characteristic function of A is denoted by χ_A and defined by

$$\chi_A(a) = \begin{cases} 1 \text{ if } a \in A \\ 0 \text{ if } a \notin A \end{cases}$$

An ordered AG-groupoid S can be considered a fuzzy subset of itself and we write $S = \chi_S$, i.e., $S(x) = \chi_S(x) = 1$ for all $x \in S$. This implies that $S(x) = 1$ for all $x \in S$.

Let $x \in S$, we define a set $A_x = \{(y, z) \in S \times S \mid x \leq yz\}$. Let μ and γ be two fuzzy subsets of S, then product of μ and γ is denoted by $\mu \circ \gamma$ and defined by:

$$(\mu \circ \gamma)(x) = \begin{cases} \vee_{(y,z)\in A_x} min\{\mu(y), \gamma(z)\} & \text{if } A_a \neq \emptyset \\ 0 & \text{if } A_a = \emptyset \end{cases}$$

Now we give the imperative properties of such ideals of an odered AG-groupoid S, which will be very helpful in later sections.

Lemma 2.1. *Let S be an ordered AG-groupoid. Then the following properties hold.*

(1) $(\mu \circ \gamma) \circ \beta = (\beta \circ \gamma) \circ \mu$,

(2) $(\mu \circ \gamma) \circ (\beta \circ \delta) = (\mu \circ \beta) \circ (\gamma \circ \delta)$ *for all fuzzy subsets μ, γ, β and δ of S.*

Proof. Let μ, γ and β be fuzzy subsets of an ordered AG-groupoid S. We have to show that $(\mu \circ \gamma) \circ \beta = (\beta \circ \gamma) \circ \mu$. Now

$$\begin{aligned} ((\mu \circ \gamma) \circ \beta)(x) &= \vee_{(y,z)\in A_x} min\{(\mu \circ \gamma)(y), \beta(z)\} \\ &= \vee_{(y,z)\in A_x} min\{\vee_{(s,t)\in A_y} min\{\mu(s), \gamma(t)\}, \beta(z)\} \\ &= \vee_{((s,t),z)\in A_x} min\{min\{\mu(s), \gamma(t)\}, \beta(z)\} \\ &= \vee_{((s,t),z)\in A_x} min\{(\mu(s), \gamma(t)), \beta(z)\} \\ &= \vee_{((z,t),s)\in A_x} min\{(\beta(z), \gamma(t)), \mu(s)\} \\ &= \vee_{((z,t),s)\in A_x} min\{min\{\beta(z), \gamma(t)\}, \mu(s)\} \\ &= \vee_{(w,s)\in A_x} min\{\vee_{(z,t)\in A_w} min\{\beta(z), \gamma(t)\}, \mu(s)\} \\ &= \vee_{(w,s)\in A_x} min\{(\beta \circ \gamma)(w), \mu(s)\} = ((\beta \circ \gamma) \circ \mu)(x) \end{aligned}$$

Similarly, we can prove (2). □

Proposition 2.1. *Let S be an ordered AG-groupoid with left identity e. Then the following assertions hold.*

(1) $\mu \circ (\gamma \circ \beta) = \gamma \circ (\mu \circ \beta)$,

(2) $(\mu \circ \gamma) \circ (\beta \circ \delta) = (\delta \circ \gamma) \circ (\beta \circ \mu)$,

(3) $(\mu \circ \gamma) \circ (\beta \circ \delta) = (\delta \circ \beta) \circ (\gamma \circ \mu)$ *for all fuzzy subsets μ, γ, β and δ of S.*

Proof. Same as Lemma 2.1. □

Theorem 2.1. *Let A and B be two non-empty subsets of an ordered AG-groupoid S. Then the following assertions hold.*

(1) *If $A \subseteq B$ then $\chi_A \subseteq \chi_B$.*

(2) $\chi_A \circ \chi_B = \chi_{(AB]}$.

(3) $\chi_A \cup \chi_B = \chi_{A\cup B}$.

(4) $\chi_A \cap \chi_B = \chi_{A\cap B}$.

Proof. Straight forward. $\square$

Theorem 2.2. *Let A be a non-empty subset of an ordered AG-groupoid S. Then the following properties hold.*

(1) *A is an AG-subgroupoid of S if and only if χ_A is a fuzzy AG-subgroupoid of S.*

(2) *A is a left (resp. right, two-sided) ideal of S if and only if χ_A is a fuzzy left (resp. right, two-sided) ideal of S.*

Proof. (1) Let A be an AG-subgroupoid of S and $a, b \in S$. If $a, b \in A$, then by definition $\chi_A(a) = 1 = \chi_A(b)$. Since $ab \in A$, A being an AG-subgroupoid of S, this implies that $\chi_A(ab) = 1$. Thus $\chi_A(ab) \geq \chi_A(a) \wedge \chi_A(b)$. Similarly, we have $\chi_A(ab) \geq \chi_A(a) \wedge \chi_A(b)$, when $a, b \notin A$. Hence χ_A is a fuzzy AG-subgroupoid of S.

Conversely, suppose that χ_A is a fuzzy AG-subgroupoid of S and let $a, b \in A$. Since $\chi_A(ab) \geq \chi_A(a) \wedge \chi_A(b) = 1$, χ_A being a fuzzy AG-subgroupoid of S. Thus $\chi_A(ab) = 1$, i.e., $ab \in A$. Hence A is an AG-subgroupoid of S.

(2) Let A be a left ideal of S and $a, b \in S$. If $a, b \in A$, then by definition $\chi_A(b) = 1$. Since $ab \in A$, A being a left ideal of S, this means that $\chi_A(ab) = 1$. Thus $\chi_A(ab) \geq \chi_A(b)$. Similarly, we have $\chi_A(ab) \geq \chi_A(b)$, when $a, b \notin A$. Therefore χ_A is a fuzzy left ideal of S.

Conversely, assume that χ_A is a fuzzy left ideal of S. Let $a, b \in A$ and $z \in S$. Since $\chi_A(zb) \geq \chi_A(b) = 1$, χ_A being a fuzzy left ideal of S. Thus $\chi_A(zb) = 1$, i.e., $zb \in A$. Therefore A is a left ideal of S. $\square$

Theorem 2.3. *Let μ be a fuzzy subset of an ordered AG-groupoid S. Then the following assertions hold.*

(1) *μ is a fuzzy AG-subgroupoid of S if and only if $\mu \circ \mu \subseteq \mu$.*

(2) *μ is a fuzzy left (resp. right) ideal of S if and only if $S \circ \mu \subseteq \mu$ (resp. $\mu \circ S \subseteq \mu$).*

Proof. (1) Suppose that μ is a fuzzy AG-groupoid of S and $x \in S$. For $\mu \circ \mu \subseteq \mu$.

If $(\mu \circ \mu)(x) = 0$, then $\mu \circ \mu \subseteq \mu$, otherwise we have

$$\begin{aligned}(\mu \circ \mu)(x) &= \vee_{(y,z)\in A_x} min\{\mu(y), \mu(z)\} \\ &\leq \vee_{(y,z)\in A_x} min\{\mu(yz)\} = \mu(x). \\ &\Rightarrow \mu \circ \mu \subseteq \mu.\end{aligned}$$

Conversely, assume that $\mu \circ \mu \subseteq \mu$. Let $y, z \in S$ such that $x \leq yz$. Now

$$\begin{aligned}
\mu(yz) &\geq \mu(x) \geq (\mu \circ \mu)(x) \\
&= \vee_{(s,t) \in A_x} min\{\mu(s), \mu(t)\} \\
&\geq \mu(y) \wedge \mu(z). \\
&\Rightarrow \mu(yz) \geq \mu(y) \wedge \mu(z).
\end{aligned}$$

Hence μ is a fuzzy AG-subgroupoid of S.

(2) Suppose that μ is a fuzzy left ideal of S and $x \in S$. If $(S \circ \mu)(x) = 0$, then $S \circ \mu \subseteq \mu$, otherwise we have

$$\begin{aligned}
(S \circ \mu)(x) &= \vee_{(y,z) \in A_x} min\{S(y), \mu(z)\} \\
&= \vee_{(y,z) \in A_x} min\{1, \mu(z)\} \\
&= \vee_{(y,z) \in A_x} \mu(z) \\
&\leq \vee_{(y,z) \in A_x} \mu(yz) = \mu(x). \\
&\Rightarrow S \circ \mu \subseteq \mu.
\end{aligned}$$

Conversely, assume that $S \circ \mu \subseteq \mu$. Let $y, z \in S$ such that $x \leq yz$. Now

$$\begin{aligned}
\mu(yz) &\geq \mu(x) \geq (S \circ \mu)(x) \\
&= \vee_{(s,t) \in A_x} min\{S(s), \mu(t)\} \\
&\geq S(y) \wedge \mu(z) = 1 \wedge \mu(z) = \mu(z). \\
&\Rightarrow \mu(yz) \geq \mu(z).
\end{aligned}$$

Therefore μ is a fuzzy left ideal of S. Similarly, we can prove (3). □

Lemma 2.2. *If μ and γ are two fuzzy AG-subgroupoids (resp. (left, right, two-sided) ideals) of an ordered AG-groupoid S, then $\mu \cap \gamma$ is also a fuzzy AG-subgroupoid (resp. (left, right, two-sided) ideal) of S.*

Proof. Let μ and γ be two fuzzy AG-subgroupoids of S. We have to show that $\mu \cap \gamma$ is also a fuzzy AG-subgroupoid of S. Now

$$\begin{aligned}
(\mu \cap \gamma)(xy) &= \mu(xy) \wedge \gamma(xy) \\
&\geq \{\mu(x) \wedge \mu(y)\} \wedge \{\gamma(x) \wedge \gamma(y)\} \\
&= \mu(x) \wedge \{\mu(y) \wedge \gamma(x)\} \wedge \gamma(y) \\
&= \mu(x) \wedge \{\gamma(x) \wedge \mu(y)\} \wedge \gamma(y) \\
&= \{\mu(x) \wedge \gamma(x)\} \wedge \{\mu(y) \wedge \gamma(y)\} \\
&= (\mu \cap \gamma)(x) \wedge (\mu \cap \gamma)(y).
\end{aligned}$$

Hence $\mu \cap \gamma$ is a fuzzy AG-subgroupoids of S. Similarly, for ideals. □

Lemma 2.3. *If μ and γ are two fuzzy AG-subgroupoids of an ordered AG-groupoid S, then $\mu \circ \gamma$ is also a fuzzy AG-subgroupoid of S.*

Proof. Let μ and γ be two fuzzy AG-subgroupoids of S. We have to show that $\mu \circ \gamma$ is also a fuzzy AG-subgroupoid of S. Now $(\mu \circ \gamma)^2 = (\mu \circ \gamma) \circ (\mu \circ \gamma) = (\mu \circ \mu) \circ (\gamma \circ \gamma) \subseteq \mu \circ \gamma$. Hence $\mu \circ \gamma$ is a fuzzy AG-subgroupoid of S. □

Remark 2.1. *If μ is a fuzzy AG-subgroupoid of an ordered AG-groupoid S, then $\mu \circ \mu$ is also a fuzzy AG-subgroupoid of S.*

Lemma 2.4. *Let S be an ordered AG-groupoid with left identity e. Then $SS = S$ and $eS = S = Se$.*

Proof. Since $SS \subseteq S$ and $x = ex \in SS$, i.e., $SS = S$. Since e is a left identity of S, i.e., $eS = S$. Now $Se = (SS)\, e = (eS)S = SS = S$. □

Lemma 2.5. *Let S be an ordered AG-groupoid with left identity e. Then every fuzzy right ideal of S is a fuzzy ideal of S.*

Proof. Let μ be a fuzzy right ideal of S and $x, y \in S$. Now $\mu\,(xy) = \mu\,((ex)\,y) = \mu\,((yx)\,e) \geq \mu\,(yx) \geq \mu\,(y)\,.$ Thus μ is a fuzzy ideal of S. □

Lemma 2.6. *If μ and γ are two fuzzy left (resp. right) ideals of an ordered AG-groupoid S with left identity e, then $\mu \circ \gamma$ is also a fuzzy left (resp. right) ideal of S.*

Proof. Let μ and γ be two fuzzy left ideals of S. We have to show that $\mu \circ \gamma$ is also a fuzzy left ideal of S. Now $S \circ (\mu \circ \gamma) = (S \circ S) \circ (\mu \circ \gamma) = (S \circ \mu) \circ (S \circ \gamma) \subseteq \mu \circ \gamma$. Hence $\mu \circ \gamma$ is a fuzzy left ideal of S. Similarly, for right ideals. □

Remark 2.2. *If μ is a fuzzy left (resp. right) ideal of an ordered AG-groupoid S with left identity e. Then $\mu \circ \mu$ is a fuzzy ideal of S.*

Lemma 2.7. *If μ and γ are two fuzzy ideals of an ordered AG-groupoid S, then $\mu \circ \gamma \subseteq \mu \cap \gamma$.*

Proof. Let μ and γ be two fuzzy ideals of S and $x \in S$. If $(\mu \circ \gamma)(x) = 0$, then $\mu \circ \gamma \subseteq \mu \cap \gamma$, otherwise we have

$$\begin{aligned}
(\mu \circ \gamma)(x) &= \vee_{(y,z)\in A_x} min\{\mu(y), \gamma(z)\} \\
&= \vee_{(y,z)\in A_x} min\{\mu(yz), \gamma(yz)\} \\
&= \vee_{(y,z)\in A_x} \{\mu(yz) \wedge \gamma(yz)\} \\
&= \vee_{(y,z)\in A_x} (\mu \cap \gamma)(yz) = (\mu \cap \gamma)(x). \\
&\Rightarrow \mu \circ \gamma \subseteq \mu \cap \gamma.
\end{aligned}$$

□

Remark 2.3. *If μ is a fuzzy ideal of an ordered AG-groupoid S, then $\mu \circ \mu \subseteq \mu$.*

Lemma 2.8. *Let S be an ordered AG-groupoid. Then $\mu \circ \gamma \subseteq \mu \cap \gamma$ for every fuzzy right ideal μ and every fuzzy left ideal γ of S.*

Proof. Same as Lemma 2.7. □

Theorem 2.4. *Let A be a non-empty subset of an ordered AG-groupoid S. Then the following conditions are true.*

(1) *A is an interior ideal of S if and only if χ_A is a fuzzy interior ideal of S.*
(2) *A is a quasi-ideal of S if and only if χ_A is a fuzzy quasi-ideal of S.*
(3) *A is a bi-ideal of S if and only if χ_A is a fuzzy bi-ideal of S.*
(4) *A is a generalized bi-ideal of S if and only if χ_A is a fuzzy generalized bi-ideal of S.*

Proof. (1) Let A be an interior ideal of S and $x, y, a \in S$. If $a \in A$, then by definition $\chi_A(a) = 1$. Since $(xa)y \in A$, A being an interior ideal of S, this means that $\chi_A((xa)y) = 1$. Thus $\chi_A((xa)y) \geq \chi_A(a)$. Similarly, we have $\chi_A((xa)y) \geq \chi_A(a)$, when $a \notin A$. Hence χ_A is a fuzzy interior ideal of S.

Conversely, suppose that χ_A is a fuzzy interior ideal of S. Let $x, y \in S$ and $a \in A$, so $\chi_A(a) = 1$. Since $\chi_A((xa)y) \geq \chi_A(a) = 1$, χ_A being a fuzzy interior ideal of S. Thus $\chi_A((xa)y) = 1$, i.e., $(xa)y \in A$. Hence A is an interior ideal of S.

(2) Let A be a quasi-ideal of S. Now

$$\begin{aligned}(\chi_A \circ S) \cap (S \circ \chi_A) &= (\chi_A \circ \chi_S) \cap (\chi_S \circ \chi_A) \\ &= \chi_{AS} \cap \chi_{SA} = \chi_{AS \cap SA} \subseteq \chi_A, \text{ by the Theorem 2.1}\end{aligned}$$

Therefore χ_A is a fuzzy quasi-ideal of S.

Conversely, assume that χ_A is a fuzzy quasi-ideal of S. Let x be an element of $AS \cap SA$. Now

$$\begin{aligned}\chi_A(x) &\supseteq ((\chi_A \circ S) \cap (S \circ \chi_A))(x) = min\{(\chi_A \circ S)(x), (S \circ \chi_A)(x)\} \\ &= min\{(\chi_A \circ \chi_S)(x), (\chi_S \circ \chi_A)(x)\} = min\{\chi_{AS}(x), \chi_{SA}(x)\} \\ &= (\chi_{AS} \cap \chi_{SA})(x) = \chi_{AS \cap SA}(x) = 1.\end{aligned}$$

This implies that $x \in A$, i.e., $AS \cap SA \subseteq A$. Therefore A is a quasi-ideal of S.

(3) Let A be a bi-ideal of S, this implies that χ_A is a fuzzy AG-subgroupoid of S by the Theorem 2.2. Let $x, y, a \in S$. If $x, y \in A$, then by definition $\chi_A(x) = 1 = \chi_A(y)$. Since $(xa)y \in A$, A being a bi-ideal of S, this means that $\chi_A((xa)y) = 1$. Thus $\chi_A((xa)y) \geq \chi_A(x) \wedge \chi_A(y)$. Similarly, we have $\chi_A((xa)y) \geq \chi_A(x) \wedge \chi_A(y)$, when $x, y \notin A$. Hence χ_A is a fuzzy bi-ideal of S.

Conversely, suppose that χ_A is a fuzzy bi-ideal of S, this means that A is an AG-subgroupoid of S by the Theorem 2.2. Let $a \in S$ and $x, y \in A$, so $\chi_A(x) = 1 = \chi_A(y)$. Since $\chi_A((xa)y) \geq \chi_A(x) \wedge \chi_A(y) = 1$, χ_A being a fuzzy interior ideal of S. Thus $\chi_A((xa)y) = 1$, i.e., $(xa)y \in A$. Hence A is a bi-ideal of S. Similarly, we can prove (4). □

Theorem 2.5. *Let μ be a fuzzy subset of an ordered AG-groupoid S. Then μ is a fuzzy interior ideal of S if and only if $(S \circ \mu) \circ S \subseteq \mu$.*

Proof. Suppose that μ is a fuzzy interior ideal of S and $x \in S$. If $((S \circ \mu) \circ S)(x) = 0$, then $(S \circ \mu) \circ S \subseteq \mu$, otherwise there exist $a, b, c, d \in S$ such that $x \leq ab$ and $a \leq cd$. Since μ is a fuzzy interior ideal of S, this implies that $\mu((cd)b) \geq \mu(d)$. Now

$$\begin{aligned}&((S \circ \mu) \circ S)(x) \\ =\;& \vee_{(a,b) \in A_x} min\{(S \circ \mu)(a), S(b)\} \\ =\;& \vee_{(a,b) \in A_x} min\{\vee_{(c,d) \in A_a} min\{S(c), \mu(d)\}, S(b)\} \\ =\;& \vee_{(a,b) \in A_x} min\{\vee_{(c,d) \in A_a} min\{1, \mu(d)\}, 1\} \\ =\;& \vee_{(a,b) \in A_x} \{\vee_{(c,d) \in A_a} \mu(d)\} \\ =\;& \vee_{((c,d),b) \in A_x} \mu(d) \leq \vee_{((c,d),b) \in A_x} \mu((cd)b) = \mu(x). \\ \Rightarrow\;& (S \circ \mu) \circ S \subseteq \mu.\end{aligned}$$

Conversely, assume that $(S \circ \mu) \circ S \subseteq \mu$ and let $y, z \in S$ such that $a \leq (xy)z$. Now

$$\begin{aligned}
\mu((xy)z) &\geq \mu(a) \geq ((S \circ \mu) \circ S)(a) \\
&= \vee_{(s,t)\in A_a} min\{(S \circ \mu)(s), S(t)\} \\
&\geq (S \circ \mu)(xy) \wedge S(z) \\
&= \vee_{(m,n)\in A_{xy}} min\{S(m), \mu(n)\} \wedge S(z) \\
&\geq \{S(x) \wedge \mu(y)\} \wedge S(z) \\
&= 1 \wedge \mu(y) \wedge 1 = \mu(y). \\
&\Rightarrow \mu((xy)z) \geq \mu(y).
\end{aligned}$$

Therefore μ is a fuzzy interior ideal of S. □

Theorem 2.6. *Let μ be a fuzzy AG-subgroupoid of an ordered AG-groupoid S. Then μ is a fuzzy bi-ideal of S if and only if $(\mu \circ S) \circ \mu \subseteq \mu$.*

Proof. Same as Theorem 2.5. □

Theorem 2.7. *Let μ be a fuzzy subset of an ordered AG-groupoid S. Then μ is a fuzzy generalized bi-ideal of S if and only if $(\mu \circ S) \circ \mu \subseteq \mu$.*

Proof. Same as Theorem 2.5. □

Lemma 2.9. *If μ and γ are two fuzzy bi- (resp. generalized bi-, quasi-, interior) ideals of an ordered AG-groupoid S, then $\mu \cap \gamma$ is also a fuzzy bi- (resp. generalized bi-, quasi-, interior) ideal of S.*

Proof. Let μ and γ be two fuzzy bi-ideals of S. This implies that μ and γ be two fuzzy AG-subgroupoids of S, then $\mu\cap\gamma$ is also a fuzzy AG-subgroupoid of S. We have to show that $(\mu\cap\gamma)((xa)y) \geq (\mu\cap\gamma)(x)\wedge(\mu\cap\gamma)(y)$. Now

$$\begin{aligned}
(\mu \cap \gamma)((xa)y) &= \mu((xa)y) \wedge \gamma((xa)y) \\
&\geq \{\mu(x) \wedge \mu(y)\} \wedge \{\gamma(x) \wedge \gamma(y)\} \\
&= \mu(x) \wedge \{\mu(y) \wedge \gamma(x)\} \wedge \gamma(y) \\
&= \mu(x) \wedge \{\gamma(x) \wedge \mu(y)\} \wedge \gamma(y) \\
&= \{\mu(x) \wedge \gamma(x)\} \wedge \{\mu(y) \wedge \gamma(y)\} \\
&= (\mu \cap \gamma)(x) \wedge (\mu \cap \gamma)(y). \\
&\Rightarrow (\mu \cap \gamma)((xa)y) \geq (\mu \cap \gamma)(x) \wedge (\mu \cap \gamma)(y).
\end{aligned}$$

Hence $\mu \cap \gamma$ is a fuzzy bi-ideal of S. $\square$

Lemma 2.10. *If μ and γ are two fuzzy bi- (resp. generalized bi-, interior) ideals of an ordered AG-groupoid S with left identity e, then $\mu \circ \gamma$ is also a fuzzy bi- (resp. generalized bi-, interior) ideal of S.*

Proof. Let μ and γ be two fuzzy bi-ideals of S. We have to show that $\mu \circ \gamma$ is also a fuzzy bi-ideal of S. Since μ and γ are fuzzy AG-subgroupoids of S, then $\mu \circ \gamma$ is also a fuzzy AG-subgroupoid of S by the Lemma 2.3. Now

$$\begin{aligned} ((\mu \circ \gamma) \circ S) \circ (\mu \circ \gamma) &= ((\mu \circ \gamma) \circ (S \circ S)) \circ (\mu \circ \gamma) \\ &= ((\mu \circ S) \circ (\gamma \circ S)) \circ (\mu \circ \gamma) \\ &= ((\mu \circ S) \circ \mu) \circ ((\gamma \circ S) \circ \gamma) \\ &\subseteq (\mu \circ \gamma). \end{aligned}$$

Therefore $\mu \circ \gamma$ is a fuzzy bi-ideal of S. $\square$

Lemma 2.11. *Every fuzzy ideal of an ordered AG-groupoid S is a fuzzy interior ideal of S. The converse is not true in general.*

Proof. Straight forward. $\square$

Proposition 2.2. *Let μ be a fuzzy subset of an ordered AG-groupoid S with left identity e. Then μ is a fuzzy ideal of S if and only if μ is a fuzzy interior ideal of S.*

Proof. Let μ be a fuzzy interior ideal of S and $x, y \in S$. Now $\mu(xy) = \mu((ex)y) \geq \mu(x)$, thus μ is a fuzzy right ideal of S. Hence μ is a fuzzy ideal of S by the Lemma 2.5. Converse is true by the Lemma 2.11. $\square$

Lemma 2.12. *Every fuzzy left (right, two-sided) ideal of an ordered AG-groupoid S is a fuzzy bi-ideal of S. The converse is not true in general.*

Proof. Suppose that μ is a fuzzy right ideal of S and $x, y, z \in S$. Now $\mu((xy)z) = \mu(xy) \geq \mu(x)$ and $\mu((xy)z) = \mu((zy)x) \geq \mu(zy) \geq \mu(z)$, this implies that $\mu((xy)z) \geq \mu(x) \wedge \mu(z)$. Hence μ is a fuzzy bi-ideal of S. $\square$

Lemma 2.13. *Every fuzzy bi-ideal of an ordered AG-groupoid S is a fuzzy generalized bi-ideal of S. The converse is not true in general.*

Proof. Obvious. $\square$

Lemma 2.14. *Every fuzzy left (right, two-sided) ideal of an ordered AG-groupoid S is a fuzzy quasi-ideal of S. The converse is not true in general.*

Proof. Straight forward. □

Proposition 2.3. *Every fuzzy quasi-ideal of an ordered AG-groupoid S is a fuzzy AG-subgroupoid of S.*

Proof. Let μ be a fuzzy quasi-ideal of S. Since $\mu \circ \mu \subseteq \mu \circ S$ and $\mu \circ \mu \subseteq S \circ \mu$, i.e., $\mu \circ \mu \subseteq \mu \circ S \cap S \circ \mu \subseteq \mu$. So μ is a fuzzy AG-subgroupoid of S. □

Proposition 2.4. *Let μ be a fuzzy right ideal and γ be a fuzzy left ideal of an ordered AG-groupoid S, respectively. Then $\mu \cap \gamma$ is a fuzzy quasi-ideal of S.*

Proof. Since $((\mu \cap \gamma) \circ S) \cap (S \circ (\mu \cap \gamma)) \subseteq (\mu \circ S) \cap (S \circ \gamma) \subseteq \mu \cap \gamma$. Therefore $\mu \cap \gamma$ is a fuzzy quasi-ideal of S. □

Lemma 2.15. *Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then every fuzzy quasi-ideal of S is a fuzzy bi-ideal of S.*

Proof. Let μ be a fuzzy quasi-ideal of S. Since $\mu \circ \mu \subseteq \mu$ by the Proposition 2.3. Now

$$\begin{aligned} (\mu \circ S) \circ \mu &\subseteq (S \circ S) \circ \mu \subseteq S \circ \mu \\ \text{and } (\mu \circ S) \circ \mu &\subseteq (\mu \circ S) \circ S = (\mu \circ S) \circ (e \circ S) \\ &= (\mu \circ e) \circ (S \circ S) = (\mu \circ e) \circ S = \mu \circ S. \\ &\Rightarrow (\mu \circ S) \circ \mu \subseteq \mu \circ S \cap S \circ \mu \subseteq \mu. \end{aligned}$$

Hence μ is a fuzzy bi-ideal of S. □

Proposition 2.5. *If μ and γ are two fuzzy quasi-ideals of an ordered AG-groupoid S with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then $\mu \circ \gamma$ is a fuzzy bi-ideal of S.*

Proof. Let μ and γ be two fuzzy quasi-ideals of S, this implies that μ and γ be two fuzzy bi-ideals of S, by the Lemma 2.15. Then $\mu \circ \gamma$ is also a fuzzy bi-ideal of S by the Lemma 2.10. □

3. Regular Ordered AG-groupoids

An ordered AG-groupoid S will be called a regular if for every $x \in S$, there exists an element $a \in S$ such that $x \leq (xa)x$. Equivalent definitions are as follows:

(1) $A \subseteq ((AS)A]$ for every $A \subseteq S$.

(2) $x \in ((xS)x]$ for every $x \in S$.

In this section, we characterize regular ordered AG-groupoids by the properties of fuzzy (left, right, quasi-, bi-, generalized bi-) ideals.

Lemma 3.1. *Every fuzzy right ideal of a regular ordered AG-groupoid S is a fuzzy ideal of S.*

Proof. Suppsoe that μ is a fuzzy right ideal of S. Let $x, y \in S$, this implies that there exists an element $a \in S$, such that $x \leq (xa)x$. Thus $\mu(xy) \geq \mu(((xa)x)y) = \mu((yx)(xa)) \geq \mu(yx) \geq \mu(y)$. Hence μ is a fuzzy ideal of S. □

Lemma 3.2. *Every fuzzy ideal of a regular ordered AG-groupoid S is a fuzzy idempotent.*

Proof. Assume that μ is a fuzzy ideal of S and $\mu \circ \mu \subseteq \mu$. We have to show that $\mu \subseteq \mu \circ \mu$. Let $x \in S$, this means that there exists an element $a \in S$ such that $x \leq (xa)x$. Thus

$$\begin{aligned}(\mu \circ \mu)(x) &= \vee_{(y,z)\in A_x} min\{\mu(y), \mu(z)\} \\ &\geq \mu(xa) \wedge \mu(x) \geq \mu(x) \wedge \mu(x) = \mu(x). \\ &\Rightarrow \mu \subseteq \mu \circ \mu.\end{aligned}$$

Therefore $\mu = \mu \circ \mu$. □

Remark 3.1. *Every fuzzy right ideal of a regular ordered AG-groupoid S is a fuzzy idempotent.*

Lemma 3.3. *Let μ be a fuzzy subset of a regular ordered AG-groupoid S. Then μ is a fuzzy ideal of S if and only if μ is a fuzzy interior ideal of S.*

Proof. Suppose that μ is a fuzzy interior ideal of S. Let $x, y \in S$, then there exists an element $a \in S$, such that $x \leq (xa)x$. Thus $\mu(xy) \geq \mu(((xa)x)y) = \mu((yx)(xa)) \geq \mu(x)$, i.e., μ is a fuzzy right ideal of S. So μ is a fuzzy ideal of S by the Lemma 3.1. Converse is true by the Lemma 2.11. □

Remark 3.2. *The concept of fuzzy (two-sided, interior) ideals coincides in regular ordered AG-groupoids.*

Proposition 3.1. *Let S be a regular ordered AG-groupoid. Then $(\mu \circ S) \cap (S \circ \mu) = \mu$ for every fuzzy right ideal μ of S.*

Proof. Assume that μ is a fuzzy right ideal of S. Then $(\mu \circ S) \cap (S \circ \mu) \subseteq \mu$, because every fuzzy right ideal of S is a fuzzy quasi-ideal of S by the Lemma 2.14. Let $x \in S$, this implies that there exists an element $a \in S$, such that $x \leq (xa)x$. Thus

$$\begin{aligned}(\mu \circ S)(x) &= \vee_{(y,z)\in A_x} min\{\mu(y), S(z)\} \\ &\geq \mu(xa) \wedge S(x) \geq \mu(x) \wedge 1 = \mu(x). \\ &\Rightarrow \mu \subseteq \mu \circ S.\end{aligned}$$

Similarly, we have $\mu \subseteq S \circ \mu$, i.e., $\mu \subseteq (\mu \circ S) \cap (S \circ \mu)$. Therefore $(\mu \circ S) \cap (S \circ \mu) = \mu$. □

Lemma 3.4. *Let S be a regular ordered AG-groupoid. Then $\mu \circ \gamma = \mu \cap \gamma$ for every fuzzy right ideal μ and every fuzzy left ideal γ of S.*

Proof. Since $\mu \circ \gamma \subseteq \mu \cap \gamma$ for every fuzzy right ideal μ and every fuzzy left ideal γ of S by the Lemma 2.8. Let $x \in S$, this means that there exists an element $a \in S$ such that $x \leq (xa)x$. Thus

$$\begin{aligned}(\mu \circ \gamma)(x) &= \vee_{(y,z)\in A_x} min\{\mu(y), \gamma(z)\} \\ &\geq \mu(xa) \wedge \gamma(x) \geq \mu(x) \wedge \gamma(x) = (\mu \cap \gamma)(x). \\ &\Rightarrow \mu \cap \gamma \subseteq \mu \circ \gamma.\end{aligned}$$

Hence $\mu \circ \gamma = \mu \cap \gamma$. □

Lemma 3.5. *Let S be an ordered AG-groupoid with left identity e and $a \in S$. Then Sa is a smallest left ideal of S containing a.*

Proof. Let $x \in Sa$ and $s \in S$, this implies that $x = s_1 a,\ s_1 \in S$. Thus

$$\begin{aligned}sx &= s(s_1a) = (es)(s_1a) = ((s_1a)s)e = ((s_1a)(es))e \\ &= ((s_1e)(as))e = (e(as))(s_1e) = (as)(s_1e) = ((s_1e)s)a \in Sa.\end{aligned}$$

Hence $sx \in Sa$ and $(Sa] \subseteq Sa$. Now $a = ea \in Sa$, so Sa is a left ideal of S containing a. Let I be another left ideal of S containing a. Since $sa \in I$, because I is a left ideal of S. But $sa \in Sa$, this means that $Sa \subseteq I$. Therefore Sa is a smallest left ideal of S containing a. □

Lemma 3.6. *Let S be an ordered AG-groupoid with left identity e. Then aS is a left ideal of S.*

Proof. Straight forward. □

Proposition 3.2. *Let S be an ordered AG-groupoid with left identity e and $a \in S$. Then $aS \cup Sa$ is a smallest right ideal of S containing a.*

Proof. We have to show that $aS \cup Sa$ is a smallest right ideal of S containing a. Now

$$\begin{aligned}(aS \cup Sa)S &= (aS)S \cup (Sa)S = (SS)a \cup (Sa)(eS) \\ &\subseteq Sa \cup (Se)(aS) = Sa \cup S(aS) \\ &= Sa \cup a(SS) \subseteq Sa \cup aS = aS \cup Sa.\end{aligned}$$

Thus $(aS \cup Sa)S \subseteq aS \cup Sa$ and also $(aS \cup Sa] \subseteq aS \cup Sa$. Therefore $aS \cup Sa$ is a right ideal of S. Since $a \in Sa$, i.e., $a \in aS \cup Sa$. Let I be another right ideal of S containing a. Now $aS \in IS \subseteq I$ and $Sa = (SS)a = (aS)S \in (IS)S \subseteq IS \subseteq I$, i.e., $aS \cup Sa \subseteq I$. Hence $aS \cup Sa$ is a smallest right ideal of S containing a. □

Theorem 3.1. *Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.*

(1) *S is a regular.*

(2) *$\mu \cap \gamma = \mu \circ \gamma$ for every fuzzy right ideal μ and every fuzzy left ideal γ of S.*

(3) *$\beta = (\beta \circ S) \circ \beta$ for every fuzzy quasi-ideal β of S.*

Proof. Suppose that (1) holds and β be a fuzzy quasi-ideal of S. Then $(\beta \circ S) \circ \beta \subseteq \beta$, because every fuzzy quasi-ideal of S is a fuzzy bi-ideal of S by the Lemma 2.15. Let $x \in S$, this implies that there exists an element a of S such that $x \leq (xa)x$. Thus

$$\begin{aligned} ((\beta \circ S) \circ \beta)(x) &= \vee_{(y,z)\in A_x} min\{(\beta \circ S)(y), \beta(z)\} \\ &\geq (\beta \circ S)(xa) \wedge \beta(x) \\ &= \vee_{(s,t)\in A_{xa}} min\{\beta(s), S(t)\} \wedge \beta(x) \\ &\geq \beta(x) \wedge S(a) \wedge \beta(x) = \beta(x). \\ &\Rightarrow \beta \subseteq (\beta \circ S) \circ \beta. \end{aligned}$$

Therefore $\beta = (\beta \circ S) \circ \beta$, i.e., (1) implies (3) . Assume that (3) holds. Let μ be a fuzzy right ideal and γ be a fuzzy left ideal of S. This means that μ and γ be fuzzy quasi-ideals of S by the Lemma 2.14, so $\mu \cap \gamma$ be also a fuzzy quasi-ideal of S. Then by our assumption, $\mu \cap \gamma = ((\mu \cap \gamma) \circ S) \circ (\mu \cap \gamma) \subseteq (\mu \circ S) \circ \gamma \subseteq \mu \circ \gamma$, i.e., $\mu \cap \gamma \subseteq \mu \circ \gamma$. Since $\mu \circ \gamma \subseteq \mu \cap \gamma$. Hence $\mu \circ \gamma = \mu \cap \gamma$, i.e., (3) $\Rightarrow$ (2). Suppose that (2) is true and $a \in S$. Then Sa is a left ideal of S containing a by the Lemma 3.5 and $aS \cup Sa$ is a right ideal of S containing a by the Proposition 3.2. So χ_{Sa} is a fuzzy left ideal and $\chi_{aS\cup Sa}$ is a fuzzy right ideal of S, by the Theorem 2.2. Then by our supposition $\chi_{aS\cup Sa} \cap \chi_{Sa} = \chi_{aS\cup Sa} \circ \chi_{Sa}$, i.e., $\chi_{(aS\cup Sa)\cap Sa} = \chi_{((aS\cup Sa)Sa]}$ by the Theorem 2.1. Thus $(aS \cup Sa) \cap Sa = ((aS \cup Sa)Sa]$. Since $a \in (aS \cup Sa) \cap Sa$, i.e., $a \in ((aS \cup Sa)Sa]$, so $a \in ((aS)(Sa) \cup (Sa)(Sa)]$. Now $(Sa)(Sa) = ((Se)a)(Sa) = ((ae)S)(Sa) = (aS)(Sa)$. This implies that

$$((aS)(Sa) \cup (Sa)(Sa)] = ((aS)(Sa) \cup (aS)(Sa)] = ((aS)(Sa)].$$

Thus $a \in ((aS)(Sa)]$. Then

$$\begin{aligned} a &\leq (ax)(ya) = ((ya)x)a = (((ey)a)x)a = (((ay)e)x)a \\ &= ((xe)(ay))a = (a((xe)y))a \in (aS)a, \text{ for any } x, y \in S. \end{aligned}$$

This means that $a \in ((aS)a]$, i.e., a is regular. Hence S is a regular, i.e., (2) $\Rightarrow$ (1) . □

Theorem 3.2. *Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.*

(1) *S is a regular.*

(2) $\mu = (\mu \circ S) \circ \mu$ *for every fuzzy quasi-ideal* μ *of* S.

(3) $\beta = (\beta \circ S) \circ \beta$ *for every fuzzy bi-ideal* β *of* S.

(4) $\delta = (\delta \circ S) \circ \delta$ *for every fuzzy generalized bi-ideal* δ *of* S.

Proof. (1) $\Rightarrow$ (4), is obvious. Since (4) $\Rightarrow$ (3) , every fuzzy bi-ideal of S is a fuzzy generalized bi-ideal of S by the Lemma 2.13. Since (3) $\Rightarrow$ (2) , every fuzzy quasi-ideal of S is a fuzzy bi-ideal of S by the Lemma 2.15. (2) $\Rightarrow$ (1) , by the Theorem 3.1. □

Theorem 3.3. *Let* S *be an ordered AG-groupoid with left identity* e, *such that* $(xe)S = xS$ *for all* $x \in S$. *Then the following conditions are equivalent.*

(1) S *is a regular.*

(2) $\mu \cap \gamma = (\mu \circ \gamma) \circ \mu$ *for every fuzzy quasi-ideal* μ *and every fuzzy ideal* γ *of* S.

(3) $\beta \cap \gamma = (\beta \circ \gamma) \circ \beta$ *for every fuzzy bi-ideal* β *and every fuzzy ideal* γ *of* S.

(4) $\delta \cap \gamma = (\delta \circ \gamma) \circ \delta$ *for every fuzzy generalized bi-ideal* δ *and every fuzzy ideal* γ *of* S.

Proof. Suppose that (1) holds. Let δ be a fuzzy generalized bi-ideal and γ be a fuzzy ideal of S. Now $(\delta \circ \gamma) \circ \delta \subseteq (S \circ \gamma) \circ S \subseteq \gamma \circ S \subseteq \gamma$ and $(\delta \circ \gamma) \circ \delta \subseteq (\delta \circ S) \circ \delta \subseteq \delta$, i.e., $(\delta \circ \gamma) \circ \delta \subseteq \delta \cap \gamma$. Let $x \in S$, this means that there exists an element $a \in S$ such that $x \leq (xa)x$. Now $xa \leq ((xa)x)a = (ax)(xa) = x((ax)a)$. Thus

$$
\begin{aligned}
((\delta \circ \gamma) \circ \delta)(x) &= \vee_{(y,z) \in A_x} min\{(\delta \circ \gamma)(y), \delta(z)\} \\
&\geq (\delta \circ \gamma)(xa) \wedge \delta(x) \\
&= \vee_{(s,t) \in A_{xa}} min\{\delta(s), \gamma(t)\} \wedge \delta(x) \\
&\geq \delta(x) \wedge \gamma((ax)a) \wedge \delta(x) \\
&\geq \delta(x) \wedge \gamma(x) = (\delta \cap \gamma)(x). \\
&\Rightarrow \delta \cap \gamma \subseteq (\delta \circ \gamma) \circ \delta.
\end{aligned}
$$

Hence $\delta \cap \gamma = (\delta \circ \gamma) \circ \delta$, i.e., (1) $\Rightarrow$ (4) . It is clear that (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Assume that (2) is true. Then $\mu \cap S = (\mu \circ S) \circ \mu$, where S itself is a fuzzy two-sided ideal, so $\mu = (\mu \circ S) \circ \mu$. Therefore S is a regular by the Theorem 3.1, i.e., (2) $\Rightarrow$ (1) . □

Theorem 3.4. *Let* S *be an ordered AG-groupoid with left identity* e, *such that* $(xe)S = xS$ *for all* $x \in S$. *Then the following conditions are equivalent.*

(1) S *is a regular.*

(2) $\mu \cap \gamma \subseteq \gamma \circ \mu$ *for every fuzzy quasi-ideal* μ *and every fuzzy right ideal* γ *of* S.

(3) $\beta \cap \gamma \subseteq \gamma \circ \beta$ *for every fuzzy bi-ideal* β *and every fuzzy right ideal* γ *of* S.

(4) $\delta \cap \gamma \subseteq \gamma \circ \delta$ *for every fuzzy generalized bi-ideal* δ *and every fuzzy right ideal* γ *of* S.

Proof. (1) $\Rightarrow$ (4), is obvious. Since (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Suppose that (2) is true, this implies that $\gamma \cap \mu = \mu \cap \gamma \subseteq \gamma \circ \mu$, where μ is a fuzzy left ideal of S. Since $\gamma \circ \mu \subseteq \gamma \cap \mu$, so $\gamma \cap \mu = \gamma \circ \mu$. Hence S is a regular by the Theorem 3.1, i.e., (2) $\Rightarrow$ (1). □

Theorem 3.5. *Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.*

(1) *S is a regular.*

(2) *$\mu \cap \gamma \cap \lambda \subseteq (\mu \circ \gamma) \circ \lambda$ for every fuzzy quasi-ideal μ, every fuzzy right ideal γ and every fuzzy left ideal λ of S.*

(3) *$\beta \cap \gamma \cap \lambda \subseteq (\beta \circ \gamma) \circ \lambda$ for every fuzzy bi-ideal β, every fuzzy right ideal γ and every fuzzy left ideal λ of S.*

(4) *$\delta \cap \gamma \cap \lambda \subseteq (\delta \circ \gamma) \circ \lambda$ for every fuzzy generalized bi-ideal δ, every fuzzy right ideal γ and every fuzzy left ideal λ of S.*

Proof. Suppose that (1) holds. Let δ be a fuzzy generalized bi-ideal, γ be a fuzzy right ideal and λ be a fuzzy left ideal of S. Let $x \in S$, this implies that there exists an element $a \in S$ such that $x \leq (xa)x$. Now

$$\begin{aligned}
x &\leq (xa)x. \\
xa &\leq ((xa)x)a = (ax)(xa) = x((ax)a). \\
(ax)a &\leq (a((xa)x))a = ((xa)(ax))a = (a(ax))(xa) \\
&= x((a(ax))a) = x(((ea)(ax))a) = x(((xa)(ae))a) \\
&= x((((ae)a)x)a) = x((nx)a) = x((nx)(ea)) = x((ae)(xn)) \\
&= x(x((ae)n)) = x(xm). \\
&\Rightarrow xa \leq x((ax)a) \leq x(x(xm)) = (ex)(x(xm)) = ((xm)x)(xe).
\end{aligned}$$

Thus

$$\begin{aligned}
((\delta \circ \gamma) \circ \lambda)(x) &= \vee_{(y,z)\in A_x} min\{(\delta \circ \gamma)(y), \lambda(z)\} \\
&\geq (\delta \circ \gamma)(xa) \wedge \lambda(x) \\
&= \vee_{(s,t)\in A_{xa}} min\{\delta(s), \gamma(t)\} \wedge \lambda(x) \\
&\geq \delta((xm)x) \wedge \gamma(xe) \wedge \lambda(x) \\
&\geq \delta(x) \wedge \gamma(x) \wedge \lambda(x) = (\delta \cap \gamma \cap \lambda)(x). \\
&\Rightarrow \delta \cap \gamma \cap \lambda \subseteq (\delta \circ \gamma) \circ \lambda.
\end{aligned}$$

Hence $(1) \Rightarrow (4)$. It is clear that $(4) \Rightarrow (3)$ and $(3) \Rightarrow (2)$. Assume that (2) holds. Then $\mu \cap S \cap \lambda \subseteq (\mu \circ S) \circ \lambda$, where μ is a right ideal of S, i.e., $\mu \cap \lambda \subseteq \mu \circ \lambda$. Since $\mu \circ \lambda \subseteq \mu \cap \lambda$, so $\mu \circ \lambda = \mu \cap \lambda$. Therefore S is a regular by the Theorem 3.1, i.e., $(2) \Rightarrow (1)$. □

4. Intra-regular Ordered AG-groupoids

An ordered AG-groupoid S will be called an intra-regular ordered AG-groupoid if for every $x \in S$ there exist elements $a, b \in S$ such that $x \leq (ax^2)b$. Equivalent definitions are as follows:

(1) $A \subseteq ((SA^2)S]$ for every $A \subseteq S$.

(2) $x \in ((Sx^2)S]$ for every $x \in S$.

In this section, we characterize intra-regular ordered AG-groupoids by the properties of fuzzy (left, right, quasi-, bi-, generalized bi-) ideals.

Lemma 4.1. *Every fuzzy left (right) ideal of an intra-regular ordered AG-groupoid S is a fuzzy ideal of S.*

Proof. Suppose that μ is a fuzzy right ideal of S. Let $x, y \in S$, this implies that there exist elements $a, b \in S$, such that $x \leq (ax^2)b$. Thus

$$\mu(xy) \geq \mu(((ax^2)b)y) = \mu((yb)(ax^2)) \geq \mu(yb) \geq \mu(y).$$

Hence μ is a fuzzy ideal of S. □

Lemma 4.2. *Let S be an intra-regular ordered AG-groupoid with left identity e. Then every fuzzy ideal of S is a fuzzy idempotent.*

Proof. Assume that μ is a fuzzy ideal of S and $\mu \circ \mu \subseteq \mu$. Let $x \in S$, this means that there exist elements $a, b \in S$, such that $x \leq (ax^2)b$. Now

$$\begin{aligned} x &\leq (ax^2)b = (a(xx))b = (x(ax))b \\ &= (x(ax))(eb) = (xe)((ax)b) = (ax)((xe)b). \end{aligned}$$

Thus

$$\begin{aligned} (\mu \circ \mu)(x) &= \vee_{(y,z) \in A_x} min\{\mu(y), \mu(z)\} \\ &\geq \mu(ax) \wedge \mu((xe)b) \geq \mu(x) \wedge \mu(x) = \mu(x). \\ &\Rightarrow \mu \subseteq \mu \circ \mu. \end{aligned}$$

Therefore $\mu = \mu \circ \mu$. □

Proposition 4.1. *Let μ be a fuzzy subset of an intra-regular ordered AG-groupoid S with left identity e. Then μ is a fuzzy ideal of S if and only if μ is a fuzzy interior ideal of S.*

Proof. Suppose that μ is a fuzzy interior ideal of S. Let $x, y \in S$, then there exist elements $a, b \in S$, such that $x \leq (ax^2)b$. Thus

$$\begin{aligned} \mu(xy) &\geq \mu(((ax^2)b)y) = \mu((yb)(ax^2)) \\ &= \mu((yb)(a(xx))) = \mu((yb)(x(ax))) \\ &= \mu((yx)(b(ax))) \geq \mu(x). \end{aligned}$$

So μ is a fuzzy right ideal of S, hence μ is a fuzzy ideal of S by the Lemma 4.1. Converse is true by the Lemma 2.11. □

Remark 4.1. *The concept of fuzzy (two-sided, interior) ideals coincides in intra-regular ordered AG-groupoids with left identity.*

Lemma 4.3. *Let S be an intra-regular ordered AG-groupoid with left identity e. Then $\gamma \cap \mu \subseteq \mu \circ \gamma$ for every fuzzy left ideal μ and every fuzzy right ideal γ of S.*

Proof. Let μ be a fuzzy left ideal and γ be a fuzzy right ideal of S. Let $x \in S$, this implies that there exist elements $a, b \in S$ such that $x \leq (ax^2)b$. Now

$$\begin{aligned} x &\leq (ax^2)b = (a(xx))b = (x(ax))b \\ &= (x(ax))(eb) = (xe)((ax)b) = (ax)((xe)b). \end{aligned}$$

Thus

$$\begin{aligned} (\mu \circ \gamma)(x) &= \vee_{(y,z)\in A_x} min\{\mu(y), \gamma(z)\} \\ &\geq \mu(ax) \wedge \gamma((xe)b) \geq \mu(x) \wedge \gamma(x) \\ &= \gamma(x) \wedge \mu(x) = (\gamma \cap \mu)(x). \\ &\Rightarrow \gamma \cap \mu \subseteq \mu \circ \gamma. \end{aligned}$$

□

Theorem 4.1. *Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.*

(1) *S is an intra-regular.*

(2) *$\gamma \cap \mu \subseteq \mu \circ \gamma$ for every fuzzy left ideal μ and every fuzzy right ideal γ of S.*

Proof. (1) ⇒ (2) is obvious by the Lemma 4.3. Suppose that (2) holds and $a \in S$. Then Sa is a left ideal of S containing a by the Lemma 3.5 and $aS \cup Sa$ is a right ideal of S containing a by the Proposition 3.2. So χ_{Sa} is a fuzzy left ideal and $\chi_{aS\cup Sa}$ is a fuzzy right ideal of S, by the Theorem 2.2. By our supposition $\chi_{aS\cup Sa}\cap\chi_{Sa}\subseteq\chi_{Sa}\circ\chi_{aS\cup Sa}$, i.e., $\chi_{(aS\cup Sa)\cap Sa}\subseteq\chi_{((Sa)(aS\cup Sa)]}$ by the Theorem 2.1. Thus $(aS\cup Sa)\cap Sa\subseteq(Sa(aS\cup Sa)]$. Since $a\in(aS\cup Sa)\cap Sa$, i.e., $a\in(Sa(aS\cup Sa)]=((Sa)(aS)\cup(Sa)(Sa)]$. Now

$$\begin{aligned}(Sa)(aS) &= (Sa)((ea)(SS))=(Sa)((SS)(ae))\\ &= (Sa)(((ae)S)S)=(Sa)((aS)S)\\ &= (Sa)((SS)a)=(Sa)(Sa).\end{aligned}$$

This implies that

$$\begin{aligned}((Sa)(aS)\cup(Sa)(Sa)] &= ((Sa)(Sa)\cup(Sa)(Sa)]\\ &= ((Sa)(Sa)]=((Sa)a)S]\\ &= (((Sa)(ea))S]=(((Se)(aa))S]\\ &= ((Sa^2)S].\end{aligned}$$

Thus $a\in(Sa^2)S$, i.e., a is an intra-regular. Hence S is an intra-regular, i.e., (2) $\Rightarrow$ (1) . □

Theorem 4.2. *Let S be an ordered AG-groupoid with left identity e, such that $(xe)S=xS$ for all $x\in S$. Then the following conditions are equivalent.*

(1) *S is an intra-regular.*

(2) *$\mu\cap\gamma=(\mu\circ\gamma)\circ\mu$ for every fuzzy quasi-ideal μ and every fuzzy ideal γ of S.*

(3) *$\beta\cap\gamma=(\beta\circ\gamma)\circ\beta$ for every fuzzy bi-ideal β and every fuzzy ideal γ of S.*

(4) *$\delta\cap\gamma=(\delta\circ\gamma)\circ\delta$ for every fuzzy generalized bi-ideal δ and every fuzzy ideal γ of S.*

Proof. Suppose that (1) holds. Let δ be a fuzzy generalized bi-ideal and γ be a fuzzy ideal of S. Now $(\delta\circ\gamma)\circ\delta\subseteq(S\circ\gamma)\circ S\subseteq\gamma\circ S\subseteq\gamma$ and $(\delta\circ\gamma)\circ\delta\subseteq(\delta\circ S)\circ\delta\subseteq\delta$, thus $(\delta\circ\gamma)\circ\delta\subseteq\delta\cap\gamma$. Let $x\in S$, this implies that there exist elements $a,b\in S$ such that $x\leq(ax^2)b$. Now

$$\begin{aligned}x &\leq (ax^2)b=(a(xx))b=(x(ax))b=(b(ax))x.\\ b(ax) &\leq b(a((ax^2)b))=b((ax^2)(ab))=b((ax^2)c)\\ &= (ax^2)(bc)=(ax^2)d=(ax^2)(ed)=(de)(x^2a)\\ &= m(x^2a)=x^2(ma)=(xx)l=(lx)x=(lx)(ex)\\ &= (xe)(xl)=x((xe)l).\end{aligned}$$

Thus

$$
\begin{aligned}
((\delta\circ\gamma)\circ\delta)(x) &= \vee_{(y,z)\in A_x} min\{(\delta\circ\gamma)(y),\delta(z)\} \\
&\geq (\delta\circ\gamma)(b(ax))\wedge\delta(x) \\
&= \vee_{(s,t)\in A_{b(ax)}} min\{\delta(s),\gamma(t)\}\wedge\delta(x) \\
&\geq \delta(x)\wedge\gamma((xe)l)\wedge\delta(x) \\
&\geq \delta(x)\wedge\gamma(x)=(\delta\cap\gamma)(x). \\
&\Rightarrow \delta\cap\gamma\subseteq(\delta\circ\gamma)\circ\delta.
\end{aligned}
$$

Hence $\delta\cap\gamma=(\delta\circ\gamma)\circ\delta$, i.e., $(1)\Rightarrow(4)$. It is clear that $(4)\Rightarrow(3)$ and $(3)\Rightarrow(2)$. Assume that (2) is true. Let μ be a fuzzy right ideal and γ be a fuzzy two-sided ideal of S. Since every fuzzy right ideal of S is a fuzzy quasi-ideal of S by the Lemma 2.14, so μ is a fuzzy quasi-ideal of S. By our assumption $\mu\cap\gamma=(\mu\circ\gamma)\circ\mu\subseteq(S\circ\gamma)\circ\mu\subseteq\gamma\circ\mu$, i.e., $\mu\cap\gamma\subseteq\gamma\circ\mu$. Therefore S is an intra-regular by the Theorem 4.1, i.e., $(2)\Rightarrow(1)$. □

Theorem 4.3. *Let S be an ordered AG-groupoid with left identity e, such that $(xe)S=xS$ for all $x\in S$. Then the following conditions are equivalent.*

(1) *S is an intra-regular.*

(2) *$\mu\cap\gamma\subseteq\gamma\circ\mu$ for every fuzzy quasi-ideal μ and every fuzzy left ideal γ of S.*

(3) *$\beta\cap\gamma\subseteq\gamma\circ\beta$ for every fuzzy bi-ideal β and every fuzzy left ideal γ of S.*

(4) *$\delta\cap\gamma\subseteq\gamma\circ\delta$ for every fuzzy generalized bi-ideal δ and every fuzzy left ideal γ of S.*

Proof. Assume that (1) holds. Let δ be a fuzzy generalized bi-ideal and γ be a fuzzy left ideal of S. Let $x\in S$, this means that there exist elements $a,b\in S$ such that $x\leq(ax^2)b$. Now $x\leq(a(xx))b=(x(ax))b=(b(ax))x$. Thus

$$
\begin{aligned}
(\gamma\circ\delta)(x) &= \vee_{(y,z)\in A_x} min\{\gamma(y),\delta(z)\} \\
&\geq \gamma(b(ax))\wedge\delta(x)\geq\gamma(x)\wedge\delta(x) \\
&= \delta(x)\wedge\gamma(x)=(\delta\cap\gamma)(x). \\
&\Rightarrow \delta\cap\gamma\subseteq\gamma\circ\delta.
\end{aligned}
$$

Hence (1) implies (4). It is clear that $(4)\Rightarrow(3)$ and $(3)\Rightarrow(2)$. Suppose that (2) holds. Let μ be a fuzzy right ideal and γ be a fuzzy left ideal of S. Since every fuzzy right ideal of S is a fuzzy quasi-ideal of S, this implies that μ is a fuzzy quasi-ideal of S. By our supposition, $\mu\cap\gamma\subseteq\gamma\circ\mu$. Thus S is an intra-regular by the Theorem 4.1, i.e., $(2)\Rightarrow(1)$. □

Theorem 4.4. *Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.*

(1) *S is an intra-regular.*

(2) *$\mu \cap \gamma \cap \lambda \subseteq (\gamma \circ \mu) \circ \lambda$ for every fuzzy quasi-ideal μ, every fuzzy left ideal γ and every fuzzy right ideal λ of S.*

(3) *$\beta \cap \gamma \cap \lambda \subseteq (\gamma \circ \beta) \circ \lambda$ for every fuzzy bi-ideal β, every fuzzy left ideal γ and every fuzzy right ideal λ of S.*

(4) *$\delta \cap \gamma \cap \lambda \subseteq (\gamma \circ \delta) \circ \lambda$ for every fuzzy generalized bi-ideal δ, every fuzzy left ideal γ and every fuzzy right ideal λ of S.*

Proof. Suppose that (1) holds. Let δ be a fuzzy generalized bi-ideal, γ be a fuzzy left ideal and λ be a fuzzy right ideal of S. Let $x \in S$, then there exist elements $a, b \in S$ such that $x \leq (ax^2)b$. Now

$$\begin{aligned} x &\leq (a(xx))b = (x(ax))b = (b(ax))x. \\ b(ax) &\leq b(a((ax^2)b)) = b((ax^2)(ab)) \\ &= b((ax^2)c) = (ax^2)(bc) = (ax^2)d \\ &= (a(xx))d = (x(ax))d = (d(ax))x. \end{aligned}$$

Thus

$$\begin{aligned} ((\gamma \circ \delta) \circ \lambda)(x) &= \vee_{(y,z) \in A_x} min\{(\gamma \circ \delta)(y), \lambda(z)\} \\ &\geq (\gamma \circ \delta)(b(ax)) \wedge \lambda(x) \\ &= \vee_{(s,t) \in A_{b(ax)}} min\{\gamma(s), \delta(t)\} \wedge \lambda(x) \\ &\geq \gamma(d(ax)) \wedge \delta(x) \wedge \lambda(x) = \gamma(x) \wedge \delta(x) \wedge \lambda(x) \\ &= (\gamma \cap \delta \cap \lambda)(x). \\ &\Rightarrow \gamma \cap \delta \cap \lambda \subseteq (\gamma \circ \delta) \circ \lambda. \end{aligned}$$

Thus (1) implies (4). Since (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Assume that (2) holds. Then $\mu \cap S \cap \lambda \subseteq (S \circ \mu) \circ \lambda$, where μ is a fuzzy left ideal of S, i.e., $\mu \cap \lambda \subseteq \mu \circ \lambda$. Thus S is an intra-regular, i.e., (2) $\Rightarrow$ (1). □

5. Regular and Intra-regular Ordered AG-groupoids

In this section, we characterize both regular and intra-regular ordered AG-groupoid by the properties of fuzzy (left, right, quasi-, bi-, generalized bi-) ideals.

Theorem 5.1. *Let S be an odered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.*

(1) *S is both a regular and an intra-regular.*

(2) $\mu \circ \mu = \mu$ *for every fuzzy bi-ideal* μ *of* S.

(3) $\mu_1 \cap \mu_2 = (\mu_1 \circ \mu_2) \cap (\mu_2 \circ \mu_1)$ *for all fuzzy bi-ideals* μ_1 *and* μ_2 *of* S.

Proof. Suppose that (1) holds. Let μ be a fuzzy bi-ideal of S and $\mu \circ \mu \subseteq \mu$. Let $x \in S$, this implies that there exists an element $a \in S$ such that $x \leq (xa)x$, also there exist elements $a, b \in S$ such that $x \leq (ax^2)b$. Now

$$\begin{aligned} x &\leq (xa)x \\ xa &\leq ((ax^2)b)a = (ab)(ax^2) = c(a(xx)) \\ &= c(x(ax)) = x(c(ax)) = x((ec)(ax)) \\ &= x((xa)(ce)) = x((xa)d) = x((da)x) \\ &= x(lx) = l(xx) = (el)(xx) = (xx)(le) \\ &= (xx)m = (mx)x. \\ mx &\leq m((ax^2)b) = (ax^2)(mb) = (a(xx))n \\ &= (x(ax))n = (x(ax))(en) = (xe)((ax)n) \\ &= (xe)((ax)(en)) = (xe)((ae)(xn)) \\ &= (xe)(x((ae)n)) = (xe)(xu) = x((xe)u) = xw. \\ &\Rightarrow xa \leq (mx)x = (xw)x. \end{aligned}$$

Thus

$$\begin{aligned} (\mu \circ \mu)(x) &= \vee_{(y,z) \in A_x} min\{\mu(y), \mu(z)\} \\ &\geq \mu((xw)x) \wedge \mu(x) \\ &\geq \mu(x) \wedge \mu(x) \wedge \mu(x) = \mu(x). \\ &\Rightarrow \mu \subseteq \mu \circ \mu. \end{aligned}$$

Hence $\mu = \mu \circ \mu$, i.e., (1) implies (2) . Assume that (2) is true. Let μ_1 and μ_2 be two fuzzy bi-ideals of S, then $\mu_1 \cap \mu_2$ and $\mu_1 \circ \mu_2$ be also fuzzy bi-ideals of S. By our assumption $\mu_1 \cap \mu_2 = (\mu_1 \cap \mu_2) \circ (\mu_1 \cap \mu_2) \subseteq \mu_1 \circ \mu_2$ and $\mu_1 \cap \mu_2 = (\mu_1 \cap \mu_2) \circ (\mu_1 \cap \mu_2) \subseteq \mu_2 \circ \mu_1$, this implies that $\mu_1 \cap \mu_2 \subseteq (\mu_1 \circ \mu_2) \cap (\mu_2 \circ \mu_1)$. Again by our supposition

$$\begin{aligned}
(\mu_1 \circ \mu_2) \cap (\mu_2 \circ \mu_1) &= ((\mu_1 \circ \mu_2) \cap (\mu_2 \circ \mu_1)) \circ ((\mu_1 \circ \mu_2) \cap (\mu_2 \circ \mu_1)) \\
&\subseteq (\mu_1 \circ \mu_2) \circ (\mu_2 \circ \mu_1) \subseteq (\mu_1 \circ S) \circ (S \circ \mu_1) \\
&= ((S \circ \mu_1) \circ S) \circ \mu_1 = (((S \circ e) \circ \mu_1) \circ S) \circ \mu_1 \\
&= (((\mu_1 \circ e) \circ S) \circ S) \circ \mu_1 = ((\mu_1 \circ S) \circ S) \circ \mu_1 \\
&= ((S \circ S) \circ \mu_1) \circ \mu_1 = (S \circ \mu_1) \circ \mu_1 \\
&= ((S \circ e) \circ \mu_1) \circ \mu_1 = ((\mu_1 \circ e) \circ S) \circ \mu_1 \\
&= (\mu_1 \circ S) \circ \mu_1 \subseteq \mu_1. \\
&\Rightarrow (\mu_1 \circ \mu_2) \cap (\mu_2 \circ \mu_1) \subseteq \mu_1.
\end{aligned}$$

Similarly, we have $(\mu_1 \circ \mu_2) \cap (\mu_2 \circ \mu_1) \subseteq \mu_2$, thus $(\mu_1 \circ \mu_2) \cap (\mu_2 \circ \mu_1) \subseteq \mu_1 \cap \mu_2$. Hence $\mu_1 \cap \mu_2 = (\mu_1 \circ \mu_2) \cap (\mu_2 \circ \mu_1)$, i.e., $(2) \Rightarrow (3)$. Suppose that (3) holds. Let μ be a fuzzy right ideal and γ be a fuzzy ideal of S, then μ and γ be also fuzzy bi-ideals of S, because every fuzzy right ideal and fuzzy ideal of S is a fuzzy bi-ideal of S. By our supposition $\mu \cap \gamma = (\mu \circ \gamma) \cap (\gamma \circ \mu)$, this implies that $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$, i.e., $\mu \cap \gamma \subseteq \mu \circ \gamma$ and $\mu \cap \gamma \subseteq \gamma \circ \mu$, where γ is also a fuzzy left ideal of S. Since $\mu \circ \gamma \subseteq \mu \cap \gamma$, thus $\mu \cap \gamma = \mu \circ \gamma$ and $\mu \cap \gamma \subseteq \gamma \circ \mu$. Hence S is both a regular and an intra-regular, i.e., $(3) \Rightarrow (1)$. □

Theorem 5.2. *Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.*

(1) *S is both a regular and an intra-regular.*

(2) *Every fuzzy quasi-ideal of S is a fuzzy idempotent.*

Proof. Suppose that S is both a regular and an intra-regular. Let μ be a fuzzy quasi-ideal of S. Then μ be a fuzzy bi-ideal of S and $\mu \circ \mu \subseteq \mu$. Let $x \in S$, this means that there exists an element $a \in S$ such that $x \leq (xa)x$, and also there exist elements $a, b \in S$ such that $x \leq (ax^2)b$. Since $x \leq (xa)x = ((xw)x)x$ by the Theorem 5.1. Thus

$$\begin{aligned}
(\mu \circ \mu)(x) &= \vee_{(y,z)\in A_x} min\{\mu(y), \mu(z)\} \\
&\geq \mu((xw)x) \wedge \mu(x) \\
&\geq \mu(x) \wedge \mu(x) \wedge \mu(x) = \mu(x). \\
&\Rightarrow \mu \subseteq \mu \circ \mu.
\end{aligned}$$

Hence $\mu = \mu \circ \mu$. Conversely, assume that every fuzzy quasi-ideal of S is a fuzzy idempotent. Let $a \in S$, then Sa is a left ideal of S containing a by the Lemma 3.5.This implies that Sa is a quasi-ideal of S, so

χ_{Sa} is a fuzzy quasi-ideal of S by the Theorem 2.4. By our assumption $\chi_{Sa} = \chi_{Sa} \circ \chi_{Sa} = \chi_{((Sa)(Sa)]}$, i.e., $Sa = ((Sa)(Sa)]$. Since $a \in Sa$, i.e., $a \in ((Sa)(Sa)]$. Thus a is both a regular and an intra-regular by the Theorems 3.1 and 4.1, respectively. Hence S is both a regular and an intra-regular. □

Theorem 5.3. *Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.*

(1) *S is both a regular and an intra-regular.*

(2) *$\mu \cap \gamma \subseteq \mu \circ \gamma$ for all fuzzy quasi-ideals μ and γ of S.*

(3) *$\mu \cap \gamma \subseteq \mu \circ \gamma$ for every fuzzy quasi-ideal μ and every fuzzy bi-ideal γ of S.*

(4) *$\mu \cap \gamma \subseteq \mu \circ \gamma$ for every fuzzy bi-ideal μ and every fuzzy quasi-ideal γ of S.*

(5) *$\mu \cap \gamma \subseteq \mu \circ \gamma$ for all fuzzy bi-ideals μ and γ of S.*

(6) *$\mu \cap \gamma \subseteq \mu \circ \gamma$ for every fuzzy bi-ideal μ and every fuzzy generalized bi-ideal γ of S.*

(7) *$\mu \cap \gamma \subseteq \mu \circ \gamma$ for every fuzzy generalized bi-ideal μ and every fuzzy quasi-ideal γ of S.*

(8) *$\mu \cap \gamma \subseteq \mu \circ \gamma$ for every fuzzy generalized bi-ideal μ and every fuzzy bi-ideal γ of S.*

(9) *$\mu \cap \gamma \subseteq \mu \circ \gamma$ for all fuzzy generalized bi-ideals μ and γ of S.*

Proof. Suppose that (1) holds. Assume that μ and γ be two fuzzy generalized bi-ideals of S. Let $x \in S$, this implies that there exists an element $a \in S$ such that $x \leq (xa)x$, and also there exist elements $a, b \in S$ such that $x \leq (ax^2)b$. Since $x \leq (xa)x = ((xw)x)x$ by the Theorem 5.1. Thus

$$
\begin{aligned}
(\mu \circ \gamma)(x) &= \vee_{(y,z) \in A_x} min\{\mu(y), \gamma(z)\} \\
&\geq \mu((xw)x) \wedge \gamma(x) \\
&\geq \mu(x) \wedge \mu(x) \wedge \gamma(x) = (\mu \cap \gamma)(x). \\
&\Rightarrow \mu \cap \gamma \subseteq \mu \circ \gamma.
\end{aligned}
$$

Hence $(1) \Rightarrow (9)$. It is clear that $(9) \Rightarrow (8) \Rightarrow (7) \Rightarrow (4) \Rightarrow (2)$ and $(9) \Rightarrow (6) \Rightarrow (5) \Rightarrow (3)$. Assume that (2) holds. Let μ be a fuzzy right ideal and γ be a fuzzy left ideal of S. Since every fuzzy right ideal and every fuzzy left ideal of S is a fuzzy quasi-ideal of S by the Lemma 2.14. By our assumption, $\mu \cap \gamma \subseteq \mu \circ \gamma$. Since $\mu \circ \gamma \subseteq \mu \cap \gamma$, so $\mu \cap \gamma = \mu \circ \gamma$, i.e., S is a regular. Again by our assumption, $\mu \cap \gamma = \gamma \cap \mu \subseteq \gamma \circ \mu$, i.e., S is an intra-regular. Hence S is both a regular and an intra-regular, i.e., $(2) \Rightarrow (1)$. In similar way, we can prove that $(3) \Rightarrow (1)$. □

Theorem 5.4. *Let S be an ordered AG-groupoid with left identity e, such that $(xe)S = xS$ for all $x \in S$. Then the following conditions are equivalent.*

(1) *S is both a regular and an intra-regular.*

(2) *$\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ for every fuzzy right ideal μ and every fuzzy left ideal γ of S.*

(3) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ *for every fuzzy right ideal* μ *and every fuzzy quasi-ideal* γ *of* S.

(4) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ *for every fuzzy right ideal* μ *and every fuzzy bi-ideal* γ *of* S.

(5) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ *for every fuzzy right ideal* μ *and every fuzzy generalized bi-ideal* γ *of* S.

(6) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ *for every fuzzy left ideal* μ *and every fuzzy quasi-ideal* γ *of* S.

(7) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ *for every fuzzy left ideal* μ *and every fuzzy bi-ideal* γ *of* S.

(8) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ *for every fuzzy left ideal* μ *and every fuzzy generalized bi-ideal* γ *of* S.

(9) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ *for all fuzzy quasi-ideals* μ *and* γ *of* S.

(10) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ *for every fuzzy quasi-ideal* μ *and every fuzzy bi-ideal* γ *of* S.

(11) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ *for every fuzzy quasi-ideal* μ *and every fuzzy generalized bi-ideal* γ *of* S.

(12) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ *for all fuzzy bi-ideals* μ *and* γ *of* S.

(13) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ *for every fuzzy bi-ideal* μ *and every fuzzy generalized bi-ideal* γ *of* S.

(14) $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$ *for all fuzzy generalized bi-ideals* μ *and* γ *of* S.

Proof. Since $\mu \cap \gamma \subseteq \mu \circ \gamma$ and $\mu \cap \gamma \subseteq \gamma \circ \mu$ for all fuzzy generalized bi-ideals μ and γ of S by the Theorem 5.3. Hence $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu)$, i.e., $(1) \Rightarrow (14)$. It is clear that $(14) \Rightarrow (13) \Rightarrow (12) \Rightarrow (9) \Rightarrow (6) \Rightarrow (2)$, $(14) \Rightarrow (11) \Rightarrow (10) \Rightarrow (9)$, $(14) \Rightarrow (8) \Rightarrow (7) \Rightarrow (6)$ and $(14) \Rightarrow (5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2)$. Assume that (2) is true. Let μ be a fuzzy right ideal and γ be a fuzzy left ideal of S. By our assumption $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu) \subseteq \gamma \circ \mu$, i.e., S is an intra-regular. Again $\mu \cap \gamma \subseteq (\mu \circ \gamma) \cap (\gamma \circ \mu) \subseteq \mu \circ \gamma$. Since $\mu \circ \gamma \subseteq \mu \cap \gamma$, so $\mu \cap \gamma = \mu \circ \gamma$, i.e., S is a regular. Hence $(2) \Rightarrow (1)$. □

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] J. R. Cho, J. Jezek and T. Kepka, Paramedial groupoids, Czech. Math. J. 49(1999), 277-290.

[2] A. Lafi, DFIG control: A fuzzy approach, Int. J. Adv. Appl. Sci. 6(2019), 107-116.

[3] S. A. Razak, D. Mohamad, I. I. Abdullah, A Group decision making problem using hierarchical based fuzzy soft matrix approach, Int. J. Adv. Appl. Sci. 4(2017) 26-32.

[4] J. Jezek and T. Kepka, Medial groupoids, Rozpravy CSAV Rada Mat. a Prir. Ved., 93/2, 1983, 93 pp.

[5] T. Kadir, In discrepancy between the traditional Fuzzy logic and inductive, Int. J. Adv. Appl. Sci. 1(2014), 36-43.

[6] N. Kausar, M. Waqar, Characterizations of non-associative rings by their intuitionistic fuzzy bi-ideals, Eur. J. Pure Appl. Math. 12(2019), 226-250.

[7] N. Kausar, Characterizations of non-associative ordered semigroups by the properties of their fuzzy ideals with thresholds $(\alpha, \beta]$, Prikl. Diskr. Mat. 43(2019), 37-59.

[8] N. Kausar, Direct product of finite intuitionistic fuzzy normal subrings over non-associative rings, Eur. J. Pure Appl. Math. 12(2)(2019), 622-648.

[9] N. Kausar, B. Islam, M. Javaid, S, Amjad, U. Ijaz, Characterizations of non-associative rings by the properties of their fuzzy ideals, J. Taibah Univ. Sci. 13(2019), 820-833.

[10] N. Kausar, B. Islam, S. Amjad, M. Waqar, Intuitionistics fuzzy ideals with thresholds$(\alpha, \beta]$ in LA-rings, Eur. J. Pure Appl. Math. 12(3)(2019), 906-943.

[11] N. Kausar, M. Waqar, Direct product of finite fuzzy normal subrings over non-associative rings, Int. J. Anal. Appl. 17(5)(2019), 752-770.

[12] M. A. Kazim and M. Naseeruddin, On almost semigroups, Alig. Bull. Math. 2(1972), 1-7.

[13] N. Kehayopulu, On left regular ordered semigroups, Math. Japon. 35(1990), 1057-1060.

[14] N. Kehayopulu, On intra-regular ordered semigroups, Semigroup Forum, 46(1993), 271-278.

[15] N. Kehayopulu, On completely regular ordered semigroups, Sci. Math. 1(1998) 27-32.

[16] N. Kehayopulu and M. Tsingelis, Fuzzy sets in ordered groupoids, Semigroup Forum, 65(2002), 128-132.

[17] N. Kehayopulu and M. Tsingelis, Fuzzy bi-ideals in ordered semigroups, Inform. Sci. 171(2005), 13-28.

[18] N. Kuroki, Fuzzy bi-ideals in semigroups, Comment. Math. Univ. St. Pauli, 28(1979), 17-21.

[19] N. Kuroki, On fuzzy semigroups, Inform. Sci. 53(1991), 203-236.

[20] J. N. Mordeson, D. S. Malik and N. Kuroki, Fuzzy semigroups, Springer, Berlin, 2003.

[21] Q. Mushtaq and S. M. Yusuf, On LA-semigroups, Alig. Bull. Math. 8(1978), 65-70.

[22] P. V. Protic and N. Stevanovic, AG-test and some general properties of Abel-Grassmann's groupoids, Pure Math. Appl. 6(1995), 371-383.

[23] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35(1971), 512-517.

[24] T. Shah, N. Kausar and I. Rehman, Intuitionistic fuzzy normal subrings over a non-associative ring, An. st. Univ. Ovidius constanta, 1(2012), 369-386.

[25] T. Shah, N. Kausar, Characterizations of non-associative ordered semigroups by their fuzzy bi-ideals, Theor. Computer Sci. 529(2014), 96-110.

[26] O. Ozer, S. Omran, On the generalized C*- valued metric spaces related with Banach fixed point theory, Int. J. Adv. Appl. Sci. 4(2017), 35-37.

[27] L. A. Zadeh, Fuzzy sets, Inform. Control, 8(1965), 338-353.