



ON GENERALIZED K-UNIFORMLY CLOSE-TO-CONVEX FUNCTIONS OF JANOWSKI TYPE

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ABSTRACT. This work is concerned with the class of analytic functions that maps open unit disk onto conic domains. Necessary condition, arc length, growth rate of coefficients, radius problems and property of some integral transformation under the class are examined.

1. INTRODUCTION

Let \mathcal{A} be the class of functions $f(z)$ whose series representation is given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

and regular in an open unit disk $\Delta = \{z: |z| < 1\}$. Let \mathcal{S} denotes the class of univalent functions in Δ and $\mathcal{C}(\rho)$, $\mathcal{S}^*(\rho)$ and $\mathcal{K}(\rho)$, $0 \leq \rho < 1$ be the well known subclasses of \mathcal{S} which consist of convex, starlike and close-to-convex functions of order ρ respectively. $\mathcal{C}(0) \equiv \mathcal{C}$, $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}(0) \equiv \mathcal{K}$ are the classes of convex, starlike and close-to-convex functions in Δ respectively. A function $f \in \mathcal{A}$ is subordinate to $g \in \mathcal{A}$ (written as $f(z) \prec g(z)$) if there exists a function $w(z)$ with $|w(z)| < 1$ and $w(0) = 0$ such that $f(z) = g(w(z))$. In addition, if $g(z)$ is univalent in Δ , then $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$ [4].

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Let \mathcal{H} be the class of functions $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ that are regular in Δ with $p(0) = 1$. Then $p \in P[A, B]$, $-1 \leq B < A \leq 1$ if and only if $p(z) \prec \frac{1+Az}{1+Bz}$, or equivalently

$$p(z) = \frac{(A + 1)h(z) - (A - 1)}{(B + 1)h(z) - (B - 1)},$$

where $h \in \mathcal{P}[1, -1] = \mathcal{P}$, the class of functions with positive real part. This class of functions was first considered and study extensively by Janowski [6].

Kanas and Wisniowska [7,8] introduced the class of k -uniformly convex functions and k -uniformly starlike function denoted by $k - UCV$ and $k - UST$ respectively, which were defined subject to the conic domain $\Omega_k, k \geq 0$, given by

$$\Omega_k = \{u + iv : u > k\sqrt{(u - 1)^2 + v^2}\} \tag{1.2}$$

This domain represents the right half plane for $k = 0$, the right branch of hyperbola for $0 < k < 1$ and an ellipse when $k > 1$. The function $p_k(z)$ plays an extremal role for all functions that maps Δ onto Ω_k and it is given by

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1, \\ 1 + \frac{2}{1-k^2} \sinh^2 \left[\left(\frac{2}{\pi} \arccos k\right) \arctan \sqrt{z}\right], & 0 < k < 1, \\ 1 + \frac{1}{1-k^2} \sin \left[\frac{\pi}{2R(t)} \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}} dx \right] + \frac{1}{k^2-1}, & k > 1, \end{cases} \tag{1.3}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0, 1), z \in \Delta$ and t is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is the complementary integral of $R(t)$ [7].

We denote by $P(p_k)$, the class of functions that are subordinate to $p_k(z)$. Ronning [20] proved that for $p \in P(p_k)$, there exists a function $h \in \mathcal{P}$ such that $p(z) = h^\gamma(z)$ and γ is given as:

$$\gamma = \frac{2}{\pi} \arctan \left(\frac{1}{k}\right) \tag{1.4}$$

It was also shown in [9] that for $p_k(z) = 1 + \delta_k z + \dots \in P(p_k)$,

$$\delta_k = \begin{cases} \frac{8(\arccos k)^2}{\pi^2(1-k^2)}, & 0 \leq k < 1, \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{\pi^2}{4(k^2-1)\sqrt{t}(1+t)R^2(t)}, & k > 1. \end{cases} \tag{1.5}$$

Very recently, K.I. Noor [18] extended the domain Ω_k to that of Janowski type, $\Omega_k[A, B]$, $-1 \leq B < A \leq 1$ and defined it as

$$\Omega_k[A, B] = \left\{w : \operatorname{Re} \left(\frac{(B - 1)h(z) - (A - 1)}{(B + 1)h(z) - (A + 1)} \right) > k \left| \frac{(B - 1)h(z) - (A - 1)}{(B + 1)h(z) - (A + 1)} - 1 \right| \right\} \tag{1.6}$$

or alternatively,

$$\begin{aligned} \Omega_k[A, B] = & \{u + iv : [(B^2 - 1)(u^2 + v^2) - 2(AB - 1)u + (A^2 - 1)]^2 \\ & > k^2[(-2(B + 1)(u^2 + v^2) + 2(A + B + 2)u - 2(A + 1))^2 + 4(A - B)^2v^2]\}. \end{aligned} \tag{1.7}$$

Geometrically, the effect of $\Omega_k[A, B]$ on Ω_k was described in [18].

We denote by $k - P[A, B]$, $-1 \leq B < A \leq 1$, the class of functions that map Δ onto $\Omega_k[A, B]$. Equivalently, we say $p \in k - P[A, B]$ if and only if

$$p(z) \prec \frac{(A + 1)p_k(z) - (A - 1)}{(B + 1)p_k(z) - (B - 1)}, \quad k \geq 0, \quad -1 \leq B < A \leq 1, \tag{1.8}$$

where p_k is given by (1.3). Also, it is worthy to note that $p \in k - P[A, B] \subset \mathcal{P}(\beta_1)$ which implies that $p(z) = (1 - \beta_1)h_1(z) + \beta_1$, (see [18]) where $h_1 \in \mathcal{P}$ and β_1 is given by

$$\beta_1 = \frac{2k + 1 - A}{2k + 1 - B} \tag{1.9}$$

We extend the class $k - P[A, B]$ as follows:

Definition 1.1. Let $p(z) \in \mathcal{H}$. Then $p \in k - P_\mu[A, B]$ if and only if for $\mu \geq 2, k \geq 0, -1 \leq B < A \leq 1$, we have

$$p(z) = \frac{\mu + 2}{4}p_1(z) - \frac{\mu - 2}{4}p_2(z), \quad p_1, p_2 \in k - P[A, B]. \tag{1.10}$$

Definition 1.2. Let $p(z) \in \mathcal{H}$. Then $p \in k - P_\mu[A, B, \alpha]$ if and only if for $\mu \geq 2, k \geq 0, -1 \leq B < A \leq 1, \alpha \in [0, 1)$, we have

$$p(z) = (1 - \alpha)h(z) + \alpha, \quad h(z) \in k - P_\mu[A, B]. \tag{1.11}$$

For $k = 0, A = 1, B = -1, \alpha = 0$, we have the class P_μ introduced and studied in [22]. Also, when $\mu = 2, \alpha = 0$, we get the class $k - P[A, B]$, which was first considered by K.I. Noor in [18]. The class $k - P_\mu[1 - 2\gamma^*, -1, 0], \gamma^* \in [0, 1)$ is the same as the class $k - P_\mu(\gamma^*)$ studied in [17].

We now define the following classes of functions.

Definition 1.3. Let $f \in \mathcal{A}$. Then $f \in k - UR_\mu[A, B, \alpha]$, $k \geq 0, \mu \geq 2, \alpha \in [0, 1), -1 \leq B < A \leq 1$, if and only if

$$\frac{zf'(z)}{f(z)} \in k - P_\mu[A, B, \alpha].$$

Definition 1.4. Let $f \in \mathcal{A}$. Then $f \in k - UV_\mu[A, B, \alpha]$, $k \geq 0, \mu \geq 2, \alpha \in [0, 1), -1 \leq B < A \leq 1$, if and only if

$$\frac{(zf'(z))'}{f'(z)} \in k - P_\mu[A, B, \alpha].$$

It is obvious to note that $f \in k - UV_\mu[A, B, \alpha] \Leftrightarrow zf'(z) \in k - UR_\mu[A, B, \alpha]$. Also, $0 - UV_\mu[1, -1, 0] = \mathcal{V}_\mu$, is the class of functions of bounded boundary rotation(see [1], [22])

Definition 1.5. Let $f \in \mathcal{A}$. Then $f \in k - \mathcal{T}_\mu[A, B, C, D, \alpha]$, $k \geq 0$, $\mu \geq 2$, $\alpha \in [0, 1)$, $-1 \leq B < A \leq 1$, $-1 \leq D < C \leq 1$ if and only if there exists $g \in k - UR_\mu[C, D, \alpha]$ such that

$$\frac{zf'(z)}{g(z)} \in k - P[A, B]$$

or equivalently as

$$\frac{f'(z)}{g'(z)} \in k - P[A, B]$$

where $g \in k - UV_\mu[C, D, \alpha]$.

We note the following special cases.

- i $0 - \mathcal{T}_2[A, B, C, D, 0] = \mathcal{K}[A, B, C, D]$, is the class of functions studied by Silvia [23].
- ii $0 - \mathcal{T}_2[1, -1, 1, -1, 0] = \mathcal{K}$, is the class of close to convex functions examined by Kaplan [10].
- iii $k - \mathcal{T}_2[A, B, C, D, 0] = k - UK[A, B, C, D]$, the class considered in [12]
- iv $1 - \mathcal{T}_2[1, -1, 1, -1, 0] = UCC$, is the class of uniformly close to convex functions explored by Kumar and Ramesha [11].
- v $1 - \mathcal{T}_2[1 - 2\rho, -1, 1, -1, 0] = UCC(\rho)$ is the class of uniformly close to convex of order ρ , $-1 \leq \rho < 1$ that was taken into account in [24].
- vi $0 - \mathcal{T}_\mu[1 - 2\rho, -1, 1 - 2\rho, -1, 0] = \mathcal{T}_\mu(\rho)$, $\rho \in [0, 1)$, is the class of functions that was studied by K.I. Noor in [16]
- vii $0 - \mathcal{T}_\mu[1, -1, 1, -1, 0] = \mathcal{T}_\mu$, is the class of generalized close-to-convex functions introduced and studied in [15].

Let

$$G(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-zu)^{-b} du, \tag{1.12}$$

where $Re a > 0$, $Re(c - a) > 0$, Γ denotes the Gamma function and $G(a, b, c; z)$ is hypergeometric function.

Unless if otherwise stated, we lay down once and for all that $k \geq 0$, $\mu \geq 2$, $\alpha \in [0, 1)$, $-1 \leq B < A \leq 1$, $-1 \leq D < C \leq 1$.

2. SOME PRELIMINARY LEMMAS

We need the following lemmas.

Lemma 2.1. [1] Every function $f \in \mathcal{V}_\mu$ is a close to convex function of order $\frac{\mu}{2} - 1$.

Lemma 2.2. [13] Let $p \in P[A, B]$. Then

$$|\arg p(z)| \leq \sin^{-1} \frac{(A - B)r}{1 - AB r^2}. \tag{2.1}$$

Lemma 2.3. [5] Let $p(z) \in \mathcal{H}$ with $\text{Re} p(z) > 0, z = re^{i\theta} (0 < r < 1)$. Then

$$\int_0^{2\pi} |p(re^{i\theta})|^\lambda d\theta < C(\lambda) \frac{1}{(1-r)^{\lambda-1}}, \tag{2.2}$$

where $C(\lambda)$ is a constant and $\lambda > 1$.

Lemma 2.4. [1] Let $g \in \mathcal{V}_\mu$. Then for $\mu > 3$ there exists $s_1 \in \mathcal{S}^*(0)$ and $p \in \mathcal{P}$ such that

$$zg'(z) = s_1(z)(p(z))^{\frac{\mu}{2}-1}$$

Lemma 2.5. [19] Let $q(z)$ be analytic in Δ with $q(0) = 1$. If $\varrho \geq 1, \text{Re } c \geq 0, 0 \leq \theta_1 < \theta_2 \leq 2\pi, z = re^{i\theta}$, then

$$\int_{\theta_1}^{\theta_2} \text{Re} \left\{ q(z) + \frac{\varrho z q'(z)}{c\varrho + q(z)} \right\} d\theta > -\sigma\pi, \quad (0 < \sigma \leq 1),$$

implies

$$\int_{\theta_1}^{\theta_2} \text{Re} q(z) d\theta > -\sigma\pi.$$

3. MAIN RESULT

In this section, we present our main work.

Theorem 3.1. Let $f \in k - UV_\mu[C, D, \alpha]$. Then $f'(z) = (\phi'(z))^{1-\beta}$, where $\phi \in \mathcal{V}_\mu, \beta = \alpha + (1 - \alpha)\beta_1$ and β_1 is given by (1.9).

Proof. Let $f \in k - UV_\mu[C, D, \alpha]$. Then

$$\frac{(zf'(z))'}{f'(z)} = (1 - \alpha) \left[\frac{\mu + 2}{4} p_1(z) - \frac{\mu - 2}{4} p_2(z) \right] + \alpha, \quad p_1, p_2 \in k - P[C, D] \subset \mathcal{P}(\beta_2).$$

where $\beta_2 = \frac{2k+1-C}{2k+1-D}$. Therefore,

$$\frac{(zf'(z))'}{f'(z)} = \frac{\mu + 2}{4} h_1(z) - \frac{\mu - 2}{4} h_2(z), \quad h_1, h_2 \in \mathcal{P}(\beta),$$

where

$$\beta = (1 - \alpha)\beta_2 + \alpha = \frac{2k + 1 - C + \alpha(C - D)}{2k + 1 - D}. \tag{3.1}$$

Thus, there exist $f_1, f_2 \in C(\beta)$ such that

$$\frac{(zf'(z))'}{f'(z)} = \frac{\mu + 2}{4} \frac{(zf_1'(z))'}{f_1'(z)} - \frac{\mu - 2}{4} \frac{(zf_2'(z))'}{f_2'(z)}$$

Integrating and using the result due to Brannan [1], we have

$$\begin{aligned} f'(z) &= \frac{(f_1'(z))^{\frac{\mu+2}{4}}}{(f_2'(z))^{\frac{\mu-2}{4}}} \\ &= \varphi'(z), \varphi \in \mathcal{V}_\mu(\beta) \\ &= (\phi'(z))^{1-\beta}, \phi \in \mathcal{V}_\mu. \end{aligned}$$

□

Theorem 3.2. Let $f \in k - \mathcal{T}_\mu[A, B, C, D, \alpha]$. Then for $\mu > 2, 0 \leq \theta_1 < \theta_2 \leq 2\pi, z = re^{i\theta}$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{(zf'(z))'}{f'(z)} d\theta > -\pi \left[(1 - \alpha) \left(\frac{C - D}{2k + 1 - D} \right) \left(\frac{\mu}{2} - 1 \right) + \gamma \right],$$

where γ is given by (1.4).

Proof. Let

$$f'(z) = g'(z)h(z), \tag{3.2}$$

where $g \in k - UV_\mu[C, D, \alpha], h \in k - P[A, B]$. From Theorem 3.1, we write

$$zg'(z) = z^\beta (z\phi'(z))^{1-\beta}.$$

By logarithmic differentiation and using Lemma 2.1, it follows that

$$\begin{aligned} \int_{\theta_1}^{\theta_2} \frac{(zg'(z))'}{g'(z)} d\theta &= \beta(\theta_2 - \theta_1) + (1 - \beta) \int_{\theta_1}^{\theta_2} \frac{(z\phi'(z))'}{\phi'(z)} d\theta \\ &> -\pi(1 - \beta) \left(\frac{\mu}{2} - 1 \right). \end{aligned} \tag{3.3}$$

Since $h \in k - P[A, B]$, there exists $h_1 \in P[A, B]$ such that $h(z) = (h_1(z))^\gamma$, where γ is given by (1.4).

Therefore,

$$f'(z) = g'(z)(h_1(z))^\gamma$$

and thus,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{(re^{i\theta} f'(re^{i\theta}))'}{f'(re^{i\theta})} d\theta = \int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{(re^{i\theta} g'(re^{i\theta}))'}{g'(re^{i\theta})} d\theta + \gamma \int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{re^{i\theta} h_1'(re^{i\theta})}{h_1(re^{i\theta})} d\theta. \tag{3.4}$$

But

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{re^{i\theta} h_1'(re^{i\theta})}{h_1(re^{i\theta})} d\theta = \arg h_1(re^{i\theta_2}) - \arg h_1(re^{i\theta_1}),$$

which implies

$$\max_{h_1 \in P[A, B]} \left| \int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{re^{i\theta} h_1'(re^{i\theta})}{h_1(re^{i\theta})} d\theta \right| \leq \max_{h_1 \in P[A, B]} |\arg h_1(re^{i\theta_2})| + \max_{h_1 \in P[A, B]} |\arg h_1(re^{i\theta_1})|. \tag{3.5}$$

Using Lemma 2.2 in (3.5), we get

$$\max_{h_1 \in P[A,B]} \left| \int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{re^{i\theta} h_1'(re^{i\theta})}{h_1(re^{i\theta})} d\theta \right| \leq \pi - 2 \cos^{-1} \frac{(A-B)r}{1-ABr^2}.$$

This means that

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{re^{i\theta} h_1'(re^{i\theta})}{h_1(re^{i\theta})} d\theta \geq -\pi. \tag{3.6}$$

Using (3.3) and (3.6) in (3.4), we have the result and this completes the proof. \square

Corollary 3.1. [14] Let $f \in k - \mathcal{T}_\mu[1, -1, 1, -1, 0]$. Then for $\mu > 2, 0 \leq \theta_1 < \theta_2 \leq 2\pi, z = re^{i\theta}$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{(zf'(z))'}{f'(z)} d\theta > -\pi \left(\frac{\mu - 2}{2(k+1)} + \gamma \right).$$

Corollary 3.2. Let $f \in 1 - \mathcal{T}_\mu[1, -1, 1, -1, 0]$. Then for $\mu > 2, 0 \leq \theta_1 < \theta_2 \leq 2\pi, z = re^{i\theta}$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{(zf'(z))'}{f'(z)} d\theta > -\pi \frac{\mu}{4}.$$

Corollary 3.3. [15] Let $f \in 0 - \mathcal{T}_\mu[1, -1, 1, -1, 0]$. Then for $0 \leq \theta_1 < \theta_2 \leq 2\pi, z = re^{i\theta}$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{(zf'(z))'}{f'(z)} d\theta > -\pi \frac{\mu}{2}.$$

Using Goodman result in [3], we have the following.

Corollary 3.4. The function $f \in k - \mathcal{T}_\mu[A, B, C, D, \alpha]$ is univalent in Δ for

$$\mu < 2 \left(\frac{1-\gamma}{1-\alpha} \right) \left(\frac{2k+1-D}{C-D} \right) + 2.$$

Remark 3.1. Let $f \in k - \mathcal{T}_\mu[A, B, C, D, \alpha]$. Then setting $\frac{zf'(z)}{f(z)} = p(z)$ in Theorem 3.2 and applying Lemma 2.5, we have

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{zf'(z)}{f(z)} d\theta > -\pi \left[(1-\alpha) \left(\frac{C-D}{2k+1-D} \right) \left(\frac{\mu}{2} - 1 \right) + \gamma \right].$$

Theorem 3.3. Let $f \in k - \mathcal{T}_\mu[A, B, C, D, \alpha]$. Then

$$|\arg f'(z)| \leq (1-\beta)\mu \sin^{-1} r + \gamma \sin^{-1} \frac{(A-B)r}{1-ABr^2},$$

where γ is given by (1.4).

Proof. Let

$$f'(z) = (\phi'(z))^{1-\beta} (h(z))^\gamma, \phi \in \mathcal{V}_\mu, h \in P[A, B],$$

where β is given by (3.1). Therefore,

$$|\arg f'(z)| \leq (1 - \beta)|\arg(\phi'(z))||\arg(h(z))^\gamma|.$$

Since it is well known in [21], that for $\phi \in \mathcal{V}_\mu$, $|\arg(\phi'(z))| \leq \mu \sin^{-1} r$. Using this results and Lemma 2.2, the proof is complete. \square

Theorem 3.4. Let $p(z) = 1 + \sum_{n=1}^\infty c_n z^n \in k - P_\mu[A, B, \alpha]$. Then for $n \geq 1$,

$$|c_n| \leq \frac{\mu(1 - \alpha)(A - B)|\delta_k|}{4},$$

where δ_k is given by (1.5).

Proof. Since $p \in k - P_\mu[A, B, \alpha]$, then we write

$$p(z) = \frac{\mu + 2}{4}[(1 - \alpha)h_1(z) + \alpha] - \frac{\mu - 2}{4}[(1 - \alpha)h_2(z) + \alpha], \quad h_1, h_2 \in k - P[A, B]. \quad (3.7)$$

Let $h_1(z) = 1 + \sum_{n=1}^\infty a_n z^n$ and $h_2(z) = 1 + \sum_{n=1}^\infty b_n z^n$. Then from (3.7), we have

$$1 + \sum_{n=1}^\infty c_n z^n = 1 + \sum_{n=1}^\infty \left[\frac{\mu + 2}{4} a_n - \frac{\mu - 2}{4} b_n \right] z^n.$$

Comparing the coefficient of z^n , we obtain

$$|c_n| \leq (1 - \alpha) \left[\frac{\mu + 2}{4} |a_n| + \frac{\mu - 2}{4} |b_n| \right].$$

It has been established in [18] that for $h(z) = 1 + \sum_{n=1}^\infty d_n z^n \in k - P[A, B]$,

$$|d_n| \leq \frac{(A - B)|\delta_k|}{2}, \quad n \geq 1$$

and δ_k is given by (1.5). Using this result, it follows that

$$|c_n| \leq \frac{\mu(1 - \alpha)(A - B)|\delta_k|}{4}$$

and this completes the proof. \square

Corollary 3.5. Let $p(z) = 1 + \sum_{n=1}^\infty c_n z^n \in 1 - P_2[1, -1, 0]$. Then for $n \geq 1$,

$$|c_n| \leq \frac{8}{\pi^2}$$

Corollary 3.6. [13] Let $p(z) = 1 + \sum_{n=1}^\infty c_n z^n \in 0 - P_2[A, B, 0]$. Then for $n \geq 1$,

$$|c_n| \leq A - B$$

Corollary 3.7. [4] Let $p(z) = 1 + \sum_{n=1}^\infty c_n z^n \in 0 - P_2[1, -1, 0]$. Then for $n \geq 1$,

$$|c_n| \leq 2.$$

Theorem 3.5. Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in k - P_{\mu}[A, B, \alpha]$. Then for $z = re^{i\theta}$ ($0 < r < 1$),

$$\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \leq \frac{1 + \left[\left(\frac{\mu(1-\alpha)(A-B)|\delta_k|}{4} \right)^2 - 1 \right] r^2}{1 - r^2}.$$

Proof. Using Parseval's identity, we have

$$\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta = 1 + \sum_{n=1}^{\infty} |c_n|^2 r^{2n}.$$

Applying Theorem 3.4, we get

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta &\leq 1 + \left[\frac{\mu(1-\alpha)(A-B)|\delta_k|}{4} \right]^2 \sum_{n=1}^{\infty} r^{2n} \\ &= \frac{1 + \left[\left(\frac{\mu(1-\alpha)(A-B)|\delta_k|}{4} \right)^2 - 1 \right] r^2}{1 - r^2}. \end{aligned}$$

□

Corollary 3.8. For $p \in 0 - P_2[A, B, 0] = P[A, B]$,

$$\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \leq \frac{1 + [(A - B)^2 - 1]r^2}{1 - r^2}.$$

Corollary 3.9. [16] For $p \in 0 - P_2[1 - 2\gamma, -1, 0] = P(\gamma)$,

$$\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \leq \frac{1 + [\mu^2(1 - \gamma)^2 - 1]r^2}{1 - r^2}.$$

Theorem 3.6. [Arc Length Problem] Let $f \in k - \mathcal{T}_{\mu}[A, B, C, D, \alpha]$. Then for $\mu > \frac{2-\gamma}{1-\beta} + 2$,

$$\mathcal{L}_r(f) \leq C(\gamma, \beta, A, B) \left(\frac{1}{1-r} \right)^{(1-\beta)(\frac{\mu}{2}+1)+\gamma-1} \quad (r \rightarrow 1),$$

where $C(\gamma, \beta, A, B) \left(\frac{1}{1-r} \right)$ is a constant that only depends on γ, β, A, B , where γ and β are respectively given by (1.4) and (3.1).

Proof. For $f \in k - \mathcal{T}_{\mu}[A, B, C, D, \alpha]$, and application of the result due to Brannan [1], we have

$$zf'(z) = z^{\beta} \left(s_1(z)(p_1(z))^{\frac{\mu}{2}-1} \right)^{1-\beta} (h_1(z))^{\gamma}, \tag{3.8}$$

where $s_1 \in \mathcal{S}^*$, $p_1 \in \mathcal{P}$ and $h_1 \in P[A, B]$. Therefore, for $z = re^{i\theta}$ ($r < 1$),

$$\mathcal{L}_r(f) = \int_0^{2\pi} |zf'(z)| d\theta$$

Using (3.8), distortion theorem for starlike function $s_1(z)$, Hölder’s inequality, Corollary 3.8 and Lemma 2.3, it follows that

$$\begin{aligned} \mathcal{L}_r(f) &\leq \frac{2\pi}{(1-r)^{2(1-\beta)}} \left(\frac{1}{2\pi} \int_0^{2\pi} |p_1(z)|^{\frac{(1-\beta)(\mu-2)}{2-\gamma}} d\theta \right)^{\frac{2-\gamma}{2}} \left(\frac{1}{2\pi} \int_0^{2\pi} |h_1(z)|^2 d\theta \right)^{\frac{\gamma}{2}} \\ &\leq \frac{2\pi}{(1-r)^{2(1-\beta)}} \left(\frac{1 + [(A-B)^2 - 1]r^2}{1-r^2} \right)^{\frac{\gamma}{2}} \left(\frac{1}{2\pi} \int_0^{2\pi} |p_1(z)|^{\frac{(1-\beta)(\mu-2)}{2-\gamma}} d\theta \right)^{\frac{2-\gamma}{2}} \\ &\leq C(\gamma, \beta, A, B, \alpha) \left(\frac{1}{1-r} \right)^{(1-\beta)(\frac{\mu}{2}+1)+\gamma-1} \quad (r \rightarrow 1), \end{aligned}$$

where

$$C(\gamma, \beta, A, B, \alpha) = [2\pi(A-B)^2]^{\frac{\gamma}{2}} (C(\lambda))^{\frac{2-\gamma}{2}}$$

and

$$\lambda = \frac{(1-\beta)(\mu-2)}{2-\gamma} > 1.$$

□

Theorem 3.7. Let $f \in k - \mathcal{T}_\mu[A, B, C, D, \alpha]$. Then for $\mu > \frac{2-\gamma}{1-\beta} + 2$,

$$a_n = O(1)n^{(1-\beta)(\frac{\mu}{2}+1)+\gamma-2} \quad (n \rightarrow \infty),$$

where $O(1)$ is a constant depending on γ, β, A and B , and γ, β are respectively given by (1.4) and (3.1).

Proof. By Cauchy Theorem, we have for $z = re^{i\theta}$

$$\begin{aligned} n|a_n| &\leq \frac{1}{2\pi r^n} \int_0^{2\pi} |zf'(z)| d\theta \\ &= \frac{1}{2\pi r^n} \mathcal{L}_r(f). \end{aligned}$$

Now, by applying Theorem 3.6 and setting $r = 1 - \frac{1}{n}$ as $n \rightarrow \infty$, we have the result. □

Theorem 3.8. Let $f \in k - \mathcal{T}_\mu[A, B, C, D, \alpha]$. Then for $\mu > \frac{2-\gamma}{1-\beta} + 2$,

$$||a_{n+1}| - |a_n|| \leq D(\gamma, \beta, \mu, A, B)n^{\eta-1},$$

where $\eta = \frac{2(\gamma-\beta-1)+\mu(1-\beta)-2}{2}$ and $D(\gamma, \beta, \mu, A, B)$ is a constant that depends only on γ, β, μ, A, B , and γ, β are given by (1.4) and (3.1) respectively.

Proof. For $f \in k - \mathcal{T}_\mu[A, B, C, D, \alpha]$, we write

$$zf'(z) = z^\beta \left(s_1(z)(p_1(z))^{\frac{\mu}{2}-1} \right)^{1-\beta} (h_1(z))^\gamma,$$

where $s_1 \in \mathcal{S}^*, p_1 \in \mathcal{P}$ and $h_1 \in P[A, B]$. Let z_1 be complex number with $|z_1| = r$. Then by Cauchy Theorem,

$$\left| z_1(n+1)|a_{n+1}| - n|a_n| \right| \leq \frac{1}{2\pi r^{n+1}} \int_0^{2\pi} |z_1 - z| \left| z^\beta \left(s_1(z)(p_1(z))^{\frac{\mu}{2}-1} \right)^{1-\beta} (h_1(z))^\gamma \right|.$$

Since $s_1 \in S^*$, then by a result due to Golusin [2],

$$|(z_1 - z)s_1| \leq \frac{2r^2}{1 - r^2},$$

we have

$$\left| z_1(n + 1)|a_{n+1}| - n|a_n| \right| \leq \frac{2r^{\beta+2}}{2\pi r^{n+1}(1 - r^2)} \int_0^{2\pi} \left| (s_1(z))^{-\beta} (p_1(z))^{\left(\frac{\mu}{2}-1\right)(1-\beta)} (h_1(z))^\gamma \right| d\theta.$$

Now, using Distortion Theorem for starlike function $s_1(z)$, Holder’s inequality, Corollary 3.8 and Lemma 2.3, we have

$$\begin{aligned} \left| z_1(n + 1)|a_{n+1}| - n|a_n| \right| &\leq \frac{2}{r^{n+1}(1 - r)^{1-2\beta}} \left(\frac{1}{2\pi} \int_0^{2\pi} |p_1(z)|^{\frac{(1-\beta)(\mu-2)}{2-\gamma}} d\theta \right)^{\frac{2-\gamma}{2}} \\ &\quad \times \left(\frac{1}{2\pi} \int_0^{2\pi} |h_1(z)|^2 d\theta \right)^{\frac{\gamma}{2}} \\ &\leq \frac{2}{r^{n+1}(1 - r)^{1-2\beta}} \left(\frac{1 + [(A - B)^2 - 1]r^2}{1 - r^2} \right)^{\frac{\gamma}{2}} \\ &\quad \times \left(\int_0^{2\pi} |p_1(z)|^{\frac{(1-\beta)(\mu-2)}{2-\gamma}} d\theta \right)^{\frac{2-\gamma}{2}} \\ &\leq \frac{D_1(\gamma, \beta, \mu, A, B)}{r^{n+1}} \left(\frac{1}{1 - r} \right)^{\frac{2\gamma-2\beta+\mu(1-\beta)-2}{2}}, \end{aligned}$$

where D_1 is a constant. Setting $r = 1 - \frac{1}{n}$ and $z_1 = \frac{n}{n+1}$, the proof is completed. □

Corollary 3.10. For $f \in k - \mathcal{T}_2[A, B, C, D, 0]$,

$$\left| |a_{n+1}| - |a_n| \right| \leq D_1(\gamma, \beta, A, B)n^{(\gamma-2\beta)-1},$$

where $D_1(\gamma, \beta, A, B)$ is a constant that depends on γ, β, A, B .

Corollary 3.11. [15] For $f \in 0 - \mathcal{T}_\mu[1, -1, 1, -1, 0]$,

$$\left| |a_{n+1}| - |a_n| \right| \leq D_1(\mu)n^{\frac{\mu}{2}-1},$$

where $D_1(\gamma, \beta, A, B)$ is a constant that depends only on μ .

Theorem 3.9. $f \in k - \mathcal{T}_\mu[A, B, C, D, \alpha]$. maps the disk $|z| < r_*$ onto a convex domain, where r_* is the smallest positive real number of the equation

$$1 - \lambda_1 r + \lambda_2 r^2 + \lambda_3 r^3 + \lambda_4 r^4 = 0 \tag{3.9}$$

where

$$\begin{aligned} \lambda_1 &= A + B + (1 - \beta)\mu + \gamma(A - B), \\ \lambda_2 &= 1 + AB + (1 - \beta)\mu(A + B), \\ \lambda_3 &= (2\beta - 1)(A + B) - (1 - \beta)\mu AB + \gamma(A - B), \\ \lambda_4 &= AB(1 - 2\beta), \end{aligned}$$

and γ, β are respectively given by (1.4) and (3.1).

Proof. We set

$$zf'(z) = z^\beta (z\phi'(z))^{1-\beta} (h(z))^\gamma,$$

where $\phi \in \mathcal{V}_\mu$ and $h \in P[A, B]$. By logarithmic differentiation, we have

$$Re \frac{(zf'(z))'}{f(z)} = \beta + (1 - \beta)Re \frac{(z\phi'(z))'}{\phi(z)} + \gamma Re \frac{zh'(z)}{h(z)}.$$

For $h \in P[A, B]$ and $\phi \in \mathcal{V}_\mu$, it is known in [6] and [22] respectively that

$$Re \frac{zh'(z)}{h(z)} \geq -\frac{(A - B)r}{(1 - Ar)(1 - Br)}$$

and

$$Re \frac{(z\phi'(z))'}{\phi(z)} \geq \frac{r^2 - \mu r + 1}{1 - r^2}.$$

Using these results, we have that

$$\begin{aligned} Re \frac{(zf'(z))'}{f(z)} &\geq \left[\beta(1 - r^2)(1 - Ar)(1 - Br) + (1 - \beta)(r^2 - \mu r + 1)(1 - Ar)(1 - Br) \right. \\ &\quad \left. - \gamma(A - B)r(1 - r^2) \right] / (1 - r^2)(1 - Ar)(1 - Br). \end{aligned}$$

If we let

$$T(r) = \beta(1 - r^2)(1 - Ar)(1 - Br) + (1 - \beta)(r^2 - \mu r + 1)(1 - Ar)(1 - Br) - \gamma(A - B)r(1 - r^2).$$

Then

$$T(1) = (1 - \beta)(2 - \mu)(1 - A)(1 - B) \leq 0, \quad \text{since } \mu \geq 2$$

and

$$T(0) = 1 > 0.$$

Thus, $r \in (0, 1)$. Hence, the theorem is proved. □

Corollary 3.12. *Let $f \in 0 - \mathcal{T}_2[1, -1, 1, -1, 0]$. Then $f \in C$ for*

$$|z| < 2 - \sqrt{3}.$$

Corollary 3.13. *Let $f \in 0 - \mathcal{T}_\mu[1, -1, 1, -1, 0]$. Then $f(z)$ maps the disc*

$$|z| < \frac{1}{2}[\mu + 2 - \sqrt{\mu^2 + 4\mu}]$$

onto a convex domain

Using the well-known distortion theorems for $\phi \in \mathcal{V}_\mu$ and $h_1 \in \mathcal{P}$, we prove the following.

Theorem 3.10. *Let $f \in k - \mathcal{T}_\mu[1, -1, C, D, \alpha]$. Then*

$$A_2(a, b, c, r_1) \leq |f(z)| \leq A_1(a, b, c, r_1),$$

where

$$A_1(a, b, c, r_1) = \frac{2^{b-1}}{a} [r_1^a G(a, b, c, -r_1) - G(a, b, c, -1)]$$

and

$$A_2(a, b, c, r_1) = \frac{2^{b-1}}{a} [G(a, b, c, -1) - r_1^{-a} G(a, b, c, -r_1^{-1})],$$

where

$$a = \left(\frac{\mu}{2} - 1\right)(1 - \beta)\gamma + 1,$$

$$b = 2\beta,$$

$$c = \left(\frac{\mu}{2} - 1\right)(1 - \beta)\gamma + 2,$$

and, γ and β are given by (1.4) and (3.1) respectively.

Proof. Let

$$f'(z) = (\phi'(z))^{1-\beta} (h_1(z))^\gamma, \phi \in \mathcal{V}_\mu, h_1 \in \mathcal{P}.$$

Then

$$\begin{aligned} |f(z)| &\leq \int_0^{|z|} |\phi(t)|^{1-\beta} |h_1(t)|^\gamma dt \\ &\leq \int_0^{|z|} \frac{(1+t)^{\left(\frac{\mu}{2}-1\right)(1-\beta)}}{(1-t)^{\left(\frac{\mu}{2}+1\right)(1-\beta)}} \frac{(1+t)^\gamma}{(1-t)^\gamma} dt \\ &= \int_0^{|z|} \left(\frac{1+t}{1-t}\right)^{\left(\frac{\mu}{2}+1\right)(1-\beta)} \frac{(1+t)^\gamma}{(1-t)^{2(1-\beta)+\gamma}} dt \end{aligned}$$

Let $\xi = \frac{1+t}{1-t}$, then $1-t = \frac{2}{1+\xi}$, $dt = \frac{2}{(1+\xi)^2} d\xi$, $(1+t) = \frac{2\xi}{1+\xi}$. Therefore,

$$\begin{aligned} |f(z)| &\leq 2^{2\beta-1} \int_1^{\frac{1+|z|}{1-|z|}} \xi^{\left(\frac{\mu}{2}-1\right)(1-\beta)+\gamma} (1+\xi)^{-2\beta} d\xi \\ &= 2^{b-1} (I_1 - I_2), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
 I_2 &= \int_0^1 \xi^{a-1}(1-\xi)^{c-a-1}(1-(1\xi))^{-b}d\xi \\
 &= \frac{1}{a}G(a, b, c, -1).
 \end{aligned}
 \tag{3.11}$$

To calculate I_1 , let $\xi = r_1u$ and $r_1 = \frac{1+r}{1-r}$. Then

$$\begin{aligned}
 I_1 &= \int_0^1 (r_1u)^{(\frac{\mu}{2}-1)(1-\beta)+\gamma}(1+r_1u)^{-2\beta}r_1du \\
 &= r_1^a \int_0^1 u^{a-1}(1+r_1u)^{-b}du \\
 &= \frac{r_1^a}{a}G(a, b, c, -r_1).
 \end{aligned}
 \tag{3.12}$$

Using (3.10), (3.11) and (3.12), the upper bound is obvious.

Now, we proceed to calculate the lower bound. Let d_r denotes the radius of the largest schlicht disk centered at the origin contained in the image of $|z| < r$ under the function $f(z)$. Then there exists a point z_0 with $|z_0| = r$, such that $|f(z_0)| = d_r$. The ray from 0 to $f(z)$ lies entirely in the image of Δ and the inverse image of this ray is a curve Γ in $|z| < r$. Thus,

$$\begin{aligned}
 d_r &= |f(z_0)| = \int_{\Gamma} |f'(z)| |d_r| \\
 &= \int_0^{|z|} |\phi'(\rho)|^{1-\beta} |h_1(\rho)|^{\gamma} d\rho \\
 &\geq \int_0^{|z|} \left(\frac{1-\rho}{1+\rho}\right)^{(\frac{\mu}{2}-1)(1-\beta)} \frac{(1-\rho)^{\gamma}}{(1+\rho)^{2(1-\beta)+\gamma}} d\rho.
 \end{aligned}$$

Let $\nu = \frac{1-\rho}{1+\rho}$. Then $d\rho = \frac{-2}{(1+\nu)^2}d\nu$ and $1-\rho = \frac{2\nu}{1+\nu}$. Going through the same process as it has been done for the upper bound, we easily obtain the lower bound and this completes the proof. \square

Theorem 3.11. Let $f, g \in k - \mathcal{T}_{\mu}[A, B, C, D, \alpha]$, η_1, c_1, δ and v be positively real, $\eta_1 < 1$, $c_1 > \eta_1$, $\eta_1 = v + \delta$. Then for the function $F(z)$ defined by

$$[F(z)]^{\eta_1} = c_1 z^{\eta_1 - c_1} \int_0^z t^{(c_1 - \delta - v) - 1} (f(t))^{\delta} (g(t))^v dt,
 \tag{3.13}$$

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{zF'(z)}{F(z)} > -\pi \left[(1-\alpha) \left(\frac{C-D}{2k+1-D} \right) \left(\frac{\mu}{2} - 1 \right) + \gamma \right],$$

where $0 \leq \theta_1 < \theta_2 \leq 2\pi$.

Proof. It has been shown in [17] that the integral transformation (3.13) is a well defined analytic function in Δ . Let $H(z) = \frac{zF'(z)}{F(z)}$. Then, differentiating (3.13), we obtain

$$F(z)^{\eta_1} [(c_1 - \eta_1) + \eta_1 H(z)] = c_1 f(z)^\delta g(z)^v.$$

Differentiating logarithmically and with some simple computations, we have

$$H(z) + \frac{\frac{1}{\eta} z H'(z)}{\frac{(c_1 - \eta_1)}{\eta_1} + H(z)} = \frac{\delta}{\eta_1} \frac{z f'(z)}{f(z)} + \frac{v}{\eta_1} \frac{z g'(z)}{g(z)}.$$

Now, since $f, g \in k - \mathcal{T}_\mu[A, B, C, D, \alpha]$, then application of Remark 3.1 gives

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{ H(z) + \frac{\frac{1}{\eta} z H'(z)}{\frac{(c_1 - \eta_1)}{\eta_1} + H(z)} \right\} > -\pi \left[(1 - \alpha) \left(\frac{C - D}{2k + 1 - D} \right) \left(\frac{\mu}{2} - 1 \right) + \gamma \right].$$

On using Lemma 2.5 with $\varrho = \frac{1}{\eta}$, $c = c_1 - \eta$, we complete the proof. □

Theorem 3.12. *Let $f, g \in k - \mathcal{T}_\mu[A, B, C, D, \alpha]$ and $-1 < \rho \leq 0$, $0 < \alpha_1 \leq 1$. Then for the function $J(z)$ defined by*

$$J(z) = \left[\left(\rho + \frac{1}{\alpha_1} \right) z^{1 - \frac{1}{\alpha_1}} \int_0^z t^{\frac{1}{\alpha_1} - 2} f(t)^\rho g(t) dt \right]^{\frac{1}{1 + \rho}}, \tag{3.14}$$

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{z J'(z)}{J(z)} > -\pi \left[(1 - \alpha) \left(\frac{C - D}{2k + 1 - D} \right) \left(\frac{\mu}{2} - 1 \right) + \gamma \right],$$

where $0 \leq \theta_1 < \theta_2 \leq 2\pi$.

Proof. Set $\frac{z J'(z)}{J(z)} = G(z)$. Then differentiating (3.14) logarithmically and with some simple calculations, we get

$$\left(\frac{1}{\alpha_1} - 1 \right) + (\rho + 1)G(z) = \frac{z^{\frac{1}{\alpha_1} - 1} f(z)^\rho g(z)}{\int_0^z t^{\frac{1}{\alpha_1} - 2} f(t)^\rho g(t) dt}.$$

This implies

$$\left(\rho + \frac{1}{\alpha_1} \right) f(z)^\rho g(z) = \left(\left(\frac{1}{\alpha_1} - 1 \right) + (\rho + 1)G(z) \right) J(z)^{\rho + 1}.$$

Differentiating logarithmically once again, we have

$$H(z) + \frac{\frac{1}{1 + \rho} z H'(z)}{\frac{(\frac{1}{\alpha_1} - 1)}{1 + \rho} + H(z)} = \frac{\rho}{1 + \rho} \frac{z f'(z)}{f(z)} + \frac{1}{1 + \rho} \frac{z g'(z)}{g(z)}.$$

Applying Lemma 2.5 with $\varrho = \frac{1}{1 + \rho}$, $c = \frac{1}{\alpha_1} - 1$, it follows that

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{z J'(z)}{J(z)} > -\pi \left[(1 - \alpha) \left(\frac{C - D}{2k + 1 - D} \right) \left(\frac{\mu}{2} - 1 \right) + \gamma \right].$$

□

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