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ON FIXED POINT THEOREM IN NON-ARCHIMEDEAN FUZZY NORMED SPACES

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Abstract. Let $(\mathfrak{X}, N)$ be a non-archimedean fuzzy normed space and $(\mathfrak{X}, \|.\|)$, a non-archimedean normed space where $\mathfrak{X}$ is a linear space over a linearly ordered non-archimedean field $\mathbb{K}$ with a non-archimedean valuation. We give a proof of the fixed point theorem in non-archimedean Fuzzy normed space.

1. Introduction

Definition 1.1 [9]: A *valuation* is a map $|\cdot|$ from a field $\mathbb{K}$ into a non-negative reals such that

(i) $|a| = 0$ if and only if $a = 0$
(ii) $|ab| = |a||b|$
(iii) $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{K}$ (triangle inequality).

When a field $\mathbb{K}$ carries an absolute value $|\cdot|$, it is called a valued field $(\mathbb{K}, |\cdot|)$. Examples of the pair $(\mathbb{K}, |\cdot|)$ is called a *valued field*. Examples of valuations are provided by the usual absolute values of $\mathbb{R}$ and $\mathbb{C}$. In the definition above, if the triangle inequality is replaced by a strong triangle inequality, i.e, $|a+b| \leq \max(|a|, |b|)$ for all $a, b \in K$, the map $|\,|$ is then called a *non-archimedean* or *ultrametric valuation*.

Theorem 1.2 [1]: Let be a Complete space, $0 < \lambda < 1$, and $f : X \to X$ be a map such that $\|f(x) - f(y)\| \leq \|x - y\|$ for all $x, y \in X$. Then there exists a unique point $x_\circ$ such that $f(x_\circ) = x_\circ$.

This fixed point in several cases have been obtained in non-archimedean normed and metric spaces (see [2], [4] [5], [6]). In this paper, we shall prove a version given in [4] for non-archimedean fuzzy normed spaces.

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2. Main Result

Definition 2.1 [4], [11]: Let $(\mathfrak{X}, \|.\|)$, a non-archimedean normed space. A function $N : \mathfrak{X} \times \mathbb{R} \to [0, 1]$ is called a nonarhimedean norm on $\mathfrak{X}$ if for all $x, y \in \mathfrak{X}$ and all $s, t \in \mathbb{R}$,

(i) $N(x, t) = 0$ for $t \leq 0$,

(ii) $N(x, t) = 1$ if and only if $x = 0$ for all $t > 0$,

(iii) $N(\lambda x, t) = N(x, \frac{t}{|\lambda|})$ for $\lambda \neq 0$

(iv) $N(x + y, \max\{s, t\}) \geq \min\{N(x, s), N(y, t)\}$

(v) $N(x, *)$ is nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$

Definition 2.2: Let $(\mathfrak{X}, N)$ be a non-archimedean fuzzy normed space. A closed ball in $(\mathfrak{X}, N)$ with centre a is the set of points $B(x, t) := \{N(x - a, t) \leq r, \}$ where $t \in \mathbb{R}^+$.

Definition 2.3: A sequence of n-closed balls $B(x_1, t) \subseteq B(x_2, t) \subseteq \cdots \subset B(x_n, t)$ is called a sequence of closed balls ordered by inclusion.

Definition 2.4: Let $(\mathfrak{X}, N)$ be a non-archimedean fuzzy normed space. Let $\{B(x_i, t)\}^n$ be a sequence of closed balls ordered by inclusion. Then, $(\mathfrak{X}, N)$ is said to be spherically complete if the sequence of closed balls $\{B(x, t)\}^n$ satisfies the finite intersection property in $(\mathfrak{X}, N)$. i.e., $\bigcap_{i=1}^{n}\{B(x_i, t)\} \neq 0$.

Theorem 2.5 [1] Let $\mathfrak{V}$ be a complete normed linear space, $0 < \alpha < 1$, and $\varphi : \mathfrak{V} \to \mathfrak{V}$ such that $\|\varphi(u) - \varphi(\nu)\| \leq \alpha\|u - \nu\|$ for all $u, \nu \in \mathfrak{V}$. Then, there exists a fixed point, $u_\circ \in \mathfrak{V}$ such that $\varphi(u_\circ) = u_\circ$.

A version of this theorem on non-archimedean normed space was proved in [6] as seen in the next result.

Proposition 2.6 [6]: Suppose that $\mathfrak{X}$ and $\mathfrak{Y}$ are non-archimedean normed over a non-archimedean field $\mathbb{K}$ with $|p| \neq 1$ for some $p \in \mathbb{N}$. Assume that $\mathfrak{X}$ or $\mathfrak{Y}$ is spherically complete. If $f : \mathfrak{X} \to \mathfrak{Y}$ is a surjective isometry, then for each $x \in \mathfrak{X}$, there exists a unique $y \in \mathfrak{X}$ such that $f(x) + f(y) = f\left(\frac{x + y}{p}\right)$.

We now state and prove a version of this result for the non-archimedean fuzzy normed spaces

Proposition 2.7: Let $(\mathfrak{X}, N)$ and $(\mathfrak{Y}, N)$ be non-archimedean fuzzy normed spaces over a non-archimedean field $\mathbb{K}$ with $|p| > 1$ for some $p \in \mathbb{N}$. Assume that $\mathfrak{X}$ or $\mathfrak{Y}$ is spherically complete. If $f : (\mathfrak{X}, N) \to (\mathfrak{Y}, N)$ is a surjective isometry, then for each $u \in \mathfrak{X}$, there exists a unique $\nu \in \mathfrak{X}$ such that $f(u) + f(\nu) = f\left(\frac{u + \nu}{p}\right)$.

Proof: First, we prove that $(\mathfrak{X}, N)$ or $(\mathfrak{Y}, N)$ is spherically complete. Suppose that $(\mathfrak{X}, N)$ is spherically complete and let $\{B(y_i, t)\}^n$ be a sequence od closed balls in $(\mathfrak{Y}, N)$ ordered by inclusion. Then by the surjectivity of f, there is a sequence $\{B(x_i, t)\}^n$ of closed balls in $(\mathfrak{X}, N)$ ordered by inclusion with

$$B(x_1, t) = f^{-1}(B(y_1, t)) \subseteq B(x_2, t) = f^{-1}(B(y_2, t)) \subseteq \cdots \subseteq B(x_n, t) = f^{-1}(B(y_n, t)).$$

Thus,

$$\bigcap^{n} f^{-1}(B(y_i, t)) \neq \phi \text{ as } \phi \neq \bigcap^{n}(B(x_i, t)) = f^{-1}(B(y_i, t))$$

because $(\mathfrak{X}, N)$ is spherically complete. Thus, $(\mathfrak{Y}, N)$ is spherically complete if $(\mathfrak{X}, N)$ is spherically complete.

Conversely, let $(\mathfrak{Y}, N)$ be spherically complete and $\{B(x,t)\}^n$ a sequence of closed balls in $(\mathfrak{X}, N)$ ordered by inclusion. Then there exists a sequence $\{f(B(x_i,t))\}^n$ of closed balls in $(\mathfrak{Y}, N)$ such that

$$f(B(x_1,t)) \subseteq f(B(x_2,t)) \subseteq \cdots f(B(x_n,t)).$$

Then,

$$B(x_1,t) = f(B(x_1,t)) \subseteq f(B(x_2,t)) = B(x_2,t) \subseteq \cdots f(B(x_n,t)) = B(x_n,t)$$

and

$$\bigcap^n B(x_i,t) \neq \phi \text{ as } \phi \neq \bigcap^n f(B(x_i,t)) = \bigcap^n (B(x_i,t))$$

because $(\mathfrak{Y}, N)$ is spherically complete. Thus, $(\mathfrak{X}, N)$ is spherically complete if $(\mathfrak{Y}, N)$ is. This implies that $(\mathfrak{X}, N)$ or $(\mathfrak{Y}, N)$ is spherically complete.

Next, we show that there exists a unique $\nu \in \mathfrak{X}$ such that $f(u) + f(\nu) = f\left(\frac{u+\nu}{p}\right)$ for each $u \in \mathfrak{X}$. To do this,let $u \in \mathfrak{X}$, and consider the mapping $\varphi : \mathfrak{X} \to \mathfrak{X} : x \mapsto px - u$. Now,

$$\begin{aligned} N(\varphi(x) - \varphi(y), t) &= N(px - u - (py - u), t) \\ &= N(px - u - py + u), t) \\ &= N((px - py), t) \\ &= N(x - y, \frac{t}{|p|}) \\ &< N(x - y, t). \end{aligned}$$

Thus, there exists $M > 1$ such that $M.N(\varphi(x) - \varphi(y), t) < N(x - y, t)$, i.e.,

$$N(\varphi(x) - \varphi(y), t) < \frac{1}{M} N(x - y, t).$$

Obviously, $0 < \frac{1}{M} < 1$, and $N(\varphi(x) - \varphi(y), t) \leq N(x - y, t)$ which implies that φ is a contractive mapping. Let $\psi : \mathfrak{Y} \to \mathfrak{Y}$ be an isometry defined by $\psi(y) = f(u) + y$. If $h = \varphi f^{-1} \psi f$. Then,

$$\begin{aligned} N(\varphi h(x) - h(y), t) &= N((\varphi f^{-1}\psi f)(x) - (\varphi f^{-1}\psi f)(y), t) \\ &= N(p(\varphi f^{-1}\psi f)(x) - u - (p(\varphi f^{-1}\psi f)(y)), t) \\ &= N(p(\varphi f^{-1}\psi f)(x) - u - p(\varphi f^{-1}\psi f)(y) + u, t) \\ &= N(p(\varphi f^{-1}\psi f)(x) - p(\varphi f^{-1}\psi f)(y), t) \\ &= N((\varphi f^{-1}\psi f)(x) - (\varphi f^{-1}\psi f)(y), \frac{t}{|p|}) \\ &< N((\varphi f^{-1}\psi f)(x) - (\varphi f^{-1}\psi f)(y), t) \\ &= N((\psi f)(x) - (\psi f)(y), t) \\ &= N(f(x) - f(y), t) \\ &= N(x - y, t). \end{aligned}$$

Similarly, there exists $K^* > 1$ such that $K^*.N(h(x) - h(y), t) \leq N(x - y, t)$. This also implies that

$$N(h(x) - h(y), t) \leq \frac{1}{K^*} N(x - y, t).$$

Since $\frac{1}{K^*} < 1$ and $N(h(x) - h(y), t) \leq N(x - y, t)$, then, h is a contraction mapping. By the fixed point theorem, h has a unique fixed point ν such that

$$\begin{aligned}(\psi f^{-1}\psi f)(\nu) &= h(\nu) \\ &= \nu.\end{aligned}$$

But

$$\begin{aligned}\psi\left(\frac{u+\nu}{p}\right) &= p.\frac{u+\nu}{p} - u \\ &= u + \nu - u \\ &= \nu.\end{aligned}$$

Therefore, $\psi(\nu) = \psi(f(\nu)) = \psi\left(\frac{u+\nu}{p}\right) = f\left(\frac{u+\nu}{p}\right)$, as f, and ψ are injections. Since $\psi(f(\nu)) = f(u) + f(\nu)$ by definition, it follows that $f(u) + f(\nu) = f\left(\frac{u+\nu}{p}\right)$. □

Remark 2.8: It is necessary for $|p| > 1$ for

(i) if $|p| > 1$, then

$$N(\varphi(u) - \varphi(\nu), p) = N(u - \nu, t)$$

as

$$N(u - \nu, \frac{t}{|p|}) = N(u - \nu, t).$$

Also,

$$N(h(u) - h(\nu), t) = N(u - \nu, t)$$

as

$$N((f^{-1}\psi f)(u) - (f^{-1}\psi f)(\nu), \frac{t}{|p|}) = N((f^{-1}\psi f)(u) - (f^{-1}\psi f)(\nu), t).$$

So, φ and h are not contraction mappings.

(ii) if $|p| = 0$, then $p = 0$ by definition of valuation. This negates the assumption that $p \in \mathbb{N}$.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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