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SOME INEQUALITIES FOR n-TIME DIFFERENTIABLE MAPPINGS USING A MULTI-STEP KERNEL WITH APPLICATIONS

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ABSTRACT. In this paper, we develop a new multi-step kernel and use it to establish new Ostrowski's type inequalities for n-time differentiable mappings, whose n-th derivatives satisfy convexity and quasi-convexity conditions. Applications of our findings to random variables and approximation of integrals are given.

1. INTRODUCTION

The classical Ostrowski inequality (1938, see [8]) is given as follows: if $x \in [x_1, x_2]$ then

$$\left| g(x) - \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} g(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{x_1+x_2}{2})^2}{(x_2 - x_1)^2} \right] (x_2 - x_1) \left\| g' \right\|_\infty , \tag{1.1}$$

where g is a differentiable function defined on a finite interval $[x_1, x_2]$, whose derivative is integrable and bounded over $[x_1, x_2]$. The constant $1/4$ is the best possible. When $x = \frac{x_1+x_2}{2}$, Inequality 1.1 reduces to the midpoint version

$$\left| g(\frac{x_1 + x_2}{2}) - \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} g(t)dt \right| \leq \frac{(x_2 - x_1)}{4} \left\| g' \right\|_\infty .$$

The importance of Ostrowski's type inequalities is due to their applications in different aspects. For generalizations and variants of Ostrowski's type inequalities, we refer the reader to [2] and [4].

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Recently, several authors have derived Ostrowski's type inequalities for n-time differentiable mappings whose n-th derivatives satisfy different types of convexity conditions, see for example [?], [3], [5], [7], [10] and [12].

In particular, Ozdemir and Yildiz, in [9], obtained the following three theorems for n-time differentiable mappings .

Theorem 1.1. *[9] Suppose that n is a positive integer, $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is n-time differentiable mapping on J° and $x_1,\ x_2 \in J^\circ$ with $x_1 < x_2$. If $g^{(n)} \in L_1([x_1, x_2])$ and $\left|g^{(n)}\right|$ is convex on $[x_1, x_2]$, then*

$$\left|\int_{x_1}^{x_2} g(x)\,dx - \sum_{k=0}^{n-1} \frac{\left(1+(-1)^k\right)(x_2-x_1)^{k+1}}{2^{k+1}(k+1)!} g^{(k)}\left(\frac{x_1+x_2}{2}\right)\right| \tag{1.2}$$
$$\leq \frac{(x_2-x_1)^{n+1}}{2^n(n+1)!}\left[\frac{\left|g^{(n)}(x_1)\right| + \left|g^{(n)}(x_2)\right|}{2}\right].$$

Theorem 1.2. *[9] Suppose that n is a positive integer, $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is n-time differentiable mapping on J° and $x_1,\ x_2 \in J^\circ$ with $x_1 < x_2$. If $g^{(n)} \in L_1([x_1, x_2])$ and $\left|g^{(n)}\right|^q$ is convex on $[x_1, x_2]$, where $q > 1$, then*

$$\left|\int_{x_1}^{x_2} g(x)\,dx - \sum_{k=0}^{n-1} \frac{\left(1+(-1)^k\right)(x_2-x_1)^{k+1}}{2^{k+1}(k+1)!} g^{(k)}\left(\frac{x_1+x_2}{2}\right)\right| \tag{1.3}$$
$$\leq \frac{(x_2-x_1)^{n+1}}{2^{n+1}n!}\left(\frac{1}{1+pn}\right)^{\frac{1}{p}}\left[\left(\frac{\left|g^{(n)}(x_1)\right|^q + 3\left|g^{(n)}(x_2)\right|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{3\left|g^{(n)}(x_1)\right|^q + \left|g^{(n)}(x_2)\right|^q}{4}\right],$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.3. *[9] Suppose that n is a positive integer, $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is n-time differentiable mapping on J° and $x_1,\ x_2 \in J^\circ$ with $x_1 < x_2$. If $g^{(n)} \in L_1([x_1, x_2])$ and $\left|g^{(n)}\right|^q$ is convex on $[x_1, x_2]$, where $q \geq 1$, then*

$$\left|\int_{x_1}^{x_2} g(x)\,dx - \sum_{k=0}^{n-1} \frac{\left(1+(-1)^k\right)(x_2-x_1)^{k+1}}{2^{k+1}(k+1)!} g^{(k)}\left(\frac{x_1+x_2}{2}\right)\right| \tag{1.4}$$
$$\leq \frac{(x_2-x_1)^{n+1}}{2^{n+1}(n+1)!}\left[\left(\frac{n+1}{2n+4}\left|g^{(n)}(x_1)\right|^q + \frac{n+3}{2n+4}\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{n+3}{2n+4}\left|g^{(n)}(x_1)\right|^q + \frac{n+1}{2n+4}\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}\right].$$

In this paper, we generalize the inequalities obtained by Ozdemir and Yildiz in [9] for convex and quasi convex functions using a multi-step kernel. Then we introduce some applications of our findings to random variables and approximation of integrals. Throught this paper, $\mathbb{R}$ denotes the set of all real numbers and $J \subset \mathbb{R}$ denotes an interval. The concepts of convex and quasi convex functions, which are well known in the literature, are given as in the following two definitions.

Definition 1.1. *[6] A function $g : J \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$g\left(tx_1 + (1-t)\, x_2\right) \leq tg\left(x_1\right) + (1-t)\, g\left(x_2\right),$$

holds for all $x_1,\ x_2 \in J$ and $t \in [0,1]$.

Definition 1.2. *[11] A function $g : J \subset \mathbb{R} \to \mathbb{R}$ is said to be quasi convex if the inequality*

$$g\left(tx_1 + (1-t)\, x_2\right) \leq \max\left\{g\left(x_1\right), g\left(x_2\right)\right\}$$

holds for all $x_1,\ x_2 \in J$ and $t \in [0,1]$.

2. Main Results

We start this section with the following two Lemmas.

Lemma 2.1. *Suppose that n and m are positive integers, $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is n-time differentiable mapping on J° and $x_1,\ x_2 \in J^\circ$ with $x_1 < x_2$. Suppose that $g^{(n)} \in L_1\left([x_1, x_2]\right)$. Then for every $s_1, s_2 \in [0,1]$ the identity*

$$\begin{aligned} & \int_{s_1}^{s_2} g\left(x_2 + t\left(x_1 - x_2\right)\right) dt \qquad (2.1) \\ = \ & \sum_{k=0}^{n-1} \frac{\left[\left(s_2 - r\right)^{k+1} g^{(k)}\left(x_2 + s_2\left(x_1 - x_2\right)\right) - \left(s_1 - r\right)^{k+1} g^{(k)}\left(x_2 + s_1\left(x_1 - x_2\right)\right)\right]}{\left(x_2 - x_1\right)^{-k}(k+1)!} \\ & + \frac{\left(x_2 - x_1\right)^n}{n!} \int_{s_1}^{s_2} (t-r)^n g^{(n)}\left(x_2 + t\left(x_1 - x_2\right)\right) dt, \end{aligned}$$

holds for each $r \in \mathbb{R}$.

Proof. Using integration by parts repeatedly, we get that

$$\begin{aligned} & \frac{1}{n!} \int_{s_1}^{s_2} (t-r)^n g^{(n)}\left(x_2 + t\left(x_1 - x_2\right)\right) dt \qquad (2.2) \\ = \ & -\sum_{k=0}^{n-1} \left.\frac{(t-r)^{k+1} g^{(k)}\left(x_2 + t\left(x_1 - x_2\right)\right)}{\left(x_2 - x_1\right)^{n-k}(k+1)!}\right|_{s_1}^{s_2} \\ & + \frac{1}{\left(x_2 - x_1\right)^n} \int_{s_1}^{s_2} g\left(x_2 + t\left(x_1 - x_2\right)\right) dt. \end{aligned}$$

Multiplying (2.2) by $\left(x_2 - x_1\right)^n$, and substituting the upper and lower integral bounds, the result follows. □

Lemma 2.2. *Suppose that n and m are positive integers, $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is n-time differentiable mapping on J° and $x_1, x_2 \in J^\circ$ with $x_1 < x_2$. If $g^{(n)} \in L_1([x_1, x_2])$ then*

$$\begin{aligned} &\int_{x_1}^{x_2} g(x)\,dx \qquad (2.3) \\ = &\sum_{l=1}^{2^{m-1}} \sum_{k=0}^{n-1} \frac{\left[1+(-1)^k\right] 2^{-m(k+1)}}{(x_2-x_1)^{-k-1}(k+1)!} g^{(k)}\left(\frac{(2l-1)x_1 + (2^m - 2l + 1)x_2}{2^m}\right) \\ &+ (x_2-x_1)^{n+1} \int_0^1 P_{n,m}(t)\, g^{(n)}(x_2 + t(x_1 - x_2))\,dt, \end{aligned}$$

where

$$P_{n,m}(t) = \frac{1}{n!}\begin{cases} t^n, & \text{if } t \in \left[0, \frac{1}{2^m}\right] \\ \left(t - 2^{1-m} l\right)^n, & \text{if } t \in \left(\frac{2l-1}{2^m}, \frac{2l+1}{2^m}\right], l = 1, 2, \cdots, 2^{m-1}-1 \\ (t-1)^n, & \text{if } t \in \left(1 - \frac{1}{2^m}, 1\right] \end{cases}.$$

Proof. Using Lemma 2.1, we have

$$\begin{aligned} &\int_0^{\frac{1}{2^m}} g(x_2 + t(x_1 - x_2))\,dt \qquad (2.4) \\ = &\sum_{k=0}^{n-1} \frac{\left[\left(\frac{1}{2^m}\right)^{k+1} g^{(k)}\left(\frac{x_1 + (2^m-1)x_2}{2^m}\right)\right]}{(x_2-x_1)^{-k}(k+1)!} \\ &+ \frac{(x_2-x_1)^n}{n!} \int_0^{\frac{1}{2^m}} P_{n,m}(t)\, g^{(n)}(x_2 + t(x_1 - x_2))\,dt, \end{aligned}$$

$$\begin{aligned} &\int_{\left(1-\frac{1}{2^m}\right)}^{1} g(x_2 + t(x_1 - x_2))\,dt \qquad (2.5) \\ = &\sum_{k=0}^{n-1} \frac{-\left(\frac{-1}{2^m}\right)^{k+1} g^{(k)}\left(\frac{(2^m-1)x_1 + x_2}{2^m}\right)}{(x_2-x_1)^{-k}(k+1)!} \\ &+ \frac{(x_2-x_1)^n}{n!} \int_{\left(1-\frac{1}{2^m}\right)}^{1} P_{n,m}(t)\, g^{(n)}(x_2 + t(x_1 - x_2))\,dt, \end{aligned}$$

and

$$\begin{aligned} &\int_{\frac{2l-1}{2^m}}^{\frac{2l+1}{2^m}} g(x_2 + t(x_1 - x_2))\,dt \qquad (2.6) \\ = &\sum_{k=0}^{n-1} \frac{\left[g^{(k)}\left(\frac{(2l+1)x_1 + (2^m-2l-1)x_2}{2^m}\right) + (-1)^k g^{(k)}\left(\frac{(2l-1)x_1 + (2^m-2l+1)x_2}{2^m}\right)\right]}{2^{m(k+1)}(x_2-x_1)^{-k}(k+1)!} \\ &+ \frac{(x_2-x_1)^n}{n!} \int_{\frac{2l-1}{2^m}}^{\frac{2l+1}{2^m}} P_{n,m}(t)\, g^{(n)}(x_2 + t(x_1 - x_2))\,dt, \end{aligned}$$

for $l = 1, 2, \cdots, 2^{m-1} - 1$. Combining (2.4), (2.5) and (2.6), the result follows. □

Theorem 2.1. *Suppose that n and m are positive integers, $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is n-time differentiable mapping on J° and x_1, $x_2 \in J^\circ$ with $x_1 < x_2$. If $g^{(n)} \in L_1([x_1, x_2])$ and $\left|g^{(n)}\right|$ is convex on $[x_1, x_2]$, then*

$$\begin{aligned} & \left| \int_{x_1}^{x_2} g(x)\,dx - \sum_{l=1}^{2^{m-1}} \sum_{k=0}^{n-1} \left[\frac{\left(1+(-1)^k\right) 2^{-m(k+1)}}{(x_2-x_1)^{-k-1}(k+1)!} \right.\right. \\ & \left.\left. \times g^{(k)}\left(\frac{(2l-1)x_1 + (2^m - 2l + 1)x_2}{2^m} \right) \right] \right| \\ \leq\; & \frac{(x_2-x_1)^{n+1}}{2^{mn}(n+1)!} \left[\frac{\left|g^{(n)}(x_1)\right| + \left|g^{(n)}(x_2)\right|}{2} \right]. \end{aligned} \tag{2.7}$$

Proof. Using convexity of $\left|g^{(n)}\right|$ on $[x_1, x_2]$, we get that

$$\begin{aligned} & \left| \int_0^1 P_{n,m}(t)\, g^{(n)}(x_2 + t(x_1 - x_2))\,dt \right| \\ \leq\; & \frac{1}{n!} \int_0^{\frac{1}{2^m}} t^n \left[(1-t)\left|g^{(n)}(x_2)\right| + t\left|g^{(n)}(x_1)\right| \right] dt \\ & + \frac{1}{n!} \sum_{l=1}^{2^{m-1}-1} \int_{\frac{2l-1}{2^m}}^{\frac{2l+1}{2^m}} \left| t - \frac{l}{2^{m-1}} \right|^n \left[(1-t)\left|g^{(n)}(x_2)\right| + t\left|g^{(n)}(x_1)\right| \right] dt \\ & + \frac{1}{n!} \int_{1-\frac{1}{2^m}}^1 (1-t)^n \left[(1-t)\left|g^{(n)}(x_2)\right| + t\left|g^{(n)}(x_1)\right| \right] dt \\ =\; & \frac{1}{2^{mn}(n+1)!} \left[\frac{\left|g^{(n)}(x_1)\right| + \left|g^{(n)}(x_2)\right|}{2} \right]. \end{aligned} \tag{2.8}$$

Using Identity 2.3 and Inequality 2.8, the result follows. □

Remark 2.1. *In Theorem 2.1,*

(1) *If $m = 1$, Inequality 2.7 reduces to Inequality 1.2.*

(2) *If $m = 2$, Inequality 2.7 reduces to*

$$\begin{aligned} & \left| \int_{x_1}^{x_2} g(x)\,dx - \sum_{k=0}^{n-1} \frac{\left(1+(-1)^k\right) 2^{-2(k+1)}}{(x_2-x_1)^{-k-1}(k+1)!} \left[g^{(k)}\left(\frac{3x_1 + x_2}{4} \right) \right.\right. \\ & \left.\left. + g^{(k)}\left(\frac{x_1 + 3x_2}{4} \right) \right] \right| \\ \leq\; & \frac{(x_2-x_1)^{n+1}}{2^{2n}(n+1)!} \left[\frac{\left|g^{(n)}(x_1)\right| + \left|g^{(n)}(x_2)\right|}{2} \right]. \end{aligned} \tag{2.9}$$

(3) *If $m = 3$, Inequality 2.7 reduces to*

$$\left|\int_{x_1}^{x_2} g(x)\,dx - \sum_{k=0}^{n-1} \frac{\left(1+(-1)^k\right)2^{-3(k+1)}}{(x_2-x_1)^{-k-1}(k+1)!}\left[g^{(k)}\left(\frac{7x_1+x_2}{8}\right)\right.\right.$$
$$\left.\left.+g^{(k)}\left(\frac{5x_1+3x_2}{8}\right)+g^{(k)}\left(\frac{3x_1+5x_2}{8}\right)+g^{(k)}\left(\frac{x_1+7x_2}{8}\right)\right]\right|$$
$$\le \frac{(x_2-x_1)^{n+1}}{2^{3n}(n+1)!}\left[\frac{\left|g^{(n)}(x_1)\right|+\left|g^{(n)}(x_2)\right|}{2}\right]. \tag{2.10}$$

Theorem 2.2. *Suppose that n and m are positive integers, $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is n-time differentiable mapping on J° and $x_1,\ x_2 \in J^\circ$ with $x_1 < x_2$. If $g^{(n)} \in L_1([x_1,x_2])$ and $\left|g^{(n)}\right|^q$ is convex on $[x_1,x_2]$, where $q > 1$, then*

$$\left|\int_{x_1}^{x_2} g(x)\,dx - \sum_{l=1}^{2^{m-1}}\sum_{k=0}^{n-1}\left[\frac{\left(1+(-1)^k\right)2^{-m(k+1)}}{(x_2-x_1)^{-k-1}(k+1)!}\right.\right.$$
$$\left.\left.\times g^{(k)}\left(\frac{(2l-1)x_1+(2^m-2l+1)x_2}{2^m}\right)\right]\right|$$
$$\le \frac{(x_2-x_1)^{n+1}}{2^{m(n+1)}n!}\left(\frac{1}{1+pn}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{1}{2^{m+1}}\left|g^{(n)}(x_1)\right|^q+\left(1-\frac{1}{2^{m+1}}\right)\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}\right.$$
$$+\left(\left(1-\frac{1}{2^{m+1}}\right)\left|g^{(n)}(x_1)\right|^q+\frac{1}{2^{m+1}}\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}$$
$$\left.+2\sum_{l=1}^{2^{m-1}-1}\left(\frac{l}{2^{m-1}}\left|g^{(n)}(x_1)\right|^q+\left(1-\frac{l}{2^{m-1}}\right)\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}\right], \tag{2.11}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Holder's inequality, we get that

$$\left|\int_0^1 P_{n,m}(t)\,g^{(n)}(x_2+t(x_1-x_2))\,dt\right| \tag{2.12}$$
$$\le \frac{1}{n!}\left(\int_0^{\frac{1}{2^m}} t^{pn}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2^m}}\left|g^{(n)}(x_2+t(x_1-x_2))\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{1}{n!}\sum_{l=1}^{2^{m-1}-1}\left(\int_{\frac{2l-1}{2^m}}^{\frac{2l+1}{2^m}}\left|t-\frac{l}{2^{m-1}}\right|^{pn}dt\right)^{\frac{1}{p}}\left(\int_{\frac{2l-1}{2^m}}^{\frac{2l+1}{2^m}}\left|g^{(n)}(x_2+t(x_1-x_2))\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{1}{n!}\left(\int_{1-\frac{1}{2^m}}^1 (1-t)^{pn}\,dt\right)^{\frac{1}{p}}\left(\int_{1-\frac{1}{2^m}}^1\left|g^{(n)}(x_2+t(x_1-x_2))\right|^q dt\right)^{\frac{1}{q}}.$$

Using convexity of $\left|g^{(n)}\right|^q$ on $[x_1, x_2]$, we find that

$$\begin{aligned}
&\left|\int_0^1 P_{n,m}(t)\, g^{(n)}(x_2 + t(x_1 - x_2))\, dt\right| \qquad (2.13)\\
\leq\ & \frac{1}{2^{m(n+1)} n!}\left(\frac{1}{1+pn}\right)^{\frac{1}{p}}\\
&\times\left[\left(\frac{1}{2^{m+1}}\left|g^{(n)}(x_1)\right|^q + \left(1 - \frac{1}{2^{m+1}}\right)\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}\right.\\
&+\left(\left(1 - \frac{1}{2^{m+1}}\right)\left|g^{(n)}(x_1)\right|^q + \frac{1}{2^{m+1}}\left|g^{(n)}(x_1)\right|^q\right)^{\frac{1}{q}}\\
&\left.+2\sum_{l=1}^{2^{m-1}-1}\left(\frac{l}{2^{m-1}}\left|g^{(n)}(x_1)\right|^q + \left(1 - \frac{l}{2^{m-1}}\right)\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Using Identity 2.3 and Inequality 2.13, the result follows. □

Remark 2.2. *In Theorem 2.2,*

(1) *If $m = 1$, Inequality 2.11 reduces to Inequality 1.3.*

(2) *If $m = 2$, Inequality 2.11 reduces to*

$$\begin{aligned}
&\left|\int_{x_1}^{x_2} g(x)\, dx - \sum_{k=0}^{n-1}\left[\frac{\left(1+(-1)^k\right)2^{-2(k+1)}}{(x_2 - x_1)^{-k-1}(k+1)!}\left(g^{(k)}\left(\frac{3x_1 + x_2}{4}\right)\right.\right.\right. \qquad (2.14)\\
&\left.\left.\left.+g^{(k)}\left(\frac{x_1 + 3x_2}{4}\right)\right)\right]\right|\\
\leq\ & \frac{(x_2 - x_1)^{n+1}}{2^{2(n+1)} n!}\left(\frac{1}{1+pn}\right)^{\frac{1}{p}}\left[2\left(\frac{\left|g^{(n)}(x_1)\right|^q + \left|g^{(n)}(x_2)\right|^q}{2}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{7\left|g^{(n)}(x_1)\right|^q + \left|g^{(n)}(x_2)\right|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{\left|g^{(n)}(x_1)\right|^q + 7\left|g^{(n)}(x_2)\right|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 2.3. *Suppose that n and m are positive integers, $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is n-time differentiable mapping on J° and $x_1, x_2 \in J^\circ$ with $x_1 < x_2$. If $g^{(n)} \in L_1([x_1, x_2])$ and $\left|g^{(n)}\right|^q$ is convex on $[x_1, x_2]$, where $q \geq 1$, then*

$$\begin{aligned}
&\left|\int_{x_1}^{x_2} g(x)\, dx - \sum_{l=1}^{2^{m-1}}\sum_{k=0}^{n-1}\left[\frac{\left(1+(-1)^k\right)2^{-m(k+1)}}{(x_2 - x_1)^{-k-1}(k+1)!}\right.\right. \qquad (2.15)\\
&\left.\left.\times g^{(k)}\left(\frac{(2l-1)x_1 + (2^m - 2l + 1)x_2}{2^m}\right)\right]\right|\\
\leq\ & \frac{(x_2 - x_1)^{n+1}}{2^{m(n+1)}(n+1)!}\\
&\times\left[\left(\frac{2^{-m}(n+1)}{(n+2)}\left|g^{(n)}(x_1)\right|^q + \left(1 - \frac{2^{-m}(n+1)}{(n+2)}\right)\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+2\sum_{l=1}^{2^{m-1}-1}\left(2^{1-m}\left|g^{(n)}(x_1)\right|^q+\left(1-2^{1-m}l\right)\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\left(1-\frac{2^{-m}(n+1)}{(n+2)}\right)\left|g^{(n)}(x_1)\right|^q+\frac{2^{-m}(n+1)}{(n+2)}\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}\Bigg].$$

Proof. Using convexity of $\left|g^{(n)}\right|^q$ on $[x_1,x_2]$, we get that

$$\left|\int_0^1 P_{n,m}(t)\,g^{(n)}(x_2+t(x_1-x_2))\,dt\right| \tag{2.16}$$
$$\leq \frac{1}{n!}\left(\int_0^{\frac{1}{2^m}}t^n dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2^m}}t^n\left|g^{(n)}(x_2+t(x_1-x_2))\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{1}{n!}\left(\int_{1-\frac{1}{2^m}}^1(1-t)^n dt\right)^{\frac{1}{p}}\left(\int_{1-\frac{1}{2^m}}^1(1-t)^n\left|g^{(n)}(x_2+t(x_1-x_2))\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{1}{n!}\sum_{l=1}^{2^{m-1}-1}\left[\left(\int_{\frac{2l-1}{2^m}}^{\frac{2l+1}{2^m}}\left|t-\frac{l}{2^{m-1}}\right|^n dt\right)^{\frac{1}{p}}\right.$$
$$\left.\times\left(\int_{\frac{2l-1}{2^m}}^{\frac{2l+1}{2^m}}\left|t-\frac{l}{2^{m-1}}\right|^n\left|g^{(n)}(x_2+t(x_1-x_2))\right|^q dt\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{2^{-m(n+1)}}{(n+1)!}\left(\frac{2^{-m}(n+1)}{(n+2)}\left|g^{(n)}(x_1)\right|^q+\left(1-\frac{2^{-m}(n+1)}{(n+2)}\right)\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\frac{2^{-m(n+1)}}{(n+1)!}\left(\left(1-\frac{2^{-m}(n+1)}{(n+2)}\right)\left|g^{(n)}(x_1)\right|^q+\frac{2^{-m}(n+1)}{(n+2)}\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\frac{2^{-m(n+1)+1}}{(n+1)!}\sum_{l=1}^{2^{m-1}-1}\left(2^{1-m}\left|g^{(n)}(x_1)\right|^q+\left(1-2^{1-m}l\right)\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}.$$

Using (2.3) and (2.16), the result follows. □

Remark 2.3. *In Theorem 2.3,*

(1) *If $m=1$, Inequality 2.15 reduces to Inequality 1.4.*

(2) *If $m=2$, Inequality 2.15 reduces to*

$$\left|\int_{x_1}^{x_2}g(x)\,dx-\sum_{k=0}^{n-1}\left[\frac{\left(1+(-1)^k\right)2^{-2(k+1)}}{(x_2-x_1)^{-k-1}(k+1)!}\left(g^{(k)}\left(\frac{3x_1+x_2}{4}\right)\right.\right.\right.$$
$$\left.\left.\left.+g^{(k)}\left(\frac{x_1+3x_2}{4}\right)\right)\right]\right| \tag{2.17}$$
$$\leq\frac{(x_2-x_1)^{n+1}}{2^{2(n+1)}(n+1)!}\left(\left(\frac{n+1}{4n+8}\right)\left|g^{(n)}(x_1)\right|^q+\left(\frac{3n+7}{4n+8}\right)\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\frac{2(x_2-x_1)^{n+1}}{2^{2(n+1)}(n+1)!}\left(\frac{\left|g^{(n)}(x_1)\right|^q+\left|g^{(n)}(x_2)\right|^q}{2}\right)^{\frac{1}{q}}$$
$$+\frac{(x_2-x_1)^{n+1}}{2^{2(n+1)}(n+1)!}\left(\left(\frac{3n+7}{4n+8}\right)\left|g^{(n)}(x_1)\right|^q+\left(\frac{n+1}{4n+8}\right)\left|g^{(n)}(x_2)\right|^q\right)^{\frac{1}{q}}.$$

Theorem 2.4. *Suppose that n and m are positive integers, $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is n-time differentiable mapping on J° and $x_1,\ x_2 \in J^\circ$ with $x_1 < x_2$. If $g^{(n)} \in L_1([x_1, x_2])$ and $\left|g^{(n)}\right|$ is quasi-convex on $[x_1, x_2]$, then*

$$\begin{aligned}
&\left|\int_{x_1}^{x_2} g(x)\,dx - \sum_{l=1}^{2^{m-1}} \sum_{k=0}^{n-1} \left[\frac{\left(1+(-1)^k\right) 2^{-m(k+1)}}{(x_2-x_1)^{-k-1}(k+1)!}\right.\right. \\
&\left.\left.\times g^{(k)}\left(\frac{(2l-1)x_1 + (2^m-2l+1)x_2}{2^m}\right)\right]\right| \\
\leq\ & \frac{(x_2-x_1)^{n+1}}{2^{mn}(n+1)!} \max\left\{\left|g^{(n)}(x_1)\right|, \left|g^{(n)}(x_2)\right|\right\}.
\end{aligned} \tag{2.18}$$

Proof. Using quasi-convexity of $\left|g^{(n)}\right|$ on $[x_1, x_2]$, we get that

$$\begin{aligned}
&\left|\int_0^1 P_{n,m}(t)\, g^{(n)}(x_2 + t(x_1-x_2))\,dt\right| \\
\leq\ & \frac{\max\left\{\left|g^{(n)}(x_1)\right|, \left|g^{(n)}(x_2)\right|\right\}}{n!}\left[\int_0^{\frac{1}{2^m}} t^n dt\right. \\
&\left. + \sum_{l=1}^{2^{m-1}-1} \int_{\frac{2l-1}{2^m}}^{\frac{2l+1}{2^m}} \left|t - \frac{l}{2^{m-1}}\right|^n dt + \int_{1-\frac{1}{2^m}}^1 (1-t)^n\,dt\right] \\
=\ & \frac{1}{2^{mn}(n+1)!} \max\left\{\left|g^{(n)}(x_1)\right|, \left|g^{(n)}(x_2)\right|\right\}.
\end{aligned} \tag{2.19}$$

Using Identity 2.3 and Inequality 2.19, the result follows. □

Theorem 2.5. *Suppose that n and m are positive integers, $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is n-time differentiable mapping on J° and $x_1,\ x_2 \in J^\circ$ with $x_1 < x_2$. If $g^{(n)} \in L_1([x_1, x_2])$ and $\left|g^{(n)}\right|^q$ is quasi-convex on $[x_1, x_2]$, where $q > 1$, then*

$$\begin{aligned}
&\left|\int_{x_1}^{x_2} g(x)\,dx - \sum_{l=1}^{2^{m-1}} \sum_{k=0}^{n-1} \left[\frac{\left(1+(-1)^k\right) 2^{-m(k+1)}}{(x_2-x_1)^{-k-1}(k+1)!}\right.\right. \\
&\left.\left.\times g^{(k)}\left(\frac{(2l-1)x_1 + (2^m-2l+1)x_2}{2^m}\right)\right]\right| \\
\leq\ & \frac{(x_2-x_1)^{n+1}}{2^{mn}n!}\left(\frac{1}{1+pn}\right)^{\frac{1}{p}} \left(\max\left\{\left|g^{(n)}(x_1)\right|^q, \left|g^{(n)}(x_2)\right|^q\right\}\right)^{\frac{1}{q}},
\end{aligned} \tag{2.20}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Holder's inequality and quasi-convexity of $\left|g^{(n)}\right|^q$ on $[x_1, x_2]$, we get that

$$\begin{aligned}
&\left|\int_0^1 P_{n,m}(t)\, g^{(n)}(x_2 + t(x_1 - x_2))\, dt\right| \qquad (2.21)\\
\leq\ & \frac{2^{-m(n+1)+1}}{n!}\left(\frac{1}{1+pn}\right)^{\frac{1}{p}}\left[\left(\max\left\{\left|g^{(n)}(x_1)\right|^q, \left|g^{(n)}(x_2)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\
&\left. + \left(2^{m-1} - 1\right)\left(\max\left\{\left|g^{(n)}(x_1)\right|^q, \left|g^{(n)}(x_2)\right|^q\right\}\right)^{\frac{1}{q}}\right]\\
=\ & \frac{1}{2^{mn} n!}\left(\frac{1}{1+pn}\right)^{\frac{1}{p}}\left(\max\left\{\left|g^{(n)}(x_1)\right|^q, \left|g^{(n)}(x_2)\right|^q\right\}\right)^{\frac{1}{q}}
\end{aligned}$$

Using (2.3) and (2.21), the result follows. □

Theorem 2.6. *Suppose that n and m are positive integers, $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is n-time differentiable mapping on J° and $x_1,\ x_2 \in J^\circ$ with $x_1 < x_2$. If $g^{(n)} \in L_1([x_1, x_2])$ and $\left|g^{(n)}\right|^q$ is quasi-convex on $[x_1, x_2]$, where $q \geq 1$, then*

$$\begin{aligned}
&\left|\int_{x_1}^{x_2} g(x)\, dx - \sum_{l=1}^{2^{m-1}}\sum_{k=0}^{n-1}\left[\frac{\left(1 + (-1)^k\right) 2^{-m(k+1)}}{(x_2 - x_1)^{-k-1}(k+1)!}\right.\right. \qquad (2.22)\\
&\left.\left.\times g^{(k)}\left(\frac{(2l-1)x_1 + (2^m - 2l + 1)x_2}{2^m}\right)\right]\right|\\
\leq\ & \frac{(x_2 - x_1)^{n+1}}{2^{mn}(n+1)!}\left(\max\left\{\left|g^{(n)}(x_1)\right|^q, \left|g^{(n)}(x_2)\right|^q\right\}\right)^{\frac{1}{q}}
\end{aligned}$$

Proof. Using quasi-convexity of $\left|g^{(n)}\right|^q$ on $[x_1, x_2]$, we get that

$$\begin{aligned}
&\left|\int_0^1 P_{n,m}(t)\, g^{(n)}(x_2 + t(x_1 - x_2))\, dt\right| \qquad (2.23)\\
\leq\ & \frac{2}{2^{mn+m}(n+1)!}\left(\max\left\{\left|g^{(n)}(x_1)\right|^q, \left|g^{(n)}(x_2)\right|^q\right\}\right)^{\frac{1}{q}}\\
+\ & \frac{2}{2^{mn+m}(n+1)!}\left(2^{m-1} - 1\right)\left(\max\left\{\left|g^{(n)}(x_1)\right|^q, \left|g^{(n)}(x_2)\right|^q\right\}\right)^{\frac{1}{q}}\\
=\ & \frac{1}{2^{mn}(n+1)!}\left(\max\left\{\left|g^{(n)}(x_1)\right|^q, \left|g^{(n)}(x_2)\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Using (2.3) and (2.23), the result follows. □

3. Some Applications

We start this section with an application to approximation of an integral. Recall that a partition D of a finite interval $[c, d]$, $c < d$, is a finite sequence of numbers $c = c_0 < c_1 < \cdots < c_n = d$.

Proposition 3.1. *Suppose that n and m are positive integers, $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is n-time differentiable mapping on J° and $c,\ d \in J^\circ$ with $c < d$. Let $D : c = c_0 < c_1 < \cdots < c_n = d$ be a partition of $[c, d]$. If*

$g^{(n)} \in L_1([c,d])$ *and* $\left|g^{(n)}\right|$ *is convex on* $[c,d]$, *then*

$$\int_c^d g(x)\,dx = A(g,D) + E(g,D), \tag{3.1}$$

where

$$\begin{aligned} &A(g,D) \\ = &\sum_{j=0}^{n-1}\sum_{l=1}^{2^{m-1}}\sum_{k=0}^{n-1}\left[\frac{\left(1+(-1)^k\right)2^{-m(k+1)}}{(c_{j+1}-c_j)^{-k-1}(k+1)!}\right. \\ &\left.\times g^{(k)}\left(\frac{(2l-1)c_j + (2^m-2l+1)c_{j+1}}{2^m}\right)\right], \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &|E(g,D)| \\ \le &\frac{1}{2^{mn+1}(n+1)!}\sum_{j=0}^{n-1}(c_{j+1}-c_j)^{n+1}\left(\left|g^{(n)}(c_j)\right| + \left|g^{(n)}(c_{j+1})\right|\right). \end{aligned} \tag{3.3}$$

Proof. For each $j = 0,1,\cdots,n-1$, applying Theorem 2.1 over the interval $[c_j, c_{j+1}]$,we have

$$\begin{aligned} &\left|\int_{c_j}^{c_{j+1}} g(x)\,dx - \sum_{l=1}^{2^{m-1}}\sum_{k=0}^{n-1}\left[\frac{\left(1+(-1)^k\right)2^{-m(k+1)}}{(c_{j+1}-c_j)^{-k-1}(k+1)!}\right.\right. \\ &\left.\left.\times g^{(k)}\left(\frac{(2l-1)c_j + (2^m-2l+1)c_{j+1}}{2^m}\right)\right]\right| \\ &\le \frac{(c_{j+1}-c_j)^{n+1}}{2^{mn}(n+1)!}\left[\frac{\left|g^{(n)}(c_j)\right| + \left|g^{(n)}(c_{j+1})\right|}{2}\right]. \end{aligned} \tag{3.4}$$

Note that

$$\begin{aligned} E(g,D) &= \int_c^d g(x)\,dx - A(g,D) \\ &= \sum_{j=0}^{n-1}\left(\left[\int_{c_j}^{c_{j+1}} g(x)\,dx - \sum_{l=1}^{2^{m-1}}\sum_{k=0}^{n-1}\frac{\left(1+(-1)^k\right)2^{-m(k+1)}}{(c_{j+1}-c_j)^{-k-1}(k+1)!}\right.\right. \\ &\left.\left.\times g^{(k)}\left(\frac{(2l-1)c_j + (2^m-2l+1)c_{j+1}}{2^m}\right)\right]\right). \end{aligned}$$

Using the triangle inequality and Inequality 3.4, we get that

$$|E(g,D)| \le \frac{1}{2^{mn+1}(n+1)!}\sum_{j=0}^{n-1}(c_{j+1}-c_j)^{n+1}\left(\left|g^{(n)}(c_j)\right| + \left|g^{(n)}(c_{j+1})\right|\right).$$

□

The second application will be devoted to random variables.

Proposition 3.2. *Let X be a random variable taking its values in the finite interval $[c,d]$, where $0<c<d$, with a probability density function $g:[c,d]\to[0,1]$. If g is n-time differentiable, $g^{(n)}\in L_1([c,d])$ and $\left|g^{(n)}\right|$ is convex on $[c,d]$, then for any positive integer m,*

$$\begin{aligned}&\left|E(X)-\left[d-\sum_{l=1}^{2^{m-1}}\sum_{k=1}^{n}\left(\frac{\left(1+(-1)^k\right)2^{-m(k+1)}}{(d-c)^{-k-1}(k+1)!}\right.\right.\right.\\&\times g^{(k-1)}\left.\left(\frac{(2l-1)c+(2^m-2l+1)d}{2^m}\right)\right)\\&\left.\left.-\frac{2^{-m+1}}{(d-c)^{-1}}\sum_{l=1}^{2^{m-1}}\Pr\left(X\le\frac{(2l-1)c+(2^m-2l+1)d}{2^m}\right)\right]\right|\\\le\;&\frac{(d-c)^{n+2}}{2^{m(n+1)}(n+2)!}\left[\frac{\left|g^{(n)}(c)\right|+\left|g^{(n)}(d)\right|}{2}\right],\end{aligned}\tag{3.5}$$

where $E(X)$ is the expectation of X.

Proof. Let $G(x)=\int_c^x g(t)\,dt$ for $x\in[c,d]$. Using integration by parts and the facts that $G'=g$ and $G(d)=1$, we get that

$$E(X)=d-\int_c^d G(t)\,dt.$$

Since $G^{(k+1)}=g^{(k)}, 0\le k\le n$, applying Theorem 2.1 on G, we have

$$\begin{aligned}&\left|\int_c^d G(t)\,dt-\sum_{l=1}^{2^{m-1}}\sum_{k=1}^{n}\left[\frac{\left(1+(-1)^k\right)2^{-m(k+1)}}{(d-c)^{-k-1}(k+1)!}\right.\right.\\&\times g^{(k-1)}\left.\left(\frac{(2l-1)c+(2^m-2l+1)d}{2^m}\right)\right]\\&\left.-\frac{2^{-m+1}}{(d-c)^{-1}}\sum_{l=1}^{2^{m-1}}\Pr\left(X\le\frac{(2l-1)c+(2^m-2l+1)d}{2^m}\right)\right|\\\le\;&\frac{(d-c)^{n+2}}{2^{m(n+1)}(n+2)!}\left[\frac{\left|g^{(n)}(c)\right|+\left|g^{(n)}(d)\right|}{2}\right].\end{aligned}\tag{3.6}$$

Thus,

$$\begin{aligned}&\left|E(X)-\left[d-\sum_{l=1}^{2^{m-1}}\sum_{k=1}^{n}\left(\frac{\left(1+(-1)^k\right)2^{-m(k+1)}}{(d-c)^{-k-1}(k+1)!}\right.\right.\right.\\&\times g^{(k-1)}\left.\left(\frac{(2l-1)c+(2^m-2l+1)d}{2^m}\right)\right)\\&\left.\left.-\frac{2^{-m+1}}{(d-c)^{-1}}\sum_{l=1}^{2^{m-1}}\Pr\left(X\le\frac{(2l-1)c+(2^m-2l+1)d}{2^m}\right)\right]\right|\\\le\;&\frac{(d-c)^{n+2}}{2^{m(n+1)}(n+2)!}\left[\frac{\left|g^{(n)}(c)\right|+\left|g^{(n)}(d)\right|}{2}\right].\end{aligned}$$

□

Remark 3.1. *In Proposition 3.2, if $m = 1$ then Identity 3.5 reduces to*

$$\begin{aligned}&\left|E(X) - d + (d-c)\Pr\left(X \le \frac{c+d}{2}\right)\right. \\ &\left. + \sum_{k=1}^{n} \frac{\left(1+(-1)^k\right)2^{-(k+1)}}{(d-c)^{-k-1}(k+1)!} g^{(k-1)}\left(\frac{c+d}{2}\right)\right| \\ \le\ & \frac{(d-c)^{n+2}}{2^{(n+1)}(n+2)!}\left[\frac{\left|g^{(n)}(c)\right| + \left|g^{(n)}(d)\right|}{2}\right]. \end{aligned} \tag{3.7}$$

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