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SOME RESULTS ABOUT A BOUNDARY VALUE PROBLEM ON MIXED CONVECTION

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Abstract. The purpose of this paper is to study the autonomous third order non linear differential equation $f''' + ff'' + g(f') = 0$ on $[0, +\infty[$ with $g(x) = \beta x(x-1)$ and $\beta > 1$, subject to the boundary conditions $f(0) = a \in \mathbb{R}$, $f'(0) = b < 0$ and $f'(t) \to \lambda \in \{0, 1\}$ as $t \to +\infty$. This problem arises when looking for similarity solutions to problems of boundary-layer theory in some contexts of fluids mechanics, as mixed convection in porous medium or flow adjacent to a stretching wall. Our goal, here is to investigate by a direct approach this boundary value problem as completely as possible, say study existence or non-existence and uniqueness solutions and the sign of this solutions according to the value of the real parameter β.

1. Introduction

In fluid mechanics, the problems are usually governed by systems of partial differential equations. In modeling of boundary layer, this is sometimes possible, and in some cases, the system of partial differential

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equations reduces to a systems involving a third order differential equation of the form

$$f''' + ff'' + g(f') = 0, \tag{1.1}$$

where the function $g : \mathbb{R} \to \mathbb{R}$ is assumed to be locally Lipschitz.

If $g(x) = 0$, the equation is the Blasius equation (1907) see [6], [15]. The case $g(x) = \beta(x^2 - 1)$ was first given by Falkner and Skan (1931) see [13]. The case $g(x) = \beta x^2$, this case occurs in the study of free convection (1966) see [3], [5], [7], [9], [12]. And for $g(x) = \beta x(x - 1)$ is the mixed convection (2003) see [1], [2], [4], [8], [10], [11], [14], [16]. In this paper is to investigate this last case with $\beta > 1$. We consider the equation

$$f''' + ff'' + \beta f'(f' - 1) = 0 \tag{1.2}$$

And we associate to equation (1.2) the boundary value problem:

$$\begin{cases} f''' + ff'' + \beta f'(f' - 1) = 0 \\ f(0) = a, \;\; a \in \mathbb{R} \\ f'(0) = b < 0 \\ f'(t) \longrightarrow \lambda \text{ as } t \longrightarrow +\infty \end{cases} \tag{$\mathcal{P}_{\beta;a,b,\lambda}$}$$

where $\lambda \in \{0, 1\}$ and $\beta > 1$. This problem arises in the study of mixed convection boundary layer near a semi-infinite vertical plate embedded in a saturated porous medium, with a prescribed power law of the distance from the leading edge for the temperature. The parameter β is a temperature power-law profile and b is the mixed convection parameter, namely $b = \frac{R_a}{Pe} - 1$, with R_a the Rayleigh number and P_e the Péclet number. The case where $a \geq 0$, $b \geq 0$, $\beta > 0$ and $\lambda \in \{0, 1\}$ was treated by Aïboudi and al.see [1], and for $a \in \mathbb{R}$, $b \leq 0$, $0 < \beta < 1$ see [2], and the results obtained generalize the ones of [11]. In [8], Brighi and Hoernel established some results about the existence and uniqueness of convex and concave solution of $(\mathcal{P}_{\beta;a,b,1})$ where $-2 < \beta < 0$ and $b > 0$. These results can be recovered from [10], where the general equation $f''' + ff'' + \mathbf{g}(f') = 0$ is studied. In [16], some theoretical results can be found about the problem $(\mathcal{P}_{\beta;0,b,1})$ with $-2 < \beta < 0$, and $b < 0$. In [14] and [16], the method used by the authors allows them to prove the existence of a convex solution for the case $a = 0$ and seems difficult to generalize for $a \neq 0$. The problem $(\mathcal{P}_{\beta;a,b,\lambda})$ with $\beta = 0$ is the well known Blasius problem. In the following, we note by f_c a solution of the problem to the initial values below and by $[0, T_c)$ the right maximal interval of its existence:

$$\begin{cases} f''' + ff'' + \beta f'(f' - 1) = 0 \\ f(0) = a \\ f'(0) = b \\ f''(0) = c \end{cases} \tag{$\mathcal{P}_{\beta;a,b,c}$}$$

To solve the boundary value problem ($\mathcal{P}_{\beta;a,b,\lambda}$) we will use the shooting method, which consists of finding the values of a reel parameter c for which the solution of (1.2) satisfying the initial conditions.

2. On Blasius Equation

In this section, we recall some results about subsolutions and supersolutions of the Blasius equation. Recall that the so-called Blasius equation is the third order ordinary differential equation $f''' + ff'' = 0$ i.e Eq.(1.1) with $g = 0$. Let us notice that, for any $\tau \in \mathbb{R}$, the function $h_\tau : t \mapsto \frac{3}{t-\tau}$ is a solution of Blasius equation on each $(-\infty, \tau)$ and $(\tau, +\infty)$. Let $I \subset \mathbb{R}$ be an interval and $f : I \longrightarrow \mathbb{R}$ be a function.

Definition 2.1. *We say that f is a subsolution (resp. a supersolution) of the Blasius equation $f''' + ff'' = 0$ if f is of class C^3 and if $f''' + ff'' \leq 0$ on I (resp.$f''' + ff'' \geq 0$ on I).*

Proposition 2.1. *Let $t_0 \in \mathbb{R}$. There does not exist no positive concave subsolution of the Blasius equation on the interval $[t_0, +\infty)$.*

Proof. See [10], Proposition 2.11. □

Proposition 2.2. *Let $t_0 \in \mathbb{R}$. There does not exist no positive convex supersolution of the Blasius equation on the interval $[t_0, +\infty)$.*

Proof. See [10], Proposition 2.5. □

3. Preliminary Results

Proposition 3.1. *Let f be a solution of the equation (1.2) on some maximal interval $I = (T_-, T_+)$ and $\beta > 1$.*

1. *If F is any anti-derivative of f on I, then $(f''e^F)' = -\beta f'(f' - 1)e^F$.*
2. *Assume that $T_+ = +\infty$ and that $f'(t) \longrightarrow \lambda \in \mathbb{R}$ as $t \to +\infty$. If moreover f is of constant sign at infinity, then $f''(t) \longrightarrow 0$ as $t \to +\infty$.*
3. *If $T_+ = +\infty$ and if $f'(t) \longrightarrow \lambda \in \mathbb{R}$ as $t \to +\infty$, then $\lambda = 0$ or $\lambda = 1$.*
4. *If $T_+ < +\infty$, then f'' and f' are unbounded near T_+.*
5. *If there exists a point $t_0 \in I$ satisfying $f''(t_0) = 0$ and $f'(t_0) = \mu$, where $\mu = 0$ or 1 then for all $t \in I$, we have $f(t) = \mu(t - t_0) + f(t_0)$.*
6. *If $f'(t) \to 0$ as $t \to +\infty$, then $f(t)$ does not tend to $-\infty$ or $+\infty$ as $t \to +\infty$.*

Proof. The first item follows immediately from equation (1.2). For the proof of items 2-5, see [3], and item 6 see [1].

□

Proposition 3.2. *Let us suppose that f be a solution of equation (1.2) on the maximal interval $I = (T_-, T_+)$*

(1) Let $H_1 = f'' + f(f' - 1)$ then $H_1' = (1-\beta)f'(f'-1)$, for all $t \in I$;

(2) Let $H_2 = 3f''^2 + \beta f'^2(2f' - 3)$ then $H_2' = -6ff''^2$, for all $t \in I$;

(3) Let $H_3 = 2ff'' - f'^2 + (2f' - \beta)f^2$ then $H_3' = 2(2-\beta)ff'^2$, for all $t \in I$;

(4) Let $H_4 = f'' + ff'$ then $H_4' = (1-\beta)f'^2 + \beta f'$, for all $t \in I$;

(5) Let $H_5 = f' + \frac{1}{2}f^2$ then $H_5' = H_4 = f'' + ff'$, for all $t \in I$.

Proof. This statements follows immediately from equation (1.2). □

4. The boundary value problem $(P_{\beta;a,b,\lambda})$

Let the boundary value problem $(P_{\beta;a,b,\lambda})$, we are interested here in a concave, convex and convex-concave solutions of a problem $(P_{\beta;a,b,\lambda})$ and there sign. We used shooting method to find these solutions, this method consists of finding the values of a parameter $c \in \mathbb{R}$ for which the solution of $(P_{\beta;a,b,c})$ satisfying the initial conditions $f'(0) = a, f'(0) = b$ and $f''(0) = c$, exists up to infinity and is such that $f'(t) \to \lambda$ as $t \to +\infty$. Define the following sets:

$$C_0 = \{c \leq 0 : f_c'' \leq 0 \text{ on } [0, T_c)\},$$

$$C_1 = \{c > 0 : f_c' \leq 0 \text{ and } f_c'' \geq 0 \text{ on } [0, T_c)\},$$

$$C_2 = \{c > 0 : \exists t_c \in [0, T_c), \exists \varepsilon_c > 0 \text{ s.t } f_c' < 0 \text{ on } (0, t_c),$$
$$f_c' > 0 \text{ on } (t_c, t_c + \varepsilon_c) \text{ and } f_c'' > 0 \text{ on } (0, t_c + \varepsilon_c)\},$$

$$C_3 = \{c > 0 : \exists s_c \in [0, T_c), \exists \varepsilon_c > 0 \text{ s.t } f_c'' > 0 \text{ on } (0, s_c),$$
$$f_c'' < 0 \text{ on } (s_c, s_c + \varepsilon_c) \text{ and } f_c' < 0 \text{ on } (0, s_c + \varepsilon_c)\}.$$

Remark 4.1. *It is easy to prove that C_0, C_1, C_2 and C_3 are disjoint nonempty open subsets of $\mathbb{R}$, $C_0 =]-\infty, 0]$ and $C_1 \cup C_2 \cup C_3 =]0, +\infty[$ (see Appendix A of [10] with $g(x) = \beta x(x-1)$ and $\beta > 0$).*

Lemma 4.1. *Let $\beta > 0$. If $c \in C_0$, then $T_c < +\infty$. Moreover, f_c is concave solution, decreasing and $f_c'(t) \to -\infty$ as $t \to T_c$.*

Proof. If $c \in C_0$, the result follows from proposition 3.1 item 1, we have $f_c''(t) < 0$ and $f_c''(t) < 0$ for all $t \in [0, +\infty)$, then f_c is a no positive concave subsolution of the Blasius equation on $[0, +\infty)$ if $a < 0$, and on $[t_0, +\infty)$ such that $f_c(t_0) = 0$ if $a > 0$, with $f_c'(t) \to -\infty$ as $t \to \mathrm{T}_c$. If we assume that $T_c = +\infty$, This leads to a contradiction with proposition 3.1, then $T_c < +\infty$. □

Lemma 4.2. *Let $\beta > 0$. Then f_c is a convex solution of the boundary value problem $(\mathcal{P}_{\beta,a,b,0})$ if and only if $c \in C_1$.*

Proof. See Appendix A of [10] with $g(x) = \beta x(x-1)$ and $\beta > 0$. □

Lemma 4.3. *Let $\beta > 0$. If $c \in C_3$, then $T_c < +\infty$. Moreover, f_c is convex-concave, decreasing and $f_c'(t) \to -\infty$ as $t \to T_c$.*

Proof. See [2], lemma 5.3. □

Remark 4.2. *From proposition 3.1 items 1,3 and 5, if $c \in C_2$, then there are only three possibilities for the solution of the initial value problem $(\mathcal{P}_{\beta;a,b,c})$:*

(1) f_c is convex and $f_c'(t) \to +\infty$ as $t \to T_c$ (with $T_c \leq +\infty$);

(2) there exists a point $t_0 \in [0, T_c)$ such that $f_c''(t_0) = 0$ and $f_c'(t_0) > 1$;

(3) f_c is a convex solution of $(\mathcal{P}_{\beta;a,b,1})$.

The next proposition shows that the case (1) cannot hold.

Proposition 4.1. *Let $\beta > 0$. There does not exit $c \geq 0$, such that f_c is convex and $f_c'(t) \to +\infty$ as $t \to T_c$ on its right maximal interval of existence $[0, T_c)$.*

Proof. Assume that f_c is convex on its right maximal interval of existence $[0, T_c)$ and $f_c'(t) \to +\infty$ as $t \to T_c$. There exist $t_0 \in [0, T_0)$, which the function H_2 is decreasing for $t > t_0$, this is a contradiction as $t \to \mathrm{T}_c$. □

Proposition 4.2. *Let $\beta > 1$. If there exist $t_0 \in [0, T_c)$ such that $f_c'(t_0) = 0$ and $f_c''(t_0) < 0$, then for all $t > t_0$, $f_c''(t) < 0$ and $f_c'(t) \neq 0$.*

Proof. Let f_c is convex on its right maximal interval of existence $[0, T_c)$, suppose there exist $t_1 > t_0$ such that $f_c'(t_1) = 0$, hence the function H_4 is decreasing on $[t_0, t_1]$, therefore $H_4(t_0) > H_4(t_1)$, we have $f_c''(t_0) > f_c''(t_1)$, which yields a contradiction. □

5. The $a < 0$ case

Proposition 5.1. *Let $\beta > 1$, the boundary value problem $(P_{\beta;a,b,1})$ has no convex solution.*

Proof. Let f_c is convex on maximal interval of existence $[0, T_c)$, such that $f_c'(t) \to 1$ as $t \to T_c$, then there exist $t_0 \in [0, T_c)$, such that $f_c'(t_0) = 0$, the function H_1 is creasing for all $t > t_0$, therefore $H_1(t) > H_1(t_0)$ for $t > t_0$, we have $f_c''(t) - f_c''(t_0) > -f_c(t)\left(f_c'(t) - 1\right) > 0$, we obtain a contradiction for t large enough because $f_c''(t) \longrightarrow 0$ and $f_c(t) > 0$. □

Proposition 5.2. *The boundary value problem $(P_{\beta;a,b,0})$ has no negative convex-concave solution.*

Proof. Let f_c is convex-concave on maximal interval of existence $[0, T_c)$, such that $f_c'(t) \to 0$ as $t \to T_c$, then there exist $t_0 \in [0, T_c)$ such that $f'_{c(}t_0) = 0$, the function H_2 is creasing for all $t > t_0$, we have $3f_c''^2(t_0) < H_2(t)$ for all $t > t_0$, $H_2(t) \to 0$ as $t \to +\infty$, a contradiction. □

Remark 5.1. *If the boundary value problem $(P_{\beta;a,b,0})$ has a convex-concave solution, then this solution changes the sign.*

Lemma 5.1. *If $c \in C_1$, then there exist c_* such that $0 < c < c_*$, $T_c = +\infty$, and the solution f_c is negative on $[0, +\infty)$.*

Proof. Let f_c is solution on maximal interval of existence $[0, T_c)$, if $c \in C_1$, then $T_c = +\infty$, the function H_2 is creasing on $[0, T_c)$, it follows that $3c^2 + \beta b^2(2b - 3) < 0$, we obtain $c < -b\sqrt{\frac{\beta(3-2b)}{3}}$, and the solution f_c is negative because $a < 0$ and $f_c' < 0$. □

Lemma 5.2. *If $c \in C_3$, then there exist c_* such that $0 < c < c_*$, $T_c < +\infty$ and the solution f_c is negative on $[0, T_c)$.*

Proof. If $c \in C_3$, then $f_c' \to -\infty$, and $T_c < +\infty$, other results same proof that lemma 5.1. □

Remark 5.2. *It follows from lemma 5.1 and lemma 5.2, there exist $c_* > 0$ such that $c > c_*$, $C_2 \neq \emptyset$ and here the solution f_c is convex-concave.*

Lemma 5.3. *Let $1 < \beta < 2$, if $c \in C_2$ and f_c is a no positive solution on maximal interval of existence $[0, T_c)$, then for all $t \in [0, T_c)$ we have $f_c(t) \leq \max\left\{a, \frac{b}{\sqrt{\beta}}\right\}$, $T_c < +\infty$ and $f_c'(t) \to -\infty$ as $t \to T_c$.*

Proof. Let $c \in C_2$ and f_c is a no positive solution on maximal interval of existence $[0, T_c)$. From the proposition 3.1, 4.2 and 5.2, there exist $t_0 \in [0, T_c)$ such that $f_c'(t_0) = 0$.
Moreover, the function H_3 is decreasing on $[0, T_c)$, we have $H_3(0) > H_3(t_0)$, it follows that, $-b^2 > 2ac - b^2 + (2b - \beta)a^2 > 2f_c(t_0)f_c''(t_0) - \beta f_c^2(t_0) > -\beta f_c^2(t_0)$, we get $f_c(t_0) < \frac{b}{\sqrt{\beta}}$ for all $t \in [0, T_c)$, the conclusion follows from that, for all $t \in [0, T_c)$, if $a < \frac{b}{\sqrt{\beta}}$ we have $f_c(t) \leq f_c(t_0)$ and, if $a > \frac{b}{\sqrt{\beta}}$ we have $f_c(t) \leq a$ with $T_c < +\infty$, and $f_c' \to -\infty$ as $t \to T_c$. □

Lemma 5.4. *If $c \in C_2$, and if there exist $t_1 \in [0, T_c)$ such that $f_c''(t_1) = 0$ and $f_c(t_1) < 0$, then $f_c'(t_1) > \frac{3}{2}$.*

Proof. If $c \in C_2$, there exist $t_0 \in [0, T_c)$ such that $f_c'(t_0) = 0$, $f_c(t_0) < 0$, and there exist $t_1 > t_0$ such that $f_c''(t_1) = 0$, we suppose $f_c(t_1) < 0$ and $f_c'(t_1) < \frac{3}{2}$, the function H_2 is creasing on $[0, t_1)$, we have $3f_c''(t_0) < \beta f_c'^2(t_1)(2f_c'(t_1) - 3)$, we obtain a contradiction. □

Remark 5.3. *Thanks to the previous lemma, if we have $f_c'(t_1) < \frac{3}{2}$ and f_c is convex-concave solution on maximal interval of existence $[0, T_c)$, then f_c changes the sign.*

Lemma 5.5. *For $1 < \beta < 2$ and $b < -1$, if $c \in C_2$ and if there exist $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$, then $f_c'(t_0) > 1$.*

Proof. Let f_c is convex-concave solution on maximal interval of existence $[0, T_c)$, $1 < \beta < 2$ and $b < -1$, if there exist $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$, the function H_3 is decreasing on $[0, t_0)$, we have $H_3(0) > H_3(t_0)$, therefore $-b^2 > -f_c'^2(t_0)$, and we obtain $f_c'(t_0) > -b > 1$. □

Lemma 5.6. *If $c \in C_2$ and there exist $t_1 \in [0, T_c)$ such that $f_c''(t_1) = 0$ and if $f_c(t_1) < 0$, then $f_c'(t_1) > \frac{-\beta}{1-\beta}$.*

Proof. If $c \in C_2$, there exist $t_0 \in [0, T_c)$ such that $f_c'(t_0) = 0$ and $f_c''(t_0) > 0$, there exist $t_1 > t_0$ such that $f_c''(t_0) = 0$, we suppose $f_c'(t_1) < \frac{-\beta}{1-\beta}$, the function H_4 is creasing on $[t_0, t_1]$, we have $f_c''(t_0) < f_c(t_1)f_c'(t_1)$, this is a contradiction. □

Theorem 5.1. *Let $\beta > 1$, $a < 0$ and $b < 0$.*

(1) The boundary value problem $(P_{\beta;a,b,0})$ has as least one negative convex solution on $[0, +\infty)$.
(2) The boundary value problem $(P_{\beta;a,b,1})$ has no convex solution on $[0, +\infty)$.
(3) The boundary value problem $(P_{\beta;a,b,+\infty})$ has no convex solution on $[0, T_c)$.

Proof. The first result follows from remark 4.1 and lemma 4.2, the second result follows from proposition 5.1 and the third result follows from proposition 4.1. □

6. The $a > 0$ case

Let $a, b \in \mathbb{R}$ with $b < 0$ and $a > 0$. We assume $\beta > 1$, and f_c be a solution of the initial value problem $(P_{\beta;a,b,c})$ on the right maximal interval of existence $[0, T_c)$, $c > 0$.

Before that, and in order to complete the study, let us divide the sets C_2 and C_3 into the following two subsets:

$$C_{2.1} = \{\text{c} \in C_2;\ f_c' > 0 \text{ on } [t_c, T_c\,)\},$$

$$C_{2.2} = \{\text{c} \in C_2;\ \exists \text{s}_c > t_c \text{ s.t } f_c' > 0 \text{ on } [t_c, s_c) \text{ and } f_c'(s_c) = 0\},$$

$$C_{3.1} = \{c \in C_3;\ f_c(s_c) < 0\},$$

$$C_{3.2} = \{c \in C_3;\ f_c(s_c) > 0\}.$$

Proposition 6.1. *If $c \in C_1 \cup C_2 \cup C_{3.1}$, then $c > -ab$*

Proof. If $c \in C_1$, $T_c = +\infty$, $f_c'(t) \to 0$ as $t \to +\infty$, the function H_4 is decreasing on $[0, +\infty)$, we have $c + ab > 0$, if $c \in C_2 \cup C_{3.1}$, there exist $t_0 \in [0, T_c)$ such that $f_c'(t_0) = 0$ or $f_c(t_0) = 0$, we have $c + ab \geq f_c''(t_0) > 0$. □

Remark 6.1. *If $c \leq -ab$ then $c \in C_{3.2}$ and $T_c < +\infty$. Thus $C_{3.2} \neq \emptyset$ and the convex part of the solution f_c is positive.*

Proposition 6.2. *If $c \in C_1 \cup C_{2.1}$ and $b > -\frac{1}{2}a^2$, then $T_c = +\infty$ and the solution f_c is positive.*

Proof. Let f_c solution of the initial value problem $(P_{\beta;a,b,c})$ on the right maximal interval of existence $[0, T_c)$, $c > 0$, if $c \in C_1 \cup C_{2.1}$, thanks to propositions 3.1 and 4.1 it follows that $T_c = +\infty$, no we suppose there exist $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$, the function H_4 is decreasing for all $t > 0$, we have $H_4(t_0) = f_c''(t_0)$, therefore H_5 is creasing on $[0, t_0)$, we obtain $b + \frac{1}{2}a^2 < f_c'(t_0) < 0$, this is a contradiction. □

Remark 6.2. *If $c \in C_{2.2}$ and $b > -\frac{1}{2}a^2$, the solution f_c is positive on $[0, t_0)$, t_0 is the point such that $t_0 > s_c$ with $f_c(t_0) = 0$ and s_c be as in definition of $C_{2.2}$.*

Lemma 6.1. *Let $\beta > 1$ and $-\frac{1}{2}a^2 < b < 0$.*
If f_c be solution of the initial value problem $(P_{\beta;a,b,c})$, on the right maximal interval of existence $[0, T_c)$ and if there exist $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$ and $f_c'(t_0) < 0$ then $f_c''(t_0) < 0$.

Proof. For contradiction, let us that $t_0 \in [0, T_c)$ with $f_c(t_0) = 0$ and $f_c'(t_0) < 0$, the function H_4 is decreasing on $[0, t_0)$ and $H_4(t_0){=}f_c''(t_0) > 0$ then for all t$\in [0, t_0)$, $H_4 > 0$, and H_5 is creasing on $[0, t_0)$, we have $b + \frac{1}{2}a^2 < f_c'(t_0) < 0$, this is a contradiction. □

Proposition 6.3. *Let $1 < \beta < 2$, $b > -\frac{1}{2}a^2$ and $c \in C_{2.2}$. For all $t \in [0, T_c\,)$, one has $f_c(t) < \sqrt{\frac{b^2 + (\beta - 2b)a^2}{\beta}}$.*

Proof. Let $c \in C_{2.2}$ and s_c be as in the definition of $C_{2.2}$, the function H_3 is creasing on $[0, s_c)$, we have:
$-b^2 + (2b - \beta)a^2 < 2ac - b^2 + (2b - \beta)a^2 < 2f_c(s_c)f_c''(s_c) - \beta f_c^2(s_c) < -\beta f_c^2(s_c)$,

which implies that $f_c(s_c) < \sqrt{\frac{b^2+(\beta-2b)a^2}{\beta}}$. From the proposition 4.2, the conclusion follows from that , for all $t \in [0, T_c\,)$, we have $f_c(t) \leq f_c(s_c)$. □

Lemma 6.2. *If $c \in C_1 \cup C_{2.1}$ and $b > -\frac{1}{2}a^2$. Then $T_c = +\infty$ and there exist $c_* > 0$ such that $c > c_*$.*

Proof. Let $c \in C_1 \cup C_{2.1}$, and $b > -\frac{1}{2}a^2$. By the definition of C_1 and $C_{2.1}$, thanks to proposition 6.2, we have $T_c = +\infty$ and f_c' is bounded. Otherwise the function H_2 is decreasing for $t > 0$, we obtain $3c^2 + \beta b^2(2b - 3) > 0$, which implies that $c > -b\sqrt{\frac{\beta(3-2b)}{3}}$. □

Remark 6.3. *There exist $c_* > 0$, if $c < c_*$, then there exist $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$, $f_c'(t_0) < 0$ and $f_c''(t_0) < 0$ say that $c \in C_{2.2} \cup C_{3.2}$, since if $c \in C_{2.1}$ then $T_c = +\infty$.*

Let us divide the set $C_{2.1}$ into the following two subsets:

$$C_{2.1.1} = \{\text{c} \in C_{2.1};\ f_c'(t) \to 0 \text{ as } t \to +\infty\},$$

$$C_{2.1.2} = \{\text{c} \in C_{2.1};\ f_c'(t) \to 1 \text{ as } t \to +\infty\}.$$

Proposition 6.4. *Let $1 < \beta < 2$, if $c \in C_1 \cup C_3 \cup C_{2.2} \cup C_{2.1.1}$. Then there exist $c_* > 0$ such that $c < c_*$.*

Proof. Let f_c solution of the initial value problem $(P_{\beta;a,b,c})$ on the right maximal interval of existence $[0, T_c)$, either there exist $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$ or $f_c'(t_0) = 0$ if $T_c < +\infty$, and if $T_c = +\infty$, we have $f_c'(t) \to 0$ as $t \to +\infty$, from proposition 3.1 item 6, it follows that the function H_3 is creasing on $[0, t_0)$ or $[0, +\infty)$, we get $2ac - b^2 + (2b - \beta)a^2 < 0$, which implies that $c < \frac{b^2+(\beta-2b)a^2}{2a}$. □

Remark 6.4. *From proposition 6.4 there exist $c_* > 0$, such that for $c \geq c_*$, then $c \in C_{2.1.2}$. Thus $C_{2.1.2} \neq \emptyset$.*

Corollary 6.1. *If $1 < \beta < 2$, $a > 0$, $b < 0$ and $b > -\frac{1}{2}a^2$, then the problem $(P_{\beta;a,b,1})$ has as least one positive convex or positive convex-concave solution on $[0, +\infty)$.*

Proof. This follows immediately from remark 6.4, lemma 6.2 and proposition 6.3. □

Theorem 6.1. *Let $\beta > 1$, $a > 0$ and $b < 0$.*

(1) *The boundary value problem $(P_{\beta;a,b,0})$ has as least one convex solution on $[0, +\infty)$ if in addition we have $b > -\frac{1}{2}a^2$ it will be no negative convex solution.*
(2) *The boundary value problem $(P_{\beta;a,b,-\infty})$ has infinity convex-concave solutions on the maximal interval of existence $[0, T_c)$ with $T_c < +\infty$, if in addition we have $b > -\frac{1}{2}a^2$ the convex part of these solutions will be no negative.*
(3) *The boundary value problem $(P_{\beta;a,b,+\infty})$ has no convex solution on $[0, T_c)$.*

Proof. The first result follows from remark 4.1, lemma 4.2 and proposition 6.2 , the second result follows from proposition 3.1, proposition 4.2 and remark 6.1, and the third result follows from proposition 4.1. □

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