GENERALIZATIONS OF MINKOWSKI AND BECKENBACH–DRESHER INEQUALITIES AND FUNCTIONALS ON TIME SCALES

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Abstract. We generalize integral forms of the Minkowski inequality and Beckenbach–Dresher inequality on time scales. Also, we investigate a converse of Minkowski's inequality and several functionals arising from the Minkowski inequality and the Beckenbach–Dresher inequality.

1. Introduction and Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The theory of time scales was introduced by Stefan Hilger [7] in order to unify the theory of difference equations and the theory of differential equations. For an introduction to the theory of dynamic equations on time scales, we refer to [3,8]. Martin Bohner and Gusein Sh. Guseinov [4,5] defined the multiple Riemann and multiple Lebesgue integration on time scales and compared the Lebesgue Δ-integral with the Riemann Δ-integral.

Let $n \in \mathbb{N}$ be fixed. For each $i \in \{1, \ldots, n\}$, let $\mathbb{T}_i$ denote a time scale and

$$\Lambda^n = \mathbb{T}_1 \times \ldots \times \mathbb{T}_n = \{t = (t_1, \ldots, t_n) : \ t_i \in \mathbb{T}_i, \ 1 \leq i \leq n\}$$

an n-dimensional time scale. Let μ_Δ be the σ-additive Lebesgue Δ-measure on Λ^n and $\mathcal{F}$ be the family of Δ-measurable subsets of Λ^n. Let $E \in \mathcal{F}$ and $(E, \mathcal{F}, \mu_\Delta)$ be a time scale measure space. Then for a Δ-measurable function $f : E \to \mathbb{R}$, the corresponding Δ-integral of f over E will be denoted according to [5, (3.18)] by

$$\int_E f(t_1, \ldots, t_n)\Delta_1 t_1 \ldots \Delta_n t_n, \quad \int_E f(t)\Delta t, \quad \int_E f \mathrm{d}\mu_\Delta, \quad \text{or} \quad \int_E f(t)\mathrm{d}\mu_\Delta(t).$$

By [5, Section 3], all theorems of the general Lebesgue integration theory, including the Lebesgue dominated convergence theorem, hold also for Lebesgue Δ-integrals on Λ^n. Here we state Fubini's theorem for integrals on time scales. It is used in the proofs of our main results.

Theorem 1.1 (Fubini's theorem). *Let $(X, \mathcal{M}, \mu_\Delta)$ and $(Y, \mathcal{L}, \nu_\Delta)$ be two finite-dimensional time scale measure spaces. If $f : X \times Y \to \mathbb{R}$ is a Δ-integrable function and if we define the functions*

$$\varphi(y) = \int_X f(x, y)\mathrm{d}\mu_\Delta(x) \quad \text{for a.e.} \quad y \in Y$$

and

$$\psi(x) = \int_Y f(x, y)\mathrm{d}\nu_\Delta(y) \quad \text{for a.e.} \quad x \in X,$$

then φ is Δ-integrable on Y and ψ is Δ-integrable on X and

$$\int_X \mathrm{d}\mu_\Delta(x) \int_Y f(x, y)\mathrm{d}\nu_\Delta(y) = \int_Y \mathrm{d}\nu_\Delta(y) \int_X f(x, y)\mathrm{d}\mu_\Delta(x). \tag{1.1}$$

Hölder's inequality and Minkowski's inequality and their converses for multiple integration on time scales were investigated in [1]. These inequalities hold for both Riemann integrals and Lebesgue integrals on time scales. For completeness, let us recall these inequalities from [1].

Theorem 1.2 (Hölder's inequality [1, Theorem 6.2]). *For $p \neq 1$, define $q = p/(p-1)$. Let $(E, \mathcal{F}, \mu_\Delta)$ be a time scale measure space. Assume w, f, g are nonnegative functions such that wf^p, wg^q, wfg are Δ-integrable on E. If $p > 1$, then*

$$\int_E w(t)f(t)g(t)\mathrm{d}\mu_\Delta(t) \leq \left(\int_E w(t)f^p(t)\mathrm{d}\mu_\Delta(t) \right)^{1/p} \times \left(\int_E w(t)g^q(t)\mathrm{d}\mu_\Delta(t) \right)^{1/q}. \tag{1.2}$$

If $0 < p < 1$ and $\int_E wg^q \mathrm{d}\mu_\Delta > 0$, or if $p < 0$ and $\int_E wf^p \mathrm{d}\mu_\Delta > 0$, then (1.2) is reversed.

Theorem 1.3 (Minkowski's inequality [1, Theorem 7.2]). *Let $(E, \mathcal{F}, \mu_\Delta)$ be a time scale measure space. For $p \in \mathbb{R}$, assume w, f, g, are nonnegative functions such that wf^p, wg^p, $w(f+g)^p$ are Δ-integrable on E. If $p \geq 1$, then*

$$\left(\int_E w(t)(f(t) + g(t))^p \mathrm{d}\mu_\Delta(t) \right)^{\frac{1}{p}} \leq \left(\int_E w(t)f^p(t)\mathrm{d}\mu_\Delta(t) \right)^{1/p} + \left(\int_E w(t)g^p(t)\mathrm{d}\mu_\Delta(t) \right)^{1/p}. \tag{1.3}$$

If $0 < p < 1$ or $p < 0$, then (1.3) is reversed provided each of the two terms on the right-hand side is positive.

Theorem 1.4 (Converse of Hölder's inequality [1, Theorem 11.3]). *For $p \neq 1$, define $q = p/(p-1)$. Let $(E, \mathcal{F}, \mu_\Delta)$ be a time scale measure space. Assume w, f, g are nonnegative functions such that wf^p, wg^q, wfg are Δ-integrable on E. Suppose*

$$0 < m \leq f(t)g^{-q/p}(t) \leq M \quad for \ all \quad t \in E.$$

If $p > 1$, then

$$\int_E w(t)f(t)g(t)\mathrm{d}\mu_\Delta(t) \geq K(p,m,M)\left(\int_E w(t)f^p(t)\mathrm{d}\mu_\Delta(t)\right)^{1/p} \times \left(\int_E w(t)g^q(t)\mathrm{d}\mu_\Delta(t)\right)^{1/q}, \quad (1.4)$$

where

$$K(p,m,M) = |p|^{1/p}|q|^{1/q}\frac{(M-m)^{1/p}|mM^p - Mm^p|^{1/q}}{|M^p - m^p|}. \quad (1.5)$$

If $0 < p < 1$ or $p < 0$, then (1.4) is reversed provided either $\int_E wg^q\mathrm{d}\mu_\Delta > 0$ or $\int_E wf^p\mathrm{d}\mu_\Delta > 0$.

In [2] Bibi et al., obtain integral forms of Minkowski's and Beckenbach–Dresher inequality on time scales. In this paper we generalize these inequalities and investigate functional obtained from our new inequalities.

2. Minkowski Inequalities

Let $U_l(x_1, x_2, \ldots, x_l)$, $V_m(x_1, x_2, \ldots, x_m)$, $G_k(x_1, x_2, \ldots, x_k)$, are real valued functions of l, m, and k variables, respectively. Let $(X, \mathcal{M}, \mu_\Delta)$ and $(Y, \mathcal{L}, \nu_\Delta)$ be time scale measure spaces. Then, throughout in the following sections, we use the following notations:

$$U_l = U_l(x) = U_l(u_1(x), u_2(x), \ldots, u_l(x)), \quad (2.1)$$

$$V_m = V_m(y) = V_m(v_1(y), v_2(y), \ldots, v_m(y)),$$

$$F_k = F_k(x,y) = F_k(f_1(x,y), f_2(x,y), \ldots, f_k(x,y)),$$

where $\{u_i(x)\}_{i=1}^l$, $\{v_i(y)\}_{i=1}^m$, $\{f_i(x,y)\}_{i=1}^k$, are defined on X, Y, and $X \times Y$, respectively. In the sequel, we assume that all occurring integrals are finite.

Theorem 2.1 (Integral Minkowski inequality). *If $p \geq 1$, then*

$$\left[\int_X \left(\int_Y F_k(x,y)V_m(y)\mathrm{d}\nu_\Delta(y)\right)^p U_l(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}} \leq \int_Y \left(\int_X F_k^p(x,y)U_l(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} V_m(y)\mathrm{d}\nu_\Delta(y) \quad (2.2)$$

holds provided all integrals in (2.2) exists. If $0 < p < 1$ and

$$\int_X \left(\int_Y F_k V_m \mathrm{d}\nu_\Delta\right)^p U_l \mathrm{d}\mu_\Delta > 0, \quad \int_Y F_k V_m \mathrm{d}\nu_\Delta > 0 \quad (2.3)$$

holds, then (2.2) is reversed. If $p < 0$ and (2.3) and

$$\int_X F_k^p U_l \mathrm{d}\mu_\Delta > 0, \quad (2.4)$$

hold, then (2.2) is reversed as well.

Proof. Let $p \geq 1$. Put

$$H(x) = \int_Y F_k(x,y)V_m(y)\mathrm{d}\nu_\Delta(y).$$

Now, by using Fubini's theorem (Theorem 1.1) and Hölder's inequality (Theorem 1.2) on time scales, we have

$$\int_X H^p(x)U_l(x)\mathrm{d}\mu_\Delta(x) = \int_X H(x)H^{p-1}(x)U_l(x)\mathrm{d}\mu_\Delta(x)$$

$$= \int_X \left(\int_Y F_k(x,y)V_m(y)\mathrm{d}\nu_\Delta(y) \right) H^{p-1}(x)U_l(x)\mathrm{d}\mu_\Delta(x)$$

$$= \int_Y \left(\int_X F_k(x,y)H^{p-1}(x)U_l(x)\mathrm{d}\mu_\Delta(x) \right) V_m(y)\mathrm{d}\nu_\Delta(y)$$

$$\leq \int_Y \left(\int_X F_k^p(x,y)U_l(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} \left(\int_X H^p(x)U_l(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{p-1}{p}} V_m(y)\mathrm{d}\nu_\Delta(y)$$

$$= \int_Y \left(\int_X F_k^p(x,y)U_l(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y)\mathrm{d}\nu_\Delta(y) \left(\int_X H^p(x)U_l(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{p-1}{p}}$$

and hence

$$\left(\int_X H^p(x)U_l(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} \leq \int_Y \left(\int_X F_k^p(x,y)U_l(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y)\mathrm{d}\nu_\Delta(y).$$

For $p < 0$ and $0 < p < 1$, the corresponding results can be obtained similarly. $\square$

Theorem 2.2 (Converse of integral Minkowski inequality). *Suppose*

$$0 < m \leq \frac{F_k(x,y)}{\int_Y F_k(x,y)V_m(y)\mathrm{d}\nu_\Delta(y)} \leq M \quad \textit{for all} \quad x \in X,\ y \in Y.$$

If $p \geq 1$, then

$$\left[\int_X \left(\int_Y F_k(x,y)V_m(y)\mathrm{d}\nu_\Delta(y) \right)^p U_l(x)\mathrm{d}\mu_\Delta(x) \right]^{\frac{1}{p}}$$

$$\geq K(p,m,M) \int_Y \left(\int_X F_k^p(x,y)U_l(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y)\mathrm{d}\nu_\Delta(y) \quad (2.5)$$

provided all integrals in (2.5) *exist, where $K(p,m,M)$ is defined by* (1.5). *If $0 < p < 1$ and* (2.3) *holds, then* (2.5) *is reversed. If $p < 0$ and* (2.3) *and* (2.4) *hold, then* (2.5) *is reversed as well.*

Proof. Let $p \geq 1$. Put

$$H(x) = \int_Y F_k(x,y)V_m(y)\mathrm{d}\nu_\Delta(y).$$

Then by using Fubini's theorem (Theorem 1.1) and the converse Hölder inequality (Theorem 1.4) on time scales, we get

$$
\begin{aligned}
\int_X H^p(x)U_l(x)\mathrm{d}\mu_\Delta(x) &= \int_X \left(\int_Y F_k(x,y)V_m(y)\mathrm{d}\nu_\Delta(y) \right) H^{p-1}(x)U_l(x)\mathrm{d}\mu_\Delta(x) \\
&= \int_Y \left(\int_X F_k(x,y)H^{p-1}(x)U_l(x)\mathrm{d}\mu_\Delta(x) \right) V_m(y)\mathrm{d}\nu_\Delta(y) \\
&\geq K(p,m,M) \int_Y \left(\int_X F_k^p(x,y)U_l(x)\mathrm{d}\mu_\Delta(x) \right)^{1/p} \\
&\quad \times \left(\int_X H^p(x)U_l(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{p-1}{p}} V_m(y)\mathrm{d}\nu_\Delta(y).
\end{aligned}
$$

Dividing both sides by $\left(\int_X H^p(x)U_l(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{p-1}{p}}$, we obtain (2.5). For $0 < p < 1$ and $p < 0$, the corresponding results can be obtained similarly.   □

Now we define the rth power mean $M^{[r]}(F_k, \mu_\Delta)$ of the function F_k with respect to the measure μ_Δ by

$$
M^{[r]}(F_k,\mu_\Delta) = \begin{cases}
\left(\dfrac{\int_X F_k^r(x,y)U_l(x)\mathrm{d}\mu_\Delta(x)}{\int_X U_l(x)\mathrm{d}\mu_\Delta(x)} \right)^{\frac{1}{r}} & \text{if} \quad r \neq 0, \\[4mm]
\exp\left(\dfrac{\int_X \log F_k(x,y)U_l(x)\mathrm{d}\mu_\Delta(x)}{\int_X U_l(x)\mathrm{d}\mu_\Delta(x)} \right) & \text{if} \quad r = 0,
\end{cases} \tag{2.6}
$$

where $\int_X U_l\mathrm{d}\mu_\Delta > 0$.

Corollary 2.1. *Let $0 < s \leq r$. Then*

$$
M^{[r]}(M^{[s]}(F_k, \mathrm{d}\nu_\Delta), \mathrm{d}\mu_\Delta) \geq K\left(\frac{r}{s}, m, M \right) M^{[s]}(M^{[r]}(F_k, \mathrm{d}\mu_\Delta), \mathrm{d}\nu_\Delta).
$$

Proof. By putting $p = r/s$ and replacing F_k by F_k^s in (2.5), raising to the power of $\frac{1}{s}$ and dividing by

$$
\left(\int_X U_l(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{r}} \left(\int_Y V_m(y)\mathrm{d}\nu_\Delta(y) \right)^{\frac{1}{s}},
$$

we get the above result.   □

3. Minkowski Functionals

In this section, we will consider some functionals which arise from the Minkowski inequality. Similar results (but not for time scales measure spaces) can be found in [9].

Let F_k and V_m be fixed functions satisfying the assumptions of Theorem 2.1. Let us consider the functional M_1 defined by

$$
\mathsf{M}_1(U_l) = \left[\int_Y \left(\int_X F_k^p(x,y)U_l(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y)\mathrm{d}\nu_\Delta(y) \right]^p - \int_X \left(\int_Y F_k(x,y)V_m(y)\mathrm{d}\nu_\Delta(y) \right)^p U_l(x)\mathrm{d}\mu_\Delta(x),
$$

where U_l is a nonnegative function on X such that all occurring integrals exist. Also, if we fix the functions F_k and U_l, then we can consider the functional

$$\mathsf{M}_2(V_m) = \int_Y \left(\int_X F_k^p(x,y) U_l(x) \mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y) \mathrm{d}\nu_\Delta(y)$$
$$- \left[\int_X \left(\int_Y F_k(x,y) V_m(y) \mathrm{d}\nu_\Delta(y) \right)^p U_l(x) \mathrm{d}\mu_\Delta(x) \right]^{\frac{1}{p}},$$

where V_m is a nonnegative function on Y such that all occurring integrals exist.

Remark 3.1.   (i) *It is obvious that M_1 and M_2 are positive homogeneous, i.e., $\mathsf{M}_1(aU_l) = a\mathsf{M}_1(U_l)$, and $\mathsf{M}_2(aV_m) = a\mathsf{M}_2(V_m)$, for any $a > 0$.*

(ii) *If $p \geq 1$ or $p < 0$, then $\mathsf{M}_1(U_l) \geq 0$, and if $0 < p < 1$, then $\mathsf{M}_1(U_l) \leq 0$.*

(iii) *If $p \geq 1$, then $\mathsf{M}_2(V_m) \geq 0$, and if $p < 1$ and $p \neq 0$, then $\mathsf{M}_2(V_m) \leq 0$.*

Theorem 3.1.   (i) *If $p \geq 1$ or $p < 0$, then M_1 is superadditive. If $0 < p < 1$, then M_1 is subadditive.*

(ii) *If $p \geq 1$, then M_2 is superadditive. If $p < 1$ and $p \neq 0$, then M_2 is subadditive.*

(iii) *Suppose U_{l1} and U_{l2} are nonnegative functions such that $U_{l2} \geq U_{l1}$. If $p \geq 1$ or $p < 0$, then*

$$0 \leq \mathsf{M}_1(U_{l1}) \leq \mathsf{M}_1(U_{l2}), \tag{3.1}$$

and if $0 < p < 1$, then (3.1) is reversed.

(iv) *Suppose V_{m1} and V_{m2} are nonnegative functions such that $V_{m2} \geq V_{m1}$. If $p \geq 1$, then*

$$0 \leq \mathsf{M}_2(V_{m1}) \leq \mathsf{M}_2(V_{m2}), \tag{3.2}$$

and if $p < 1$ and $p \neq 0$, then (3.2) is reversed.

Proof. First we show (i). We have

$$\mathsf{M}_1(U_{l1} + U_{l2}) - \mathsf{M}_1(U_{l1}) - \mathsf{M}_1(U_{l2})$$
$$= \left[\int_Y \left(\int_X f^p(x,y)(U_{l1}+U_{l2})(x) \mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y) \mathrm{d}\nu_\Delta(y) \right]^p$$
$$- \int_X \left(\int_Y F_k(x,y) V_m(y) \mathrm{d}\nu_\Delta(y) \right)^p (U_{l1}+U_{l2})(x) \mathrm{d}\mu_\Delta(x)$$
$$- \left[\int_Y \left(\int_X F_k^p(x,y) U_{l1}(x) \mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y) \mathrm{d}\nu_\Delta(y) \right]^p$$
$$+ \int_X \left(\int_Y F_k(x,y) V_m(y) \mathrm{d}\nu_\Delta(y) \right)^p U_{l1}(x) \mathrm{d}\mu_\Delta(x)$$
$$- \left[\int_Y \left(\int_X F_k^p(x,y) U_{l2}(x) \mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y) \mathrm{d}\nu_\Delta(y) \right]^p$$

$$+ \int_X \left(\int_Y F_k(x,y) V_m(y) \mathrm{d}\nu_\Delta(y) \right)^p U_{l2}(x) \mathrm{d}\mu_\Delta(x)$$

$$= \left[\int_Y \left(\int_X F_k^p(x,y)(U_{l1}+U_{l2})(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y)\mathrm{d}\nu_\Delta(y) \right]^p$$

$$- \left[\int_Y \left(\int_X F_k^p(x,y) U_{l1}(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y)\mathrm{d}\nu_\Delta(y) \right]^p$$

$$- \left[\int_Y \left(\int_X F_k^p(x,y) U_{l2}(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y)\mathrm{d}\nu_\Delta(y) \right]^p.$$

Using the Minkowski inequality (1.3) for integrals (Theorem 1.3) with p replaced by $1/p$, we have

$$\mathsf{M}_1(U_{l1}+U_{l2}) - \mathsf{M}_1(U_{l1}) - \mathsf{M}_1(U_{l2}) \begin{cases} \geq 0 & \text{if} \quad p \geq 1 \text{ or } p < 0, \\ \leq 0 & \text{if} \quad 0 < p \leq 1. \end{cases} \tag{3.3}$$

So, M_1 is superadditive for $p \geq 1$ or $p < 0$, and it is subadditive for $0 < p \leq 1$. The proof of (ii) is similar: After a simple calculation, we have

$$\mathsf{M}_2(V_{m1}+V_{m2}) - \mathsf{M}_2(V_{m1}) - \mathsf{M}_2(V_{m2})$$

$$= \left[\int_X \left(\int_Y F_k(x,y) V_{m1}(y) \mathrm{d}\nu_\Delta(y) \right)^p U_l(x) \mathrm{d}\mu_\Delta(x) \right]^{\frac{1}{p}}$$

$$+ \left[\int_X \left(\int_Y F_k(x,y) V_{m2}(y) \mathrm{d}\nu_\Delta(y) \right)^p U_l(x) \mathrm{d}\mu_\Delta(x) \right]^{\frac{1}{p}}$$

$$- \left[\int_X \left(\int_Y F_k(x,y)(V_{m1}+V_{m2})(y)\mathrm{d}\nu_\Delta(y) \right)^p U_l(x)\mathrm{d}\mu_\Delta(x) \right]^{\frac{1}{p}}.$$

Using the Minkowski inequality (2.2) for integrals (Theorem 2.1), we have that this is nonnegative for $p \geq 1$ and nonpositive for $p < 1$ and $p \neq 0$. Now we show (iii). If $p \geq 1$ or $p < 0$, then using superadditivity and positivity of M_1, $U_{l2} \geq U_{l1}$ implies

$$\mathsf{M}_1(U_{l2}) = \mathsf{M}_1(U_{l1}+(U_{l2}-U_{l1})) \geq \mathsf{M}_1(U_{l1}) + \mathsf{M}_1(U_{l2}-U_{l1}) \geq \mathsf{M}_1(U_{l1}),$$

and the proof of (3.1) is established. If $0 < p < 1$, then using subadditivity and negativity of M_1, $U_{l2} \geq U_{l1}$ implies

$$\mathsf{M}_1(U_{l2}) \leq \mathsf{M}_1(U_{l1}) + \mathsf{M}_1(U_{l2}-U_{l1}) \leq \mathsf{M}_1(U_{l1}).$$

The proof of (iv) is similar.   □

Remark 3.2. *Put $X, Y \subseteq \mathbb{N}$, then for fixed F_k and U_l, the function M_2 has the form*

$$\mathsf{M}_2(V_{m1}) = \sum_{j \in Y} V_{m1}(j) \left(\sum_{i \in X} U_l(i) F_k(i,j)^p \right)^{1/p} - \left(\sum_{i \in X} U_l(i) \left(\sum_{j \in Y} V_{m1}(j) F_k(i,j) \right)^p \right)^{1/p},$$

where $f(i,j) = F_k(i,j) \geq 0$. If $p \geq 1$, then the mapping M_2 is superadditive, and $V_{m2}(j) \geq V_{m1}(j)$ for all $j \in Y$ implies

$$0 \leq \sum_{j \in Y} V_{m1}(j) \left(\sum_{i \in X} U_l(i) F_k(i,j)^p \right)^{1/p} - \left(\sum_{i \in X} U_l(i) \left(\sum_{j \in Y} V_{m1}(j) F_k(i,j) \right)^p \right)^{1/p}$$

$$\leq \sum_{j \in Y} V_{m2}(j) \left(\sum_{i \in X} U_l(i) F_k(i,j)^p \right)^{1/p} - \left(\sum_{i \in X} U_l(i) \left(\sum_{j \in Y} V_{m2}(j) F_k(i,j) \right)^p \right)^{1/p}$$

provided all occurring sums are finite.

Corollary 3.1.   (i) Suppose U_{l1} and U_{l2} are nonnegative functions such that $CU_{l2} \geq U_{l1} \geq cU_{l2}$, where $c, C \geq 0$. If $p \geq 1$ or $p < 0$, then

$$c\mathsf{M}_1(U_{l2}) \leq \mathsf{M}_1(U_{l1}) \leq C\mathsf{M}_1(U_{l2}),$$

and if $0 < p < 1$, then the above inequality is reversed.

(ii) Suppose V_{m1} and V_{m2} are nonnegative functions such that $CV_{m2} \geq V_{m1} \geq cV_{m2}$, where $c, C \geq 0$. If $p \geq 1$, then

$$c\mathsf{M}_2(V_{m2}) \leq \mathsf{M}_2(V_{m1}) \leq C\mathsf{M}_2(V_{m2}),$$

and if $p < 1$ and $p \neq 0$, then the above inequality is reversed.

Corollary 3.2. If V_{m1} and V_{m2} are nonnegative functions such that $V_{m2} \geq V_{m1}$, then

$$M^{[0]} \left(\int_Y F_k(x,y) V_{m1}(y) \mathrm{d}\nu_\Delta(y), \mu_\Delta \right) - \int_Y M^{[0]}(F_k, \mu_\Delta) V_{m1}(y) \mathrm{d}\nu_\Delta(y)$$

$$\leq M^{[0]} \left(\int_Y F_k(x,y) V_{m2}(y) \mathrm{d}\nu_\Delta(y), \mu_\Delta \right) - \int_Y M^{[0]}(F_k, \mu_\Delta) V_{m2}(y) \mathrm{d}\nu_\Delta(y), \quad (3.4)$$

where $M^{[0]}(F_k, \mu_\Delta)$ is defined in (2.6).

The next result gives another property of M_1, but a similar result can also be stated for M_2.

Theorem 3.2. Let $\varphi : [0, \infty) \to [0, \infty)$ be a concave function. Suppose U_{l1} and U_{l2} are nonnegative functions such that

$$\varphi \circ U_{l1}, \quad \varphi \circ U_{l2}, \quad \varphi \circ (\alpha U_{l1} + (1-\alpha)U_{l2})$$

are Δ-integrable for $\alpha \in [0,1]$. If $p \geq 1$, then

$$\mathsf{M}_1(\varphi \circ (\alpha U_{l1} + (1-\alpha)U_{l2})) \geq \alpha \mathsf{M}_1(\varphi \circ U_{l1}) + (1-\alpha)\mathsf{M}_1(\varphi \circ U_{l2}),$$

and if $0 < p < 1$, then the above inequality is reversed.

Proof. We show this only for $p \geq 1$ as the other case follows similarly. Since φ is concave, we have

$$\varphi(\alpha U_{l1} + (1-\alpha)U_{l2})) \geq \alpha\varphi(U_{l1}) + (1-\alpha)\varphi(U_{l2}).$$

Now, from (3.1) and (3.3), we have

$$\mathsf{M}_1(\varphi \circ (\alpha U_{l1} + (1-\alpha)U_{l2})) \geq \mathsf{M}_1(\alpha(\varphi \circ U_{l1}) + (1-\alpha)(\varphi \circ U_{l2}))$$

$$\geq \mathsf{M}_1(\alpha(\varphi \circ U_{l1})) + \mathsf{M}_1((1-\alpha)(\varphi \circ U_{l2}))$$

$$\geq \alpha\mathsf{M}_1(\varphi \circ U_{l1}) + (1-\alpha)\mathsf{M}_1(\varphi \circ U_{l2}),$$

and the proof is established. $\square$

Let F_k, U_l and V_m be fixed functions satisfying the assumptions of Theorem 2.1. Let us define functionals M_3 and M_4 by

$$\mathsf{M}_3(A) = \left[\int_Y \left(\int_A F_k^p(x,y)U_l(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} V_m(y)\mathrm{d}\nu_\Delta(y)\right]^p - \int_A \left(\int_Y F_k(x,y)V_m(y)\mathrm{d}\nu_\Delta(y)\right)^p U_l(x)\mathrm{d}\mu_\Delta(x)$$

and

$$\mathsf{M}_4(B) = \int_B \left(\int_X F_k^p(x,y)U_l(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} V_m(y)\mathrm{d}\nu_\Delta(y) - \left[\int_X \left(\int_B F_k(x,y)V_m(y)\mathrm{d}\nu_\Delta(y)\right)^p U_l(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}},$$

where $A \subseteq X$ and $B \subseteq Y$.

The following theorem establishes superadditivity and monotonicity of the mappings M_3 and M_4.

Theorem 3.3.   (i) *Suppose $A_1, A_2 \subseteq X$ and $A_1 \cap A_2 = \emptyset$. If $p \geq 1$ or $p < 0$, then*

$$\mathsf{M}_3(A_1 \cup A_2) \geq \mathsf{M}_3(A_1) + \mathsf{M}_3(A_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

(ii) *Suppose $A_1, A_2 \subseteq X$ and $A_1 \subseteq A_2$. If $p \geq 1$ or $p < 0$, then*

$$\mathsf{M}_3(A_1) \leq \mathsf{M}_3(A_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

(iii) *Suppose $B_1, B_2 \subseteq Y$ and $B_1 \cap B_2 = \emptyset$. If $p \geq 1$, then*

$$\mathsf{M}_4(B_1 \cup B_2) \geq \mathsf{M}_4(B_1) + \mathsf{M}_4(B_2),$$

and if $p < 1$ and $p \neq 0$, then the above inequality is reversed.

(iv) *Suppose $B_1, B_2 \subseteq Y$ and $B_1 \subseteq B_2$. If $p \geq 1$, then*

$$\mathsf{M}_4(B_1) \leq \mathsf{M}_4(B_2),$$

and if $p < 1$ and $p \neq 0$, then the above inequality is reversed.

The proof of Theorem 3.3 is omitted as it is similar to the proof of Theorem 3.1.

Remark 3.3. *For $p \geq 1$, if S_m is a subset of Y with m elements and if $S_m \supseteq S_{m-1} \supseteq \ldots \supseteq S_2$, then we have*

$$\mathsf{M}_4(S_m) \geq \mathsf{M}_4(S_{m-1}) \geq \ldots \geq \mathsf{M}_4(S_2) \geq 0$$

and $\mathsf{M}_4(S_m) \geq \max\{\mathsf{M}_4(S_2) : S_2$ is any subset of S_m with 2 elements$\}$.

4. Beckenbach–Dresher Inequalities

Let U_l, V_m, F_k be defined as in (4.1). Let $\mathcal{F}_n(x_1, x_2, \ldots, x_n)$, $\mathcal{G}_t(x_1, x_2, \ldots, x_t)$ are real valued functions of n, and t variables, respectively. Let $(X, \mathcal{M}, \mu_\Delta)$, $(X, \mathcal{M}, \lambda_\Delta)$ and $(Y, \mathcal{L}, \nu_\Delta)$ be time scale measure spaces. Then, throughout in the following sections, we use the following notations:

$$\mathcal{W}_n = \mathcal{W}_n(x) = \mathcal{W}_n(w_1(x), w_2(x), \ldots, w_n(x)), \tag{4.1}$$

$$\mathcal{G}_t = \mathcal{G}_t(x, y) = \mathcal{G}_t(g_1(x, y), g_2(x, y), \ldots, g_t(x, y)),$$

where U_l and $\mathcal{W}_n$ are nonnegative functions on X, V_m is a nonnegative function on Y, F_k is a nonnegative function on $X \times Y$ with respect to the measure $(\mu_\Delta \times \nu_\Delta)$, and $\mathcal{G}_t$ is a nonnegative function on $X \times Y$ with respect to the measure $(\lambda_\Delta \times \nu_\Delta)$. In the sequel, we assume that all occurring integrals are finite.

Theorem 4.1. *If*

$$s \geq 1, \quad q \leq 1 \leq p, \quad and \quad q \neq 0 \tag{4.2}$$

or

$$s < 0, \quad p \leq 1 \leq q, \quad and \quad p \neq 0, \tag{4.3}$$

then

$$\frac{\left[\int_X \left(\int_Y F_k(x, y) V_m(y) \mathrm{d}\nu_\Delta(y)\right)^p U_l(x) \mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y \mathcal{G}_t(x, y) V_m(y) \mathrm{d}\nu_\Delta(y)\right)^q \mathcal{W}_n(x) \mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}}$$

$$\leq \int_Y \frac{\left(\int_X F_k^p(x, y) U_l(x) \mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}}{\left(\int_X \mathcal{G}_t^q(x, y) \mathcal{W}_n(x) \mathrm{d}\lambda_\Delta(x)\right)^{\frac{s-1}{q}}} V_m(y) \mathrm{d}\nu_\Delta(y) \tag{4.4}$$

provided all occurring integrals in (4.4) exist. If

$$0 < s \leq 1, \quad p \geq 1, \quad q \leq 1, \quad and \quad q \neq 0, \tag{4.5}$$

then (4.4) is reversed.

Proof. Assume (4.2) or (4.3). By using the integral Minkowski inequality (2.2) and Hölder's inequality (1.2), we have

$$
\frac{\left[\int_X \left(\int_Y F_k(x,y)V_m(y)\mathrm{d}\nu_\Delta(y)\right)^p U_l(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y \mathcal{G}_t(x,y)V_m(y)\mathrm{d}\nu_\Delta(y)\right)^q \mathcal{W}_n(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}}
$$

$$
\leq \frac{\left[\int_Y \left(\int_X F_k^p(x,y)U_l(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} V_m(y)\mathrm{d}\nu_\Delta(y)\right]^s}{\left[\int_Y \left(\int_X \mathcal{G}_t^q(x,y)\mathcal{W}_n(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{1}{q}} V_m(y)\mathrm{d}\nu_\Delta(y)\right]^{s-1}}
$$

$$
= \left[\int_Y \left(\left(\int_X F_k^p(x,y)U_l(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}\right)^{\frac{1}{s}} V_m(y)\mathrm{d}\nu_\Delta(y)\right]^s
$$

$$
\times \left[\int_Y \left(\left(\int_X \mathcal{G}_t^q(x,y)\mathcal{W}_n(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{1-s}{q}}\right)^{\frac{1}{1-s}} V_m(y)\mathrm{d}\nu_\Delta(y)\right]^{1-s}
$$

$$
\leq \int_Y \left(\int_X F_k^p(x,y)U_l(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}} \left(\int_X \mathcal{G}_t^q(x,y)\mathcal{W}_n(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{1-s}{q}} V_m(y)\mathrm{d}\nu_\Delta(y).
$$

If (4.5) holds, then the reversed inequality in (4.4) can be proved in a similar way. □

5. Beckenbach–Dresher Functionals

Let F_k, $\mathcal{G}_t$, U_l, $\mathcal{W}_n$ be fixed functions satisfying the assumptions of Theorem 4.1. We define the Beckenbach–Dresher functional $\mathsf{BD}(V_m)$ by

$$
\mathsf{BD}(V_m) = \int_Y \frac{\left(\int_X F_k^p(x,y)U_l(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}}{\left(\int_X \mathcal{G}_t^q(x,y)\mathcal{W}_n(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{s-1}{q}}} V_m(y)\mathrm{d}\nu_\Delta(y)
$$

$$
- \frac{\left[\int_X \left(\int_Y F_k(x,y)V_m(y)\mathrm{d}\nu_\Delta(y)\right)^p U_l(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y \mathcal{G}_t(x,y)V_m(y)\mathrm{d}\nu_\Delta(y)\right)^q \mathcal{W}_n(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}},
$$

where we suppose that all occurring integrals exist.

Theorem 5.1. *If* (4.2) *or* (4.3) *holds, then*

$$
\mathsf{BD}(V_{m1} + V_{m2}) \geq \mathsf{BD}(V_{m1}) + \mathsf{BD}(V_{m2}). \tag{5.1}
$$

If $V_{m2} \geq V_{m1}$*, then*

$$
\mathsf{BD}(V_{m1}) \leq \mathsf{BD}(V_{m2}). \tag{5.2}
$$

If $C, c \geq 0$ *and* $CV_{m2} \geq V_{m1} \geq cV_{m2}$*, then*

$$
C\mathsf{BD}(V_{m2}) \geq \mathsf{BD}(V_{m1}) \geq c\mathsf{BD}(V_{m1}). \tag{5.3}
$$

If (4.5) *holds, then* (5.1)*,* (5.2) *and* (5.3) *are reversed.*

Proof. Assume (4.2) or (4.3). Then we have

$$
\begin{aligned}
&\mathsf{BD}(V_{m1} + V_{m2}) - \mathsf{BD}(V_{m1}) - \mathsf{BD}(V_{m2}) \\
&= \frac{\left[\int_X \left(\int_Y F_k(x,y)V_{m1}(y)\mathrm{d}\nu_\Delta(y)\right)^p U_l(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y \mathcal{G}_t(x,y)V_{m1}(y)\mathrm{d}\nu_\Delta(y)\right)^q \mathcal{W}_n(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \\
&\quad + \frac{\left[\int_X \left(\int_Y F_k(x,y)V_{m2}(y)\mathrm{d}\nu_\Delta(y)\right)^p U_l(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y \mathcal{G}_t(x,y)V_{m2}(y)\mathrm{d}\nu_\Delta(y)\right)^q \mathcal{W}_n(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \\
&\quad - \frac{\left[\int_X \left(\int_Y F_k(x,y)V_{m1}(y)\mathrm{d}\nu_\Delta(y) + \int_Y F_k(x,y)V_{m2}(y)\mathrm{d}\nu_\Delta(y)\right)^p U_l(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y \mathcal{G}_t(x,y)V_{m1}(y)\mathrm{d}\nu_\Delta(y) + \int_Y \mathcal{G}_t(x,y)V_{m2}(y)\mathrm{d}\nu_\Delta(y)\right)^q \mathcal{W}_n(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \\
&\geq \quad 0,
\end{aligned}
$$

where in the last inequality we used (4.4) from Theorem 4.1. Using Theorem 4.1 again, $V_{m2} \geq V_{m1}$ implies

$$
\mathsf{BD}(V_{m2}) = \mathsf{BD}(V_{m1} + (V_{m2} - V_{m1})) \geq \mathsf{BD}(V_{m1}) + \mathsf{BD}(V_{m2} - V_{m1}) \geq \mathsf{BD}(V_{m1}).
$$

The proof of (5.3) is similar. If (4.5) holds, then the reversed inequalities of (5.1), (5.2) and (5.3) can be proved in a similar way. $\square$

Let F_k, $\mathcal{G}_t$, U_l, V_m, $\mathcal{W}_n$ be fixed functions. We define a functional BD_1 by

$$
\mathsf{BD}_1(A) = \int_A \frac{\left(\int_X F_k^p(x,y)U_l(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}}{\left(\int_X \mathcal{G}_t^q(x,y)\mathcal{W}_n(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{s-1}{q}}} V_m(y)\mathrm{d}\nu_\Delta(y)
$$
$$
- \frac{\left[\int_X \left(\int_A F_k(x,y)V_m(y)\mathrm{d}\nu_\Delta(y)\right)^p U_l(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_A \mathcal{G}_t(x,y)V_m(y)\mathrm{d}\nu_\Delta(y)\right)^q \mathcal{W}_n(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}},
$$

where $A \subseteq Y$.

For BD_1, the following result holds.

Theorem 5.2.   (i) *Suppose $A_1, A_2 \subseteq Y$ and $A_1 \cap A_2 = \emptyset$. If (4.2) or (4.3) holds, then*

$$
\mathsf{BD}_1(A_1 \cup A_2) \geq \mathsf{BD}_1(A_1) + \mathsf{BD}_1(A_2),
$$

and if (4.5) holds, then the above inequality is reversed.

(ii) *Suppose $A_1, A_2 \subseteq Y$ and $A_1 \subseteq A_2$. If (4.2) or (4.3) holds, then*

$$
\mathsf{BD}_1(A_1) \leq \mathsf{BD}_1(A_2),
$$

and if (4.5) holds, then the above inequality is reversed.

The proof of Theorem 5.2 is omitted as it is similar to the proof of Theorem 5.1.

Remark 5.1. *If $S_k \subseteq X$ has k elements and if $S_m \supseteq S_{m-1} \supseteq \ldots \supseteq S_2$, then* (4.2) *or* (4.3) *implies*

$$\mathsf{BD}_1(S_m) \geq \mathsf{BD}_1(S_{m-1}) \geq \cdots \geq \mathsf{BD}_1(S_2) \geq 0$$

and $\mathsf{BD}_1(S_m) \geq \max\{\mathsf{BD}_1(S_2) : S_2$ *is any subset of* S_m *with* 2 *elements*$\}$, *while* (4.5) *implies the reversed inequalities with* max *replaced by* min.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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