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GENERATING SETS AND A STRUCTURE OF THE WREATH PRODUCT OF GROUPS WITH NON-FAITHFUL GROUP ACTION

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Abstract. Given a permutational wreath product sequence of cyclic groups, we investigate its minimal generating set, the minimal generating set for its commutator and some properties of its commutator subgroup. We generalize the result presented in the book of J. Meldrum [11] also the results of A. Woryna [4]. The quotient group of the restricted and unrestricted wreath product by its commutator is found. The generic sets of commutator of wreath product were investigated. The structure of wreath product with non-faithful group action is investigated. We strengthen the results from the author [17, 19] and construct the minimal generating set for the wreath product of both finite and infinite cyclic groups, in addition to the direct product of such groups. We generalise the results of Meldrum J. [11] about commutator subgroup of wreath products since, as well as considering regular wreath products, we consider those which are not regular (in the sense that the active group $\mathcal{A}$ does not have to act faithfully). The commutator of such a group, its minimal generating set and the center of such products has been investigated here. The minimal generating sets for new class of wreath-cyclic geometrical groups and for the commutator of the wreath product are found.

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1. Introduction

The form of commutator presentation [11] has been given here in the form of wreath recursion [10] and additionally, its commutator width has been studied. The results about commutators's structure given in [11] were improved.

Lucchini A. [6] previously investigated a case of the generating set of $C_p^{n-1} \wr G$, where G denotes a finite n-generated group, p is a prime which does not divide the order $|G|$ and C_p denotes the cyclic group of order p. The results of Lucchini A. [6] tell us that the wreath product $C_p^{n-1} \wr G$ is also n-generated. We firstly consider the active group G which is cyclic and then generalize this wreath product for both iterated wreath products and for the direct product of iterated wreath products of cyclic groups. It should be noted that to some extent a similar question for iterated wreath product was studied was studied by Bondarenko I [3]. One of the goal of our research is to study the center and commutator subgroup of wreath product with non-faithful action of active group on the set. Also as the goal of our paper is the minimal generating set and upper bound of minimal size of the generating set of the commutator subgroup of such class of group. The structure of center and quotient group by its commutator subgroup for a of such non-regular wreath product were still not investigated.

2. Prelimenaries

Let G be a group. We denote by $d(G)$ the minimal number of generators of the group G [3, 6]. The *commutator width* of G [14], denoted $cw(G)$, is defined to be the least integer n, such that every element of G' is a product of at most n commutators if such an integer exists, and otherwise is $cw(G) = \infty$.

The estimations of the upper bound of generating set of commutator subgroup were given by [14]. The property of commutator widths for groups and elements has proven to be important and in particular, its connections with stable commutator length and bounded cohomology has become significant.

Meldrum J. [11] briefly considered one form of commutators of the wreath product $A \wr B$. In order to obtain a more detailed description of this form, we take into account the commutator width ($cw(G)$) as presented in work of Muranov A. [12].

The form of commutator presentation [11] has been given here in the form of wreath recursion [10] and additionally, its commutator width has been studied.

The subtree of X^* (or $\mathbb{T}$) which is induced by the set of vertices $\cup_{i=0}^{k} X^i$ is denoted by $X^{[k]}$ (or $\mathbb{T}_k$). Denote the restriction of the action of an automorphism $g \in AutX^*$ to the subtree $X^{[l]}$ by $g_{(v)}|_{X^{[l]}}$. It should be noted that a restriction $g_{(v)}|_{X^{[1]}}$ is called the *vertex permutation* (v.p) of g in a vertex v.

3. Minimal generating set of direct product of wreath products of cyclic groups

This work strengthens previous results by the author [17] and will additionally consider a new class of groups. This class is precisely the *wreath-cyclic* groups and will be denoted by $\Im$. Let $G \in \Im$, then this class is constructed by formula:

$$G = (\overset{n_0}{\underset{j_0=0}{\wr}} C_{k_{j_0}}) \times (\overset{n_1}{\underset{j_1=0}{\wr}} C_{k_{j_1}}) \times \cdots \times (\overset{n_l}{\underset{j_l=0}{\wr}} C_{k_{j_l}}), 1 \leq k_{j_i} < \infty, n_i < \infty,$$

where the orders of C_{i_j} are denoted by i_j.

It should be noted that at the end of this product, a semidirect product could arise with a given homomorphism ϕ, which is defined by a free action on the set $\mathbb{Z}$. In other words, one would obtain a group of the form $\left(\prod\limits_{i=1}^{k} G_i\right)^n \ltimes_\phi \mathbb{Z}$.

Note that the last group here is isomorphic to one of the fundamental orbital groups $O_f(f)$ of the Morse function f. Namely, we have $\pi_0\left(S, f\left|_{\partial M}\right.\right)$ [21].

Consider now the group $H = \overset{n}{\underset{j=1}{\wr}} C_{i_j}$, whose orders i_j for all C_{i_j} are mutually coprime for all $j > 1$ and whose number of cyclic factors in the wreath product is finite. We will call such group H *wreath-cyclic*.

Note that the multiplication rule of automorphisms g, h which are presented in the form of wreath recursion [13] $g = (g_{(1)}, g_{(2)}, \ldots, g_{(d)})\sigma_g$, $h = (h_{(1)}, h_{(2)}, \ldots, h_{(d)})\sigma_h$, is given precisely by the formula:

$$g \cdot h = (g_1 h_{\sigma_g(1)}, g_2 h_{\sigma_g(2)}, \ldots, g_d h_{\sigma_g(d)})\sigma_g\sigma_h.$$

In the general case, if an active group is not cyclic, then the cycle decomposition of an n-tuple for automorphism sections will induce the corresponding decomposition of the σ_g. If σ is v.p of automorphism g at v_{ij} and all the vertex permutations below v_{ij} are trivial, then we do not distinguish σ from the section $g_{v_{ij}}$ of g which is defined by it. That is to say, we can write $g_{v_{ij}} = \sigma = (v_{ij})g$ as proposed by Bartholdi L., Grigorchuk R. and Šuni Z. [1].

We now make use of both rooted and directed automorphisms as introduced by Bartholdi L., Grigorchuk R. and Šuni Z. [1]. Recall that we denote a truncated tree by $\mathbb{T}$.

Definition 3.1. *An automorphism of $\mathbb{T}$ is said to be rooted if all of its vertex permutations corresponding to non-empty words are trivial.*

Let $l = x_1 x_2 x_3 \cdots$ be an infinite ray in $\mathbb{T}$.

Definition 3.2. *The automorphism g of $\mathbb{T}$ is said to be directed along the infinite ray l if all vertex permutations along l and all vertex permutations corresponding to vertices whose distance to the ray l is at least two are trivial. In such case, we say that l is the spine of g (as exemplified in Figure 1).*

It should be noted that because we consider only truncated trees and truncated automorphisms here and for convenience, we will say rooted automorphism instead of truncated rooted automorphism.

We reformulate and generalize the result of A. Woryna [4] about a minimal generating set of iterated wreath product. Also we make the statement more general after this theorem.

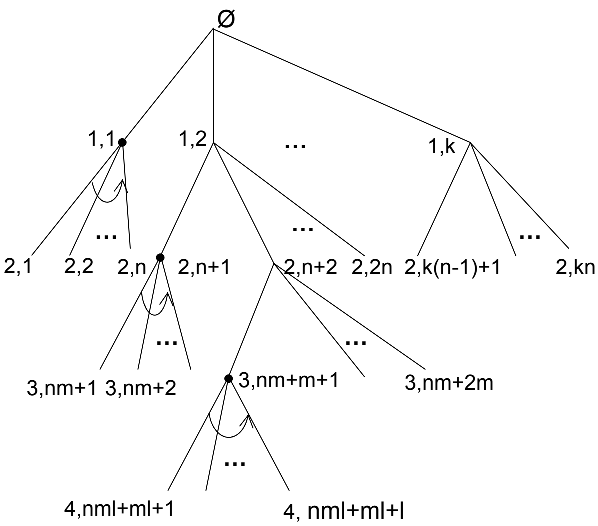

Fig. 1. Directed automorphism

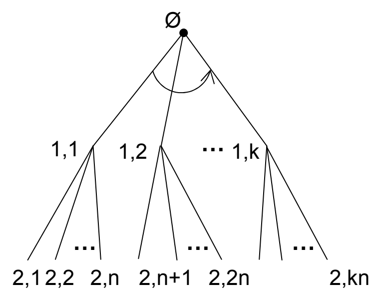

Fig. 2. Rooted automorhism

Theorem 3.1. *If orders of cyclic groups $\mathbb{C}_{n_i}$, $\mathbb{C}_{n_j}$ are mutually coprime $i \neq j$, then the group $G = C_{i_1} \wr C_{i_2} \wr \cdots \wr C_{i_m}$ admits two generators, namely β_0, β_1.*

Proof. Construct the generators of $\underset{j=0}{\overset{n}{\wr}} C_{i_j}$ as a rooted automorphism β_0 (Figure 2) and a directed automorphism β_1 [1] along a path l (Figure 1) on a rooted labeled truncated tree T_X.

We consider the group $G = C_{i_1} \wr C_{i_2} \wr \cdots \wr C_{i_m}$. Construct the generating set of $C_{i_1} \wr C_{i_2} \wr \cdots \wr C_{i_m}$, where the active group is on the left. Denote by $lcm_1 = lcm(i_2, i_3, \ldots, i_m)$ the least common multiplier of the orders by $i_2, i_3, \ldots, i_m$. In a similar fashion, we denote

$$lcm_k = lcm(i_1, i_2, \ldots, i_{k-1}, i_{k+1}, \ldots, i_m)$$

similarly.

We utilise a presentation of those wreath product elements from a tableaux of Kaloujnine L. [9] which has the form $\sigma = [a_1, a_2(x),\ a_3\,(x_1, x_2)\,, \ldots]$. Additionally, we use a subgroup of tableau with length n which has the form

$$\sigma_{(n)} = [a_1, a_2(x_1), \ldots a_n(x_1, \ldots, x_n)]\,.$$

The tableaux which has first n trivial coordinates was denoted in [20] by

$${}^{(n)}\sigma = [e, \ldots, e, \alpha_{n+1}(x_1, \ldots, x_n), \alpha_{n+1}(x_1, \ldots, x_{n+1}), \ldots]\,.$$

The canonical set of generators for the wreath product of $C_p \wr \cdots \wr C_p \wr C_p$ was used by Dmitruk Y. and Sushchanskii V. [7] and additionally utilized by the author [16]. This set has form

$$\sigma'_1 = [\pi_1,\ e,\ \ e, \ldots, e]\,, \sigma'_2 = [e_1,\ \pi_2,\ \ e, \ldots, e]\,, \ldots, \sigma'_n = [e_1,\ e, \ldots, e, \pi_n]\,. \tag{3.1}$$

We split such a tableau into sections with respect to (3.1), where the i-th section corresponds to portrait of α at i-th level. The first section corresponds to an active group and the crown of wreath product G, the second section is separated with a semicolon to a base of the wreath product. The sections of the base of wreath product are divided into parts by semicolon and these parts correspond to groups C_{i_j} which form the base of wreath product. The l-th section of of a tableau presentation of automorphism β_1 corresponds to portrait of automorphism β_1 on level X^l.

The portrait of automorphisms β_1 on level X^l is characterised by the sequence $(e, \ldots, e, \pi_l, e, \ldots, e)$, where coordinate π_l is the vertex number of unique non trivial v.p on X^l, the sequence has $i_0 i_1 \ldots i_{l-1}$ coordinates. Therefore, our first generator has the form $\beta_0 = [\pi_1, e, e, \ldots, e]$, which is the rooted automorphism. The second generator has the form

$$\beta_1 = \left[e; \underbrace{\pi_2,\ e, e, \ldots, e}_{i_1}; \overbrace{\underbrace{e, e, \ldots, e}_{i_2}, \pi_3, e, \ldots, e}^{i_1 i_2}; \underbrace{\overbrace{e, \ldots, e}^{i_2 i_3 + i_3}, \pi_4, e, \ldots, e}_{i_1 i_2 i_3};\ e, \ldots\ , e \right],$$

It should be noted that after the last (fourth) semicolon (or in other words before π_5) there are $i_2 i_3 i_4 + i_3 i_4 + i_4$ trivial coordinates. There are $i_2 i_3 i_4 i_5 + i_3 i_4 i_5 + i_4 i_5 + i_5$ trivial coordinates before π_6 (or in other words after the fifth semicolon but before π_6). In a section after $k-1$ semicolon the coordinate of a non-trivial element π_k is calculated in a similar way. We know from [20] that $\beta_1$is generator of ${}^{(2)}G$, i.e. 2-base of G. Recall that ${}^{(k)}G$ calls k-th base of G. The subgroup ${}^{(k)}G$ is a subgroup of all tableaux of form ${}^{(k)}u$ with $u \in G$.

Let $C_n = \langle \pi_n \rangle$ and set $\sigma_1 = \beta_0$. We have to show that our generating set $\{\beta_0, \beta_1\}$ generates the whole canonical generating set. For this, we obtain the second new generator σ_2 in form of the tableau

$$\sigma_2^{lcm_2} = \beta_1^{lcm_2} = \left[e; \underbrace{\pi_2^{lcm_2},\ e, e, \ldots, e}_{i_1}; \underbrace{e, e, \ldots, e}_{i_1 i_2}; \underbrace{e, e, \ldots, e}_{i_1 i_2 i_3};\ e, \ldots\ , e \right].$$

Because $\mathrm{ord}(\pi_1) = i_1$ and $(i_1,\ lcm_1) = 1$, we find that the element $\pi_1^{lcm_1}$ is generator of C_{i_1} since $\mathrm{ord}(\pi_1) = \mathrm{ord}(\pi_1^{lcm_1})$. We obtain that

$$\sigma_2 = \left(\beta_1^{lcm_2}\right)^{lcm_2^{-1} (mod\, i_2)},$$

which corresponds to generator σ_2 of canonical generating set (3.1). Observe that $b_3 = \sigma_1^{-1}\beta_1$ is generator of ${}^{(3)}G$, i.e. it is precisely a 3-base of G.

It is known [20] that the generator σ_2 precisely generates the group that is isomorphic to the group $[U]_2$ for all 2-nd coordinate tableaux. From the same principle, one can obtain that

$$\sigma_3 = \beta_1^{lcm_3} = \left[e; \underbrace{e,\ e, e, \dots, e}_{i_1}; \overbrace{\underbrace{e, e, \dots, e}_{i_2}, \pi_3^{lcm_3}, e, \dots, e}^{i_1 i_2}; \underbrace{\overbrace{e, \dots, e}^{i_2 i_3}, e, e, \dots, e}_{i_1 i_2 i_3};\ e\ \dots\ e \right].$$

This generator σ_3 generates the group which is isomorphic to the group of all $(2i_1+2)$-th coordinate tableaux, which is precisely $[U]_{2i_1+2}$ [20]. Making use of the same principle allows us to express all the σ_i from our canonical generating set.

Note that if it were a self-similar group, then it would be more useful to present it in terms of wreath recursion, as the set where β_0 is the rooted automorphism. Given a permutational representation of C_{i_j} we can present our group by wreath recursion. We present β_1 by wreath recursion as $\beta_1 = (\pi_2, \beta_2, e, e, \dots, e)$. It would be written in form $\sigma_1^{lcm_2} = {\beta_1}^{lcm_2} = ({\pi_2}^{lcm_2}, {\beta_2}^{lcm_2}, e, e, \dots, e) = ({\pi_2}^{lcm(2)}, e, e, \dots, e)$, since $\mathrm{ord}(\pi_2) = i_2$ and $(i_2,\ lcm_2) = 1$ then the element $\pi_2^{lcm_2}$ is generator of C_{i_2} too, because $\mathrm{ord}(\pi_2) = \mathrm{ord}(\pi_2^{lcm_2})$.

We then obtain the second generator σ_2 of canonical generating set by exponentiation $\left(\beta_1^{lcm_2}\right)^{lcm_2^{-1} (mod\, i_2)} = (\pi_2, e, \dots, e)$. Since we have obtained $\sigma_2 = (\pi_2, e, \dots, e)$, we can express $\sigma_2^{-1} = \left(\pi_2^{-1}, e, \dots, e\right)$, where π_2 is a state of σ_2.

Consider an alternative recursive constructed generating set which consists of nested automorphism β_1 states which are $\beta_2,\ \beta_3, \dots, \beta_m$ and the automorphism β_0. The state β_2 is expressed as follows ${\sigma_2}^{-1}\beta_1 = (e, \beta_2, e, \dots, e)$.

It should be noted that a second generator of a recursive generating set could be constructed in an other way, namely

$$\beta'_2 = {\beta_1}^{i_2} = ({\pi_2}^{i_2}, {\beta_2}^{i_2}, e, e, \dots, e) = (e, {\beta_2}^{i_2}, \dots, e, e),$$

where β_2 is the state in a vertex of the second level X^2.

We can then express the next state β_2 of β_1 by multiplying $\sigma_2^{-1}\beta_1 = (e, \beta_2, e, \dots, e)$. Therefore, by a recursive approach, we obtain $\beta_2 = (\pi_3, \beta_3, e, \dots, e)$ and analogously we obtain $\beta_2^{lcm_3} = \sigma_3^{lcm_3} = (\pi_3^{lcm_3}, e, \dots, e)$. Similarly, we obtain

$$\beta_{k-1}^{lcm_k} = \sigma_k^{lcm_k} = \left(\pi_k^{lcm_k}, e, \dots, e\right)$$

and $\sigma_k = \left(\beta_{k-1}^{lcm_k}\right)^{lcm_k^{-1} (\mathrm{mod}\ i_k)} = (\pi_k, e, \dots, e)$. The k-th generator of the recursive generating set can therefore be expressed as $\sigma_k^{-1}\beta_{k-1} = (e, \beta_k, e, \dots, e)$.

The last generator of our generating set has another structure, namely $\sigma_m = (\pi_m, e, \dots, e)$ which concludes the proof. $\square$

Let $\mathop{\wr}\limits_{j=0}^{n} C_{i_j}$ be generated by β_0 and β_1 and $\mathop{\wr}\limits_{l=0}^{m} C_{k_l} = \langle \alpha_0, \alpha_1 \rangle$. Denote an order of g by $|g|$.

Theorem 3.2. *If* $(|\alpha_0|, |\beta_0|) = 1$ *and* $(|\alpha_1|, |\beta_1|) = 1$ *or* $(|\alpha_0|, |\beta_1|) = 1$ *and* $(|\alpha_1|, |\beta_0|) = 1$, *then there exists a generating set of 2 elements for the wreath-cyclic group*

$$G = (\mathop{\wr}_{j=0}^{n} C_{i_j}) \times (\mathop{\wr}_{l=0}^{m} C_{k_l}),$$

where i_j *are orders of* C_{i_j}.

Proof. The generators α_1 and β_1 are directed automorphisms, $\alpha_0, \ \beta_0$ are rooted automorphisms [1]. The structure of tableaux are described above in Theorem 1. In case $(|\alpha_0|, |\beta_0|) = 1$ are mutually coprime and $(|\alpha_1|, |\beta_1|) = 1$ are mutually coprime, then we group generator α_0 and β_0 in vector that is first generator of direct product $(\mathop{\wr}_{j=0}^{n} C_{i_j}) \times (\mathop{\wr}_{l=0}^{m} C_{k_l})$. Therefore, the first generator of G has form $(\alpha_0, \ \beta_0)$ and the second generator has form of vector $(\beta_1, \ \alpha_1)$. The generator α_1 has a similar structure.

In order to express the generator σ_2 of the canonical set (3.1) from $\langle \alpha_0, \beta_1 \rangle$ we change the exponent from β_1 to lcm_2. Analogously, we obtain $\sigma_k = {\beta_1}^{lcm_k}$ which concludes the proof. □

4. Center and commutator subgroup of wreath product their minimal generating sets

Let us find upper bound of generators number for G'. Let $\mathcal{A}$ be a group and $\mathcal{B}$ a permutation group, i.e. a group $\mathcal{A}$ acting upon a set X, where the active group $\mathcal{A}$ can act not faithfully. Consider the set of all pairs $\{(a, f), f : X \to h, a \in \mathcal{A}\}$. We define a product on this set as

$$\{(a_1, f_1)(a_2, f_2) := (a_1 a_2, f_1 f_2^{a_1})\},$$

where $f_1^{a_2}(x) = f_1(a_2(x))$.

Theorem 4.1. *If* $W = (\mathcal{A}, X) \wr (\mathcal{B}, Y)$, *where* $|X| = n, \ |Y| = m$ *and active group* $\mathcal{A}$ *acts on* X *transitively, then*

$$d\,(W') \le (n-1)d(\mathcal{B}) + d(\mathcal{B}') + d(\mathcal{A}').$$

Proof. The generators of W' in form of tableaux [2]: ${a'}_i = (a_i; e, e, e, \dots, e)$, $t_1 = (e; h_{j_1}, e, e, \dots, c_{j_1}), \dots$, $t_k = (e; e, e, e, \dots, h_{j_k}, e, \dots, c_{j_k})$, $t_l = (e; e, e, e, \dots, h_{j_l}, c_{j_l})$, where $h_j, c_{j_l} \in S_B$, $\mathcal{B} = \langle S_B \rangle$, $a_i \in S_A$, $A = \langle S_A \rangle$. Note that, on a each coordinate of tableau, that presents a commutator of $[a; h_1, \dots, h_n]$ and $[b; g_1, \dots, g_n]$, $a, \, b \in \mathcal{A}, h_i, g_j \in \mathcal{B}$ can be product of form $a_1 a_2 a_1^{-1} a_2^{-1} \in \mathcal{A}'$ and $h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)} \in \mathcal{B}$, according to Corollary 4.9 [11]. This products should satisfy the following condition:

$$\prod_{i \in X}^{n} h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)} \in \mathcal{B}'. \tag{4.1}$$

That is to say that the product of coordinates of wreath product base is an element of commutator $\mathcal{B}'$. As it was described above it is subdirect product of $\underbrace{\mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}}_{n}$ with the additional condition (4.1). This is the case because not all element of the subdirect product are independent because the elements must be chosen in such a way that (4.1) holds. We may rearrange the factors in the product in the following way:

$$\prod_{i=1}^{n} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} = (\prod_{i=1}^{n} h_i g_i h_i^{-1} g_i^{-1})[g,h] \in \mathcal{B}'.$$

where $[g,h]$ is a commutator in case $cw(B)=1$. We express this element from $\mathcal{B}'$ as commutator $[g,h]$ if $cw(B)=1$. In the general case, we would have $\prod_{j=1}^{cw(B)}[g_j,h_j]$ instead of this element. This commutator are formed as product of commutators of rearranged elements of $\prod_{i=1}^{n} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1}$. Therefore, we have a subdirect product of n the copies of the group B which has been equipped by condition (4.1). The multiplier $\prod_{j=1}^{cw(B)}[g_j,h_j]$ from $\mathcal{B}'$, which has at least $d(\mathcal{B}')$ generators

$$\prod_{i=1}^{n} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} = (\prod_{i=1}^{n} h_i g_i h_i^{-1} g_i^{-1}) \prod_{j=1}^{cw(B)}[g_j,h_j] \in \mathcal{B}'.$$

Since $(\prod_{i=1}^{n} h_i g_i h_i^{-1} g_i^{-1}) = e$ and the product $\prod_{j=1}^{cw(B)}[g_j,h_j]$ belongs to $\mathcal{B}'$, then condition (4.1) holds. The assertion of a theorem on a recursive principle is easily generalized on multiple wreath product of groups.

Thus minimal total amount consists of at least $d(\mathcal{B}')$ generators for $n-1$ factors of group $\mathcal{B}$, $d(\mathcal{B}')$ generators for the dependent factor from $\mathcal{B}'$ and $d(\mathcal{A})$ generators of the group $\mathcal{A}$.

It should be noted that not all the elements of commutator subgroup, that has structure of the subdirect product, are independent by (4.1), at least one of them must be chosen carefully such that would be (4.1) satisfied. This implies the estimation $d(W') \le (n-1)d(B) + d(B')$.

Thus minimal total amount consists of at least $d(\mathcal{B}')$ generators for $n-1$ factors of group $\mathcal{B}$, $d(\mathcal{B}')$ generators for the dependent factor from $\mathcal{B}'$ and $d(\mathcal{A})$ generators of the group $\mathcal{A}$ which concludes the proof. □

We shall consider special case when a passive group $(\mathcal{B}, Y)$ of W is a perfect group. Since we obtain a direct product of $n-1$ the copies of the group B then according to Corollary 3.2. of Wiegold J. [22] $d(\mathcal{B}^n) \le d(\mathcal{B}) + n - 1$ [22]. More exact upper bound give us Theorem A. [22], which use s a the size of the smallest simple image of G.

Therefore, in this case our upper bound has the form

$$d(W') \le c\log_s n + d(\mathcal{B}') + d(\mathcal{A}').$$

Now we consider non-regular wreath product, where active group can be both as infinite as finite and consider a center of such group. We generalize a result of Meldrum J. [11] because we consider not only the permutation wreath product groups, but the group $\mathcal{A}$ does not have to act on the set X faithfully, hence

$(\mathcal{A}, X) \wr \mathcal{B}$ is not regular wreath product, where $\mathcal{B}$ is a passive group. Recall that an action is said to be faithful if for every $g \in G$, there exists x from G-space X such that $x^g \neq x$.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be $\mathcal{A}$-space. If an non faithfully action by conjugation determines a shift of copies of $\mathcal{B}$ from direct product $\mathcal{B}^n$ then we have not standard wreath product $(\mathcal{A}, X) \wr \mathcal{B}$ that is semidirect product of $\mathcal{A}$ and $\prod\limits_{x_i \in X} \mathbb{B}$ that is $\mathcal{A} \ltimes_{\varphi} (\mathcal{B})^n$ and the following Proposition holds. Let $\mathcal{K} = ker(\mathcal{A}, X)$ that is subgroup of $\mathcal{A}$ that acts on X as a pointwise stabilizer, that is kernel of action of $\mathcal{A}$ on X.

Denote by $Z(\tilde{\triangle}(\mathcal{B}))$ the subgroup of diagonal [5] $Fun(X, Z(B))$ of functions $f : X \to Z(B)$ which are constant on each orbit of action of A on X for unrestricted wreath product, and denote by $Z(\triangle(\mathcal{B}^n))$ the subgroup of diagonal $Fun(X, Z(\mathcal{B}^n))$ of functions with the same property for restricted wreath product, where n is number of non-trivial coordinates in base of wreath product.

Proposition 4.1. *A center of the group $(\mathcal{A}, X) \wr \mathcal{B}$ is direct product of normal closure of center of diagonal of $Z(\mathcal{B}^n)$ i.e. $(E \times Z(\triangle(\mathcal{B}^n)))$, trivial an element, and intersection of $(\mathcal{K}) \times E$ with $Z(\mathcal{A})$. In other words,*

$$Z((\mathcal{A}, X) \wr \mathcal{B}) = \langle (1;\ \underbrace{h, h, \ldots, h}_{n}),\ e,\ Z(\mathcal{K}, X) \wr \mathcal{E} \rangle \simeq \langle Z(\mathcal{A}) \cap \mathcal{K}) \times Z(\triangle(\mathcal{B}^n)) \rangle,$$

where $h \in Z(\mathcal{B})$, $|X| = n$.

For restricted wreath product with n non-trivial coordinate: $Z((\mathcal{A}, X) \wr \mathcal{B}) =$
$\langle (1;\ \ldots, h, h, \ldots, h, \ldots),\ e,\ Z(\mathcal{K}, X) \wr \mathcal{E} \rangle \simeq (Z(\mathcal{A}) \cap \mathcal{K}) \times Z(\triangle(\mathcal{B}^n)).$

In case of unrestricted wreath product we have: $Z((\mathcal{A}, X) \wr \mathcal{B}) =$
$\langle (1;\ \ldots, h_{-1}, h_0, h_1, \ldots, h_i, h_{i+1}, \ldots,),\ e,\ Z(\mathcal{K}, X) \wr \mathcal{E} \rangle \simeq (Z(\mathcal{A}) \cap \mathcal{K}) \times Z(\tilde{\triangle}(\mathcal{B})).$

Proof. The elements of center subgroup have to satisfy the condition: $f : X \to B$ such is constant on each orbit $\mathcal{O}_j$ of action $\mathcal{A}$ on X i.e. $f(x) = b_i$ for any $x \in \mathcal{O}_j$. Also every b_x: $b_x \in Z(\mathcal{B})$. Indeed the elements of form $(1;\ \underbrace{h, h, \ldots, h}_{n})$ will not be changed by action of conjugation of any element from $\mathcal{A}$ because any permutation elements coordinate of diagonal of $\mathcal{B}^n$ does not change it. Also h commutes with any element of base of $(\mathcal{A}, X) \wr \mathcal{B}$ because h from centre of $\mathcal{B}$. Since the action is defined by shift on finite set X, $|X| = n$ is not faithfully, then its kernel $\mathcal{K} \neq E$ which confirms the proposition. Also elements of subgroup $(\mathcal{A}, X) \wr \mathcal{E})$ belongs to $Z((\mathcal{A}, X) \wr \mathcal{B})$ iff it acts trivial on X. □

This is generalization of Theorem 4.2 from the book [11] because action of $\mathcal{A}$ is not faithfully.

Example 4.1. *If $\mathcal{A} = \mathbb{Z}$ then a centre $Z((\mathbb{Z}, X) \wr \mathcal{B}) =$*
$\langle (1;\ \underbrace{h, h, \ldots, h}_{n}),\ e,\ n\mathbb{Z} \ltimes \mathcal{E} : h \in Z(\triangle(\mathcal{B}^n)) \rangle$. *Since the action defined by shift on finite set X is not faithfully, and its kernel is isomorphic to $n\mathbb{Z}$ because cyclic shift on n coordinates is invariant on X.*

Generating set for commutator subgroup $(\mathbb{Z}_n \wr \mathbb{Z}_m)'$, where $\mathbb{Z}_n,\ \mathbb{Z}_m$ have presentation in additive form, is the following:

$$h_1 = (0; 1, 0, \ldots, m-1)\,,$$

$$h_2 = (0; 0, 1, 0, \ldots, m-1)\,,$$

$$\vdots$$

$$h_{n-1} = (0; 0, \ldots, 1, m-1)\,.$$

Thus, it consist of n tableaux of form $h_i = (h_{i1}, \ldots, h_{im})$ and relations for coordinate of any tableau $h_i, i \in \{1, \ldots, n-1\}$ is

$$h_{i1} + \cdots + h_{in-1} \equiv 0(mod\ m).$$

According to Theorem 3, for wreath product of abelian groups presented in multiplicative form, this relation has the form

$$\prod_{i=1}^{n} h_i f_{i_{\pi_a}} h^{-1}_{i_{\pi_a \pi_b}} f^{-1}_{i_{\pi_a \pi_b \pi_a^{-1}}} [h, f] = \prod_{i=1}^{n} (h_i f_{i_{\pi_a}} h^{-1}_{i_{\pi_a \pi_b}} f^{-1}_{i_{\pi_a \pi_b \pi_a^{-1}}} \prod_{j=i+1}^{i+2} [h_j, f_{j_{\pi_a}}]) = e.$$

Example 4.2. *If $G = \mathbb{Z}_n \wr \mathbb{Z}_m$ is standard wreath product, then $d(G') = n - 1$.*

Let $G = Z \wr_X Z$ and $G = A \wr_X B$ be a restricted wreath product, where only n non-trivial elements in coordinates of base of wreath product which are indexed by elements from X, in degenerated case $\mid X \mid= n$. Z acts on X by left shift. Also A acts transitively from left.

Remark 4.1. *The quotient group of a restricted wreath products $G = Z \wr_X Z$ by a commutator subgroup is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. In previous conditions if $G = A \wr_X B$ then, $G/G' = A/A' \times B/B'$. If $G = Z_n \wr Z_m$, where $(m,\ n) = 1$, then $d(G/G') = 1$. If $G = Z \wr Z$ is an unrestricted regular wreath product then $G/G' \simeq Z \times E \simeq Z$.*

Proof. Consider the element of $G = A \wr_X \mathcal{B}$, where A can be Z which acts on X by left shift, then elements of commutator subgroup has form:
$[e; \ldots, h_{-n}, \ldots, h_0, h_1, \ldots, h_n, \ldots,]$, where $h_i \in B$. According to Corollary 4.9 [11] the commutator of elements $h = [a; h_1, \ldots, h_n]$, $g = [b; g_1, \ldots, g_n]$, $g, h \in G$ satisfies the condition (4.1), which for case where B is abelian such: $\prod_{i=1}^{n} h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)} = e$, where g_i, h_i are non trivial coordinates from base of group, $a,\ b \in A$, $g_i, h_j \in B$. The commutator with the shifted coordinate $h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)}$ appears within the i-th coordinate position due to action of A. According to Corollary 4.9 [11] the set of elements satisfying condition (4.1) forms a commutator. Also the equivalent condition can be formulated:

$$\prod_{i=1}^{n} h_i g_i h_i^{-1} g_i^{-1} \in \mathcal{B}', \tag{4.2}$$

Therefore, if $\mathcal{B}$ is abelian an element h of G belongs to G' iff h satisfy a condition: $\prod_{i=1}^{n} h_i = e$.

For unrestricted wreath product to show that all base of wreath product is in the commutator subgroup we choose an element $[e; \ldots, h_{-1}, h_0, h_1, \ldots]$, where h_i is variable, and form a commutator which is an arbitrary element $[e; \ldots, g_{-1}, g_0, g_1, \ldots]$ of wreath product base:

$[e; \ldots, h_{-1}, h_0, h_1, \ldots][\sigma; e, e, \ldots, e][e; \ldots, h_{-1}^{-1}, h_0^{-1}, h_1^{-1}, \ldots][\sigma^{-1}; e, e, \ldots, e] =$

$= [e; \ldots, g_{-1}, g_0, g_1, \ldots]$. For convenience we present Z in additive form. Then to previous equality holds the following equations have to be satisfied: $h_0 - h_1 = g_0, h_1 - h_2 = 0, h_2 - h_3 = 0, \ldots$ it implies that $h_1 = h_0 - 1$, $h_2 = h_1$, $h_3 = h_2$, ... $h_i + 1 = h_i$. Therefore $h_i = 0$, $i \geq 1$. From other side we have $h_{-1} - h_0 = g_0, h_{-1} - h_{-2} = 0, h_{-2} - h_{-3} = 0, \ldots$ so $h_{-i} = g_0$, for all $i < 0$. That is impossible in the restricted case but possible in the unrestricted. As a corollary $G/G' \simeq Z \times Z$ for restricted case. Thus, for unrestricted case all base of G is in G' as a corollary $G/G' \simeq Z \times E$.

Thus, this group is a subdirect product of $\underbrace{B \times B \times \cdots \times B}_{n}$ with the additional condition (4.2) where, because for any element of the subgroup of coordinates there exists a surjective homomorphism acting upon B, we can conclude that G' must be a subdirect product. The commutator subgroup is the kernel of homomorphism $\varphi : G \twoheadrightarrow G/G'$. More precisely,

$$G = (Z, X) \wr (Z, Y) \twoheadrightarrow G/G' \simeq Z/Z' \times Z/Z' = \mathbb{Z} \times \mathbb{Z}.$$

In case $G = A \wr \mathcal{B}$ the $ker\varphi$ has the same structure, the homomorphism φ maps those elements of B^n, as base of G, which satisfy $\prod_{i=1}^{n} h_i = e$, i.e. the elements of B' in e of the group G/G'. Thus, $ker\varphi = G'$. To show that the properties of injectivity and surjectivity hold for this homomorphism, we chose the elements from G which have the form $[e; e, \ldots e, h, e, \ldots, e]$ that can be generator in canonical form of generating set of wreath product (3.1), where $h \notin G'$, corresponding to a a specimen from the quotient group B/B'. Also we chose independently, an element of the form $[a; e, \ldots, e, \ldots, e]$ corresponding to a specimen of the quotient group A/A'. Therefore, we must have a one-to-one correspondence between G/G' and $A/A' \times B/B'$. In this case, we obtain ${}^{G}/_{G'} \simeq \left[{}^{A}/_{A'} \times {}^{B}/_{B'}\right]$. The basic property of homomorphism for generators in canonical form (3.1) is obviously accomplished.

In the scenario when the action of Z upon the n elements from the set is isomorphic to the action of Z_n elements on the set or the action of the Z_n elements on itself. In case $G = Z \wr Z$ we have ${}^{G}/_{G'} \simeq [\mathbb{Z} \times \mathbb{Z}]$.

For the group $G = Z_n \wr Z_m$ the same is true with ${}^{G}/_{G'} \simeq [\mathbb{Z}_n \times \mathbb{Z}_m]$ and dependently of fact of $(m, n) = 1$ or not can admits one or two generators. □

Let $f : M \to R$ now be a C^{∞} Morse function. Let $\mathcal{D}(M)$ be a group of diffeomorphisms which preserve the Morse function [21] f on M (Möbius). Consider a group H of automorphisms of critical sets X_i on M which are induced by the action of diffeomorphisms h of a group $D(M)$ which preserve the Morse function f.

In other words, the h here are from the stabilizer $S(f) \triangleleft D(M)$. We note that the generators with stabilizers with the right action by diffeomorphisms $\pi_0 S(f|_{X_i,} \partial X_i)$ are τ_i. The generators of the cyclic group Z which define a shift are ρ. Since the group action is continuous, this implies that the ρ can realize only cyclic shifts, else one would change the domains of of simple connectedness X_i (critical sets) order.

The group $H \simeq \mathbb{Z} \ltimes (\mathbb{Z})^n = \langle \rho, \ \tau \rangle$ with defined above homomorphism in $Aut Z^n$ has two generators and non trivial relations [18]

$\left\langle \rho, \tau_1, \ldots, \tau_n \left| \rho \tau_{i(\bmod n)} \rho^{-1} = \tau_{i+1(\bmod n)} \right., \ \tau_i \tau_j = \tau_j \tau_i, \ i, \ j \leq n \right\rangle.$

Corollary 4.1. *A center of the group $H = \mathbb{Z} \ltimes_{\varphi} (\mathbb{Z})^n$ is a normal closure of sets: diagonal of Z^n, trivial an element and subgroup that is kernel of action by conjugation of elements of Z^n that is ($\langle \rho^{2n} \rangle \simeq 2n\mathrm{Z}$). In other words,*

$$Z(H) = \langle (1; \ \underbrace{h, h, \ldots, h}_{n}), \ e, \ 2n\mathrm{Z} \ltimes \mathrm{E} \simeq 2n\mathbb{Z} \times \mathbb{Z}. \rangle,$$

where $h, \ g \in \mathrm{Z}$.

Proof. Since the action is defined by conjugation and relation $\rho^{2n} \tau_i \rho^{-2n} = \tau_i$ holds then the element (ρ^{2n}, e) commutates with every (e, τ_i). The stabilizer of such an action over the $\mathbb{Z}$-space $X = \{x_1, x_2, \ldots, x_{2n}\}$ is the subgroup $2n\mathsf{Z}$.

So subgroup stabilize all x_i of Z-space M. Other words subgroup $\langle \rho^{2n} \rangle$ belongs to kernel of action ϕ. Besides the element $(1; \ \underbrace{h, h, \ldots, h}_{n})$ will not be changed by action of conjugation of any element from H because any permutation elements coordinate of diagonal of Z^n does not change it.

Thus, $Z(H) \simeq 2n\mathbb{Z} \times \mathbb{Z}$. □

Corollary 4.2. *The centre of a group of the form $\mathbb{Z} \ltimes_{\phi} (\mathcal{B})^n \simeq (\mathbb{Z}, X) \wr \mathcal{B}$ generates, by normal closure of: center of diagonal of $\mathcal{B}^n$, trivial an element, and $n\mathrm{Z} \underset{X}{\wr} \mathcal{E}$.*

5. Conclusion

The minimal generating set for wreath-cyclic groups have been constructed. The investigation of structure of wreath product that described in book of Meldrum [11] was generalized on case of non-faithful group action of an active group. The center of wreath product, where active group action is non-faithfully. New estimations of the upper bound of generating set of commutator subgroup was obtained.

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References

[1] Laurent Bartholdi, Rostislav I Grigorchuk, and Zoran Šuni. Branch groups. In Handbook of algebra, volume 3, Elsevier, 2003, 989–1112.

[2] Agnieszka Bier and Vitaliy Sushchansky. Kaluzhnin's representations of sylow p-subgroups of automorphism groups of p-adic rooted trees. Algebra Discr. Math., 19 (2015), 19-38.

[3] Ievgen V Bondarenko. Finite generation of iterated wreath products. Arch. Math., 95 (4) (2010), 301–308.

[4] A. Woryna, The rank and generating set for iterated wreath products of cyclic groups, Commun. Algebra, 39 (7) (2011), 2622-2631.

[5] John Dixon and Brian Mortimer. Permutation groups, Springer Science & Business Media, Volume 163, 1996.

[6] Andrea Lucchini. Generating wreath products and their augmentation ideals. Rend. Semin. Mat. Univ. Padova, 98 (1997), 67–87.

[7] Yu V Dmitruk and VI Sushchanskii. Structure of sylow 2-subgroups of the alternating groups and normalizers of sylow subgroups in the symmetric and alternating groups. Ukr. Math. J., 33 (1981), 235–241.

[8] I. Martin Isaacs. Commutators and the commutator subgroup. Amer. Math. Mon., 9 (1977), 720–722.

[9] Léo Kaloujnine. Sur les p-groupes de sylow du groupe symétrique du degré p^m. C. R. Acad. Sci., 221 (1945), 222–224.

[10] Yaroslav Lavrenyuk. On the finite state automorphism group of a rooted tree. Algebra Discr. Math., 2002 (2002), 79–87.

[11] John DP Meldrum. Wreath products of groups and semigroups, volume 74. CRC Press, 1995.

[12] Alexey Muranov. Finitely generated infinite simple groups of infinite commutator width. Int. J. Algebra Comput., 17 (2007), 607–659.

[13] Volodymyr Nekrashevych. Self-similar groups, Mathematical Surveys and Monographs, Amer. Math. Soc., New York, Volume 117, 2005.

[14] Nikolay Nikolov. On the commutator width of perfect groups. Bull. Lond. Math. Soc., 36 (2004), 30–36.

[15] Vladimir Sharko. Smooth and topological equivalence of functions on surfaces. Ukr. Math. J., 55 (2003), 832–846.

[16] Ruslan Skuratovskii. Corepresentation of a sylow p-subgroup of a group s n. Cybern. Syst. Anal., 45(1) (2009), 25–37.

[17] Ruslan Skuratovskii. Minimal generating sets for wreath products of cyclic groups, groups of automorphisms of Ribe graph and fundumental groups of some Morse functions orbits. In Algebra, Topology and Analysis (Summer School), (2016), 121–123.

[18] Ruslan Skuratovskii. Minimal Generating Set and a Structure of the Wreath Product of Groups, and the Fundamental Group of the Orbit Morse Function. arXiv:1901.00061 [math.GR], (2019), 1-14.

[19] Ruslan Skuratovskii. Minimal generating sets of cyclic groups wreath product (in russian). In International Conference, Mal'tsev meetting, (2018), 118.

[20] Vitaly Ivanovich Sushchansky. Normal structure of the isometric metric group spaces of p-adic integers. Algebraic structures and their application. Kiev, (1988), 113–121.

[21] Sergiy Maksymenko. Deformations of functions on surfaces by isotopic to the identity diffeomorphisms. arXiv:1311.3347 [math.GT], (2013).

[22] James Wiegold. Growth sequences of finite groups. J. Aust. Math. Soc., 17(2)(1974), 133–141.