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A NOTE ON GENERALIZED INDEXED PRODUCT SUMMABILITY

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Abstract. In the past, many researchers like Szasz, Rajgopal, Parameswaran, Ramanujan, Das, Sulaiman, have established results on products of two summability methods. In the present article, we have established a result on generalized indexed product summability which not only generalizes the result of Misra et al [2] and Paikray et al [3] but also the result of Sulaiman [7].

1. Introduction

If we look back to the history, it is found that, in 1952, Szasz [8] published some results on products of summability methods. Subsequently, Rajgopal [5] in 1954, Parameswaran [4] in 1957, Ramanujan [6] in 1958 etc. published some more results on products of summability methods. Later Das [1] in 1969 proved a result on absolute product summability. In 2008, Sulaiman [7] published a result on indexed product summability of an infinite series. The result of Sulaiman was then extended by Paikray et al.[3] in 2010 and Misra et al [2] in 2011.

Let $\sum a_n$ be an infinite series with the sum of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real

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constants such that

$$P_n = p_0 + p_1 + p_2 + ... + p_n \to \infty \text{ as } n \to \infty \ (P_{-i} = p_{-i} = 0). \tag{1.1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{n}\sum_{\nu=0}^{n} p_\nu s_\nu \tag{1.2}$$

defines the (R, p_n) transform of $\{s_n\}$ generated by $\{p_n\}$.

The series $\sum a_n$ is said to be summable $|R, p_n|_k, k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|t_n - t_{n-1}|^k < \infty. \tag{1.3}$$

Similarly, the sequence-to-sequence transformation

$$T_n = \frac{1}{n}\sum_{\nu=0}^{n} p_{n-\nu} s_\nu \tag{1.4}$$

defines the (N, p_n) transform of $\{s_n\}$ generated by $\{p_n\}$.

Let $\{\tau_n\}$ be the sequence of (N, q_n) transform of the (N, p_n) transform of $\{s_n\}$, generated by the sequence $\{q_n\}$ and $\{p_n\}$ respectively.That is

$$\tau_n = \frac{1}{Q_n}\sum_{r=0}^{n} q_{n-r}\frac{1}{P_r}\sum_{\nu=0}^{r} p_{r-\nu} s_\nu$$

Then the series $\sum a_n$ is said to be summable $|(N, q_n)(N, p_n)|_k, k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|\tau_n - \tau_{n-1}|^k < \infty, \tag{1.5}$$

and the series $\sum a_n$ is said to be summable $|(N, q_n)(N, p_n), \delta|_k, k \geq 1, 1 \geq \delta k \geq 0$ if

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1}|\tau_n - \tau_{n-1}|^k < \infty. \tag{1.6}$$

Similarly, if $\{\alpha_n\}$ is a sequence of positive numbers, then the series $\sum a_n$ is said to be summable $|(N, q_n)(N, p_n), \alpha_n|_k, k \geq 1$, if

$$\sum_{n=1}^{\infty} {\alpha_n}^{k-1}|\tau_n - \tau_{n-1}|^k < \infty, \tag{1.7}$$

and the series $\sum a_n$ is summable $|(N, q_n)(N, p_n), \alpha_n; \delta|_k, k \geq 1, 1 \geq \delta k \geq 0$, if

$$\sum_{n=1}^{\infty} {\alpha_n}^{\delta k + k - 1}|\tau_n - \tau_{n-1}|^k < \infty. \tag{1.8}$$

For, μ a real number, the series $\sum a_n$ is summable $|(N, q_n)(N, p_n), \alpha_n, \delta, \mu|_k, k \geq 1, 1 \geq \delta k \geq 0$, if

$$\sum_{n=1}^{\infty} {\alpha_n}^{\mu(\delta k + k - 1)}|\tau_n - \tau_{n-1}|^k < \infty. \tag{1.9}$$

We assume through out this paper that $Q_n = q_0+q_1+...+q_n \to \infty$ as $n \to \infty$ and $P_n = p_0+p_1+...+p_n \to \infty$ as $n \to \infty$.

2. Known Theorems

In 2008, Sulaiman [7] has proved the following theorem.

Theorem 2.1. *Let $k \geq 1$ and $\{\lambda_n\}$ be a sequence of constants. Let us define*

$$f_\nu = \sum_{r=\nu}^{n} \frac{q_r}{p_r},\ F_\nu = \sum_{r=\nu}^{n} p_r f_r \tag{2.1}$$

Let $p_n Q_n = O(P_n)$ such that

$$\sum_{n=\nu+1}^{\infty} \frac{n^{k-1} {q_n}^k}{{Q_n}^k Q_{n-1}} = O\left(\frac{(\nu q_\nu)^{k-1}}{Q_\nu^{k-1}}\right). \tag{2.2}$$

Then the sufficient condition for the implication $\sum a_n$ is summable $|R, r_n|_k \Rightarrow \sum a_n\lambda_n$ is summable $|(R, q_n)(R, p_n)|_k$ are

$$|\lambda_\nu| F_\nu = O\left(Q_\nu\right), \tag{2.3}$$

$$|\lambda_\nu| = O\left(Q_\nu\right), \tag{2.4}$$

$$p_\nu R_\nu |\lambda_\nu| = O\left(Q_\nu\right), \tag{2.5}$$

$$p_\nu q_\nu R_\nu |\lambda_\nu| = O\left(Q_\nu Q_{\nu-1} r_\nu\right), \tag{2.6}$$

$$p_n q_n R_n |\lambda_n| = O\left(P_n Q_n r_n\right), \tag{2.7}$$

$$R_{\nu-1} |\Delta\lambda_\nu| F_{\nu-1} = O\left(Q_\nu r_\nu\right), \tag{2.8}$$

and

$$R_{\nu-1} |\Delta\lambda_\nu| = O\left(Q_\nu r_\nu\right), \tag{2.9}$$

where $R_n = r_1 + r_2 + ... + r_n$.

Subsequently Paikray et al [3] generalized the above theorem by replacing the (R, p_n) summability by A summability. He proved:

Theorem 2.2. *Let $k \geq 1$ and $\{\lambda_n\}$ be a sequence of constants. Let us define*

$$f_\nu = \sum_{r=\nu}^{n} q_r a_{r\nu},\ F_\nu = \sum_{r=\nu}^{n} f_r \tag{2.10}$$

Then the sufficient condition for the implication $\sum a_n$ *is summable* $|R, r_n|_k \Rightarrow \sum a_n \lambda_n$ *is summable* $|(R, q_n)(A)|_k$ *are*

$$\sum_{n=\nu+1}^{m+1} \frac{n^{k-1} {q_n}^k}{{Q_n}^k Q_{n-1}} = O\left(\frac{1}{{\lambda_\nu}^k}\right), \tag{2.11}$$

$$\left(\sum_{r=\nu}^{n} {q_r}^{\frac{k}{k-1}}\right) = O(q_\nu), \tag{2.12}$$

$$\left(\sum_{r=\nu}^{n} a_{r,\nu}^k\right) = O\left(\nu^{k-1}\right), \tag{2.13}$$

$$R_\nu = O\left(r_\nu\right), \tag{2.14}$$

$$\frac{q_n}{Q_n} = O\left(1\right), \tag{2.15}$$

$$\frac{q_n \lambda_n a_{n,n}}{Q_{n-1}} = O\left(1\right), \tag{2.16}$$

$$\frac{\left(\Delta \lambda_\nu\right)^k}{{q_\nu}^{k-1}} = O\left(\nu^{k-1}\right), \tag{2.17}$$

$$\frac{\Delta \lambda_\nu}{\lambda_\nu} = O\left(1\right), \tag{2.18}$$

and

$$\frac{{\lambda_\nu}^k}{{q_\nu}^{k-1}} = O\left(\nu^{k-1}\right), \tag{2.19}$$

where $R_n = r_1 + r_2 + ... + r_n$.

In 2011, Misra et al [2], generalize the above theorems and proved the following theorem.

Theorem 2.3. *For the sequences of real constants* $\{p_n\}$ *and* $\{q_n\}$ *and the sequence of positive numbers* $\{\alpha_n\}$*, we define*

$$f_\nu = \sum_{i=\nu}^{n} \frac{q_{n-i} p_{i-\nu}}{P_i} \text{ and } F_\nu = \sum_{i=\nu}^{n} f_i \tag{2.20}$$

Let

$$Q_n = O\left(q_n P_n\right) \tag{2.21}$$

and

$$\sum_{n=\nu+1}^{m+1} \frac{\{f(\alpha_n)\}^k (\alpha_n)^{k-1} {q_n}^k}{{Q_n}^k Q_{n-1}} = O\left(\frac{(\nu q_\nu)^{k-1}}{Q_\nu^k}\right) \text{ as } m \to \infty. \tag{2.22}$$

Then for any sequence $\{r_n\}$ *and* $\{\lambda_n\}$*, the sufficient conditions for the implication* $\sum a_n$ *is summable* $|R, r_n|_k \Rightarrow \sum a_n\lambda_n$ *is summable* $|(N, q_n)(N, p_n), \alpha_n; f|_k, k \geq 1,$ *are*

$$|\lambda_\nu| F_\nu = O\left(Q_\nu\right), \tag{2.23}$$

$$|\lambda_n| = O\left(Q_n\right), \tag{2.24}$$

$$R_\nu F_\nu |\lambda_\nu| = O\left(Q_\nu r_\nu\right), \tag{2.25}$$

$$q_n R_n F_n |\lambda_n| = O\left(Q_n Q_{n-1} r_n\right), \tag{2.26}$$

$$R_{\nu-1} F_{\nu+1} |\Delta\lambda_\nu| = O\left(Q_\nu r_\nu\right), \tag{2.27}$$

$$R_{\nu-1} |\Delta\lambda_\nu| = O\left(Q_\nu r_\nu\right), \tag{2.28}$$

$$q_n R_n |\lambda_n| = O\left(Q_n Q_{n-1} r_n\right), \tag{2.29}$$

$$\sum_{n=1}^{\infty} n^{k-1} |t_n|^k = O(1), \tag{2.30}$$

and

$$\sum_{n=2}^{\infty} \{f(\alpha_n)\}^k (\alpha_n)^{k-1} |t_n|^k = O(1), \tag{2.31}$$

where $R_n = r_1 + r_2 + ... + r_n$.

In what follows, we established a theorem on generalized product summability of the infinite series $\sum a_n\lambda_n$ in the following form:

3. Main Theorem

Theorem 3.1. *For 'μ' a real number, the sequences of real constants* $\{p_n\}$ *and* $\{q_n\}$ *and the sequence of positive numbers* $\{\alpha_n\}$*, we define*

$$f_\nu = \sum_{i=\nu}^{n} \frac{q_{n-i} p_{i-\nu}}{P_i} \text{ and } F_\nu = \sum_{i=\nu}^{n} f_i \tag{3.1}$$

Let

$$Q_n = O\left(q_n P_n\right) \tag{3.2}$$

and

$$\sum_{n=\nu+1}^{\infty} \frac{{\alpha_n}^{\mu(k\delta+k-1)} {q_n}^k}{{Q_n}^k Q_{n-1}} = O\left(\frac{(\nu q_\nu)^{k-1}}{Q_\nu^k}\right) \text{ as } m \to \infty. \tag{3.3}$$

Then for any sequence $\{r_n\}$ *and* $\{\lambda_n\}$*, the sufficient conditions for the implication* $\sum a_n$ *is summable* $|R, r_n|_k \Rightarrow \sum a_n\lambda_n$ *is summable* $|(N, q_n)(N, p_n), \alpha_n, \delta, \mu|_k, k \geq 1,$ *are*

$$|\lambda_\nu| F_\nu = O\left(Q_\nu\right), \tag{3.4}$$

$$|\lambda_n| = O\left(Q_n\right), \tag{3.5}$$

$$R_\nu F_\nu |\lambda_\nu| = O\left(Q_\nu r_\nu\right), \tag{3.6}$$

$$q_n R_n F_n |\lambda_n| {\alpha_n}^{\mu\delta} = O\left(Q_n Q_{n-1} r_n\right), \tag{3.7}$$

$$R_{\nu-1} F_{\nu+1} |\Delta\lambda_\nu| = O\left(Q_\nu r_\nu\right), \tag{3.8}$$

$$R_{\nu-1} |\Delta\lambda_\nu| = O\left(Q_\nu r_\nu\right), \tag{3.9}$$

$$q_n R_n |\lambda_n| {\alpha_n}^{\mu\delta} = O\left(Q_n Q_{n-1} r_n\right), \tag{3.10}$$

$$\sum_{n=1}^{\infty} n^{k-1} |t_n|^k = O(1), \tag{3.11}$$

and

$$\sum_{n=2}^{\infty} (\alpha_n)^{\mu(k-1)} |t_n|^k = O(1), \tag{3.12}$$

where $R_n = r_1 + r_2 + ... + r_n$.

4. Proof of Theorem 3.1

Let $\{{t_n}'\}$ be the (R, r_n) transform of the series $\sum a_n$. Then

$${t_n}' = \frac{1}{R} \sum_{\nu=0}^{n} r_\nu s_\nu$$

$$t_n = {t_n}' - t'_{n-1} = \frac{r_n}{R_n R_{n-1}} \sum_{\nu=1}^{n} R_{\nu-1} a_\nu$$

Let $\{s_n\}$ be the sequence of partial sums of the series $\sum a_n\lambda_n$ and $\{\tau_n\}$ be the sequence of $(N, q_n)(N, p_n)$-transform of the series $\sum a_n\lambda_n$. Then

$$\begin{aligned} \tau_n &= \frac{1}{Q_n} \sum_{r=0}^{n} q_{n-r} \frac{1}{P_r} \sum_{\nu=0}^{r} p_{r-\nu} s_\nu \\ &= \frac{1}{Q_n} \sum_{\nu=0}^{n} s_\nu \sum_{r=\nu}^{n} \frac{q_{n-\nu} p_{r-\nu}}{P_r} \\ &= \frac{1}{Q_n} \sum_{\nu=0}^{n} f_\nu s_\nu \end{aligned} \tag{4.1}$$

Hence

$$
\begin{aligned}
T_n &= \tau_n - \tau_{n-1} \\
&= \frac{1}{Q_n}\sum_{\nu=0}^{n} f_\nu s_\nu - \frac{1}{Q_{n-1}}\sum_{\nu=0}^{n-1} f_\nu s_\nu \\
&= -\frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n} f_\nu s_\nu + \frac{f_n s_n}{Q_{n-1}} \\
&= -\frac{q_n}{Q_nQ_{n-1}}\sum_{r=0}^{n} f_r \sum_{\nu=0}^{r} a_\nu\lambda_\nu + \frac{f_n}{Q_{n-1}}\sum_{\nu=0}^{n} a_\nu\lambda_\nu \\
&= -\frac{q_n}{Q_nQ_{n-1}}\sum_{r=0}^{n} a_r\lambda_r \sum_{\nu=0}^{r} f_\nu + \frac{f_n}{Q_{n-1}}\sum_{\nu=0}^{n} a_\nu\lambda_\nu \qquad (4.2)\\
&= -\frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=1}^{n} R_{\nu-1}a_\nu\left(\frac{\lambda_\nu}{R_{\nu-1}}\sum_{r=\nu}^{n} f_r\right) + \frac{q_0p_0}{P_nQ_{n-1}}\sum_{\nu=1}^{n} R_{\nu-1}a_\nu\left(\frac{\lambda_\nu}{R_{\nu-1}}\right) \\
&= -\frac{q_n}{Q_nQ_{n-1}}\left[\sum_{\nu=1}^{n-1}\left(\sum_{r=1}^{\nu} R_{r-1}a_r\right)\Delta\left(\frac{\lambda_\nu}{R_{\nu-1}}\sum_{r=\nu}^{n} f_r\right) + \left(\sum_{\nu=1}^{n} R_{\nu-1}a_\nu\right)\frac{\lambda_n}{R_{n-1}}f_n\right] \\
&\quad + \frac{q_0p_0}{P_nQ_{n-1}}\left[\sum_{\nu=1}^{n-1}\left(\sum_{r=1}^{\nu} R_{r-1}a_r\right)\Delta\left(\frac{\lambda_\nu}{R_{\nu-1}}\right) + \left(\sum_{\nu=1}^{n} R_{\nu-1}a_\nu\right)\frac{\lambda_n}{R_{n-1}}\right] \\
&= -\frac{q_n}{Q_nQ_{n-1}}\left[\sum_{\nu=1}^{n-1}\left\{\lambda_\nu F_\nu t_\nu + \frac{R_{\nu-1}}{r_\nu}f_\nu\lambda_\nu t_\nu + \frac{R_{\nu-1}}{r_\nu}\left(\Delta\lambda_\nu\right)F_{\nu+1}t_\nu\right\} + \frac{R_n}{r_n}\lambda_n F_n t_n\right] \\
&\quad + \frac{q_0p_0}{P_nQ_{n-1}}\left[\sum_{\nu=1}^{n-1}\left\{\lambda_\nu t_\nu + \frac{R_{\nu-1}}{r_\nu}\left(\Delta\lambda_\nu\right)t_\nu\right\} + \frac{R_n}{r_n}\lambda_n t_n\right]
\end{aligned}
$$

$$
= \sum_{i=1}^{7} T_{n,i}, \texttt{say}. \tag{4.3}
$$

In order to prove this theorem, using (4.3) and Minokowski's inequality, it is sufficient to show that

$$
\sum_{n=1}^{\infty} {\alpha_n}^{\mu(\delta k+k-1)}|T_{n,i}|^k < \infty \texttt{ for } i = 1,2,3,4,5,6,7.
$$

On applying Holder's inequality, we have

$$
\begin{aligned}
&\sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)}|T_{n,1}|^k \\
&= \sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)}\left|\frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=1}^{n-1}\lambda_\nu F_\nu t_\nu\right|^k
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)} \frac{{q_n}^k}{{Q_n}^k Q_{n-1}} \sum_{\nu=1}^{n-1} \frac{|\lambda_\nu|^k {F_\nu}^k |t_\nu|^k}{{q_\nu}^{k-1}} \left(\frac{1}{Q_{n-1}} \sum_{\nu=1}^{n-1} q_\nu \right)^{k-1} \\
&= O(1) \sum_{\nu=1}^{m} \frac{1}{{q_\nu}^{k-1}} |\lambda_\nu|^k {F_\nu}^k |t_\nu|^k \sum_{n=\nu+1}^{m+1} \frac{{\alpha_n}^{\mu(\delta k+k-1)} {q_n}^k}{{Q_n}^k Q_{n-1}} \\
&= O(1) \sum_{\nu=1}^{m} \frac{1}{{q_\nu}^{k-1}} |\lambda_\nu|^k {F_\nu}^k |t_\nu|^k \frac{(\nu q_\nu)^{k-1}}{{Q_\nu}^k}, \text{ using (3.2)} \\
&= O(1) \sum_{\nu=1}^{m} \nu^{k-1} |t_\nu|^k \left(\frac{|\lambda_\nu| F_\nu}{Q_\nu} \right)^k \\
&= O(1) \sum_{\nu=1}^{m} \nu^{k-1} |t_\nu|^k \text{ using (3.4)} \\
&= O(1) \text{ as } m \to \infty.
\end{aligned}
$$

Next

$$
\begin{aligned}
&\sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)} |T_{n,2}|^k \\
&= \sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)} \Big| \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=1}^{n-1} \frac{R_{\nu-1}}{r_\nu} f_\nu \lambda_\nu t_\nu \Big|^k \\
&\leq \sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)} \frac{{q_n}^k}{{Q_n}^k Q_{n-1}} \sum_{\nu=1}^{n-1} \frac{{R_\nu}^k {F_\nu}^k |\lambda_\nu|^k |t_\nu|^k}{{q_\nu}^{k-1} {r_\nu}^k} \left(\frac{1}{Q_{n-1}} \sum_{\nu=1}^{n-1} q_\nu \right)^{k-1} \\
&= O(1) \sum_{\nu=1}^{m} \frac{{R_\nu}^k {F_\nu}^k |\lambda_\nu|^k |t_\nu|^k}{{q_\nu}^{k-1} {r_\nu}^k} \sum_{n=\nu+1}^{m+1} \frac{{\alpha_n}^{\mu(\delta k+k-1)} {q_n}^k}{{Q_n}^k Q_{n-1}} \\
&= O(1) \sum_{\nu=1}^{m} \nu^{k-1} |t_\nu|^k \left(\frac{R_\nu F_\nu |\lambda_\nu|}{r_\nu Q_\nu} \right)^k \\
&= O(1) \sum_{\nu=1}^{m} \nu^{k-1} |t_\nu|^k \text{ using (3.6)} \\
&= O(1) \text{ as } m \to \infty.
\end{aligned}
$$

Further

$$
\begin{aligned}
&\sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)} |T_{n,3}|^k \\
&= \sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)} \Big| \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=1}^{n-1} \frac{R_{\nu-1}}{r_\nu} F_{\nu+1} (\Delta \lambda_\nu) t_\nu \Big|^k \\
&\leq \sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)} \frac{{q_n}^k}{{Q_n}^k Q_{n-1}} \sum_{\nu=1}^{n-1} \frac{(R_{\nu-1})^k (F_{\nu+1})^k |\Delta \lambda_\nu|^k |t_\nu|^k}{{q_\nu}^{k-1} {r_\nu}^k} \left(\frac{1}{Q_{n-1}} \sum_{\nu=1}^{n-1} q_\nu \right)^{k-1}
\end{aligned}
$$

$$= O(1)\sum_{\nu=1}^{m}\frac{(R_{\nu-1})^k(F_{\nu+1})^k|\Delta\lambda_\nu|^k|t_\nu|^k}{{q_\nu}^{k-1}{r_\nu}^k}\sum_{n=\nu+1}^{m+1}\frac{{\alpha_n}^{\mu(\delta k+k-1)}{q_n}^k}{{Q_n}^kQ_{n-1}} \text{ using (3.3)}$$

$$= O(1)\sum_{\nu=1}^{m}\nu^{k-1}|t_\nu|^k\left(\frac{R_{\nu-1}F_{\nu+1}|\Delta\lambda_\nu|}{r_\nu Q_\nu}\right)^k$$

$$= O(1)\sum_{\nu=1}^{m}\nu^{k-1}|t_\nu|^k \text{ using (3.7)}$$

$$= O(1) \text{ as } m\to\infty.$$

Again,

$$\sum_{n=2}^{m+1}{\alpha_n}^{\mu(\delta k+k-1)}|T_{n,4}|^k$$

$$= \sum_{n=2}^{m+1}{\alpha_n}^{\mu(\delta k+k-1)}\left|\frac{q_n}{Q_nQ_{n-1}}\frac{R_n\lambda_nf_nt_n}{r_n}\right|^k$$

$$\leq \sum_{n=2}^{m+1}{\alpha_n}^{\mu(\delta k+k-1)}|t_n|^k\left(\frac{q_nR_nF_n|\lambda_n|}{Q_nQ_{n-1}r_n}\right)^k$$

$$= \sum_{n=2}^{m+1}{\alpha_n}^{\mu(k-1)}|t_n|^k\left(\frac{q_nR_nF_n|\lambda_n|{\alpha_n}^{\mu\delta}}{Q_nQ_{n-1}r_n}\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}{\alpha_n}^{\mu(k-1)}|t_n|^k, \text{ using (3.7)}$$

$$= O(1) \text{ as } m\to\infty.$$

Next,

$$\sum_{n=2}^{m+1}{\alpha_n}^{\mu(\delta k+k-1)}|T_{n,5}|^k$$

$$= \sum_{n=2}^{m+1}{\alpha_n}^{\mu(\delta k+k-1)}\left|\frac{p_0q_0}{P_nQ_{n-1}}\sum_{\nu=1}^{n-1}\lambda_\nu t_\nu\right|^k$$

$$\leq O(1)\sum_{n=2}^{m+1}{\alpha_n}^{\mu(\delta k+k-1)}\frac{1}{{P_n}^kQ_{n-1}}\sum_{\nu=1}^{n-1}\frac{|\lambda_\nu|^k|t_\nu|^k}{{q_\nu}^{k-1}}\left(\frac{1}{Q_{n-1}}\sum_{\nu=1}^{n-1}q_\nu\right)^{k-1}$$

$$= O(1)\sum_{\nu=1}^{m}\frac{|\lambda_\nu|^k|t_\nu|^k}{{q_\nu}^{k-1}}\sum_{n=\nu+1}^{m+1}\frac{{\alpha_n}^{\mu(\delta k+k-1)}}{{P_n}^kQ_{n-1}}$$

$$= O(1)\sum_{\nu=1}^{m}\frac{|\lambda_\nu|^k|t_\nu|^k}{{q_\nu}^{k-1}}\sum_{n=\nu+1}^{m+1}\frac{{\alpha_n}^{\mu(\delta k+k-1)}{q_n}^k}{{Q_n}^kQ_{n-1}} \text{ using (3.2)}$$

$$= O(1)\sum_{\nu=1}^{m}\nu^k|t_\nu|^k\left(\frac{|\lambda_\nu|}{Q_\nu}\right)^k$$

$$= O(1)\sum_{\nu=1}^{m}\nu^k|t_\nu|^k \text{ using (3.6)}$$

$$= O(1) \text{ as } m\to\infty.$$

Again,

$$
\begin{aligned}
&\sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)}|T_{n,6}|^k \\
&= \sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)}\Big|\frac{p_0 q_0}{P_n Q_{n-1}}\sum_{\nu=1}^{n-1}\frac{R_{\nu-1}}{r_\nu}(\Delta\lambda_\nu)t_\nu\Big|^k \\
&\leq O(1)\sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)}\frac{1}{{P_n}^k Q_{n-1}}\sum_{\nu=1}^{n-1}\frac{(R_{\nu-1})^k|\Delta\lambda_\nu|^k|t_\nu|^k}{{r_\nu}^k {q_\nu}^{k-1}}\left(\frac{1}{Q_{n-1}}\sum_{\nu=1}^{n-1} q_\nu\right)^{k-1} \\
&= O(1)\sum_{\nu=1}^{m}\frac{(R_{\nu-1})^k|\Delta\lambda_\nu|^k|t_\nu|^k}{{r_\nu}^k {q_\nu}^{k-1}}\sum_{n=\nu+1}^{m+1}\frac{{\alpha_n}^{\mu(\delta k+k-1)}}{{P_n}^k Q_{n-1}} \\
&= O(1)\sum_{\nu=1}^{m}\nu^{k-1}|t_\nu|^k\left(\frac{R_{\nu-1}|\Delta\lambda_\nu|}{r_\nu Q_\nu}\right)^k \\
&= O(1)\sum_{\nu=1}^{m}\nu^{k-1}|t_\nu|^k \ \texttt{using (3.9)} \\
&= O(1) \ \texttt{as} \ m\to\infty.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)}|T_{n,7}|^k \\
&= \sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)}\Big|\frac{p_0 q_0}{P_n Q_{n-1}}\frac{R_n}{r_n}\lambda_n t_n\Big|^k \\
&= O(1)\sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)}|t_n|^k\left(\frac{R_n|\lambda_n|}{P_n Q_{n-1} r_n}\right)^k \\
&= O(1)\sum_{n=2}^{m+1} {\alpha_n}^{\mu(\delta k+k-1)}|t_n|^k\left(\frac{q_n R_n|\lambda_n|}{Q_n Q_{n-1} r_n}\right)^k \\
&= O(1)\sum_{n=2}^{m+1} {\alpha_n}^{\mu(k-1)}|t_n|^k\left(\frac{q_n R_n|\lambda_n|{\alpha_n}^{\mu\delta}}{Q_n Q_{n-1} r_n}\right)^k \\
&= O(1)\sum_{n=2}^{m+1} {\alpha_n}^{\mu(k-1)}|t_n|^k, \ \texttt{using (3.10)} \\
&= O(1) \ \texttt{as} \ m\to\infty.
\end{aligned}
$$

This completes the proof of the theorem.

5. Conclusion

For $\mu = 1$, the summability method $|(N, q_n)(N, p_n), \alpha_n, \delta, \mu|_k$ reduces to the summability method $|(N, q_n)(N, p_n), \alpha_n, \delta|_k$. For, $f(\alpha_n) = (\alpha_n)^\delta$ and $\delta \geq 0$, $|(N, q_n)(N, p_n), \alpha_n, \delta; f|_k$ - summability reduces to $|(N, q_n)(N, p_n), \alpha_n, \delta|_k$ - summability. Again, for $\delta = 0$, $|(N, q_n)(N, p_n), \alpha_n, \delta|_k$ - summability reduces to $|(N, q_n)(N, p_n), \alpha_n|_k$- summability and for $\alpha_n = n$, $|(N, q_n)(N, p_n), \alpha_n|_k$- summability reduces to $|(N, q_n)(N, p_n)|_k$-summability. When $p_n = 1 = q_n$, $|(N, q_n)(N, p_n)|_k$-summability is same as $|(R, q_n)(R, p_n)|_k$-summability. Also, $|(R, q_n)(R, p_n)|_k$-summability reduces to $|(R, q_n)(A)|_k$-summability when (R, p_n)-summability is replaced by A- summability. From the above results and discussions, we are in a conclusion that our results are more generalized and in particular generalizes the results of Sulaiman [7], Paikray et al [3] and Misra et al [2].

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] Das, G., Tauberian theorems for absolute Norlund summability, Proc. Lond. Math. Soc. 19 (2) (1969), 357-384.

[2] Misra, M., Padhy, B.P., Buxi, S.K. and Misra, U.K., On indexed product summability of an infinite series, J. Appl. Math. Bioinform. 1 (2) (2011), 147-157.

[3] Paikray, S.K., Misra, U.K. and Sahoo, N.C., Product Summability of an Infinite Series, Int. J. Computer Math. Sci. 1 (7) (2010), 853-863.

[4] Parameswaran, M.R., Some product theorems in summability, Math. Z. 68 (1957), 19-26.

[5] Rajgopal, C.T., Theorems on product of two summability methods with applications, J. Indian Math. Soc. 18 (1) (1954), 88-105.

[6] Ramanujan, M.S., On products of summability methods, Math. Z. 69 (1) (1958), 423-428.

[7] Sulaiman, W.T., A Note on product summability of an infinite series, Int. J. Math. Sci. 2008 (2008), Article ID 372604.

[8] Szasz, O., On products of summability methods, Proc. Amer. Math. Soc. 3 (2) (1952), 257-263.