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NONLINEAR (m,p)-ISOMETRIC AND $(2,p)$-CONCAVE MAPPINGS ON COMPLEX NORMED SPACES

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Abstract. Let S be a self mapping on a complex normed space $\mathcal{X}$. In this paper, we study the class of mappings satisfying the following condition

$$\sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}\left\|S^kx-S^ky\right\|^p=0,$$

for all $x,y\in X$, where m is a positive integer. We prove some of the properties of these classes of mappings.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. In the 1990s, Agler and Stankus [1] studied the following operator. For an operator $T\in\mathcal{B}(\mathcal{H})$ and a positive integer m, define

$$B_m(T):=\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^k. \tag{1.1}$$

An operator $T\in\mathcal{B}(\mathcal{H})$ is said to be m-contractive $\big($respectively, m-expansive and m-isometric $\big)$ if $B_m(T)\le 0$ $\big($respectively, $B_m(T)\le 0$ and $B_m(T)=0$ $\big)$ for some positive integer m. Clearly, T is an

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m-contractive if and only if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^k h\|^2 \geq 0, \quad \forall\, h \in \mathcal{H},$$

T is an m-expansive if and only if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^k h\|^2 \leq 0, \quad \forall\, h \in \mathcal{H},$$

and T is an m-isometry if and only if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^k h\|^2 = 0, \quad \forall\, h \in \mathcal{H}.$$

Agler and Stankus [1–3] developed a theory for m-isometric operators with rich connections to Toeplitz operators, classical function theory and nonstationary stochastic processes. The topics related to m-isometries are currently being studied intensively (see e.g., [4, 17, 22]). In [5, 11, 14, 20] certain types of operators (composition, multiplication, shift) were considered and some conditions under which these operators are m-isometries were given;

Let $\mathbb{L}^2(\mathbb{T})$ be the set of square integrable measurable functions on of the unit circle $\mathbb{T} = \partial\mathbb{D}$ and $\mathbb{H}^2(\mathbb{T})$ be the corresponding Hardy space. Let $\mathbb{L}^\infty(\mathbb{T})$ be the set of bounded measurable functions on $\mathbb{T}$ and let $\mathbb{H}^\infty(\mathbb{T}) := \mathbb{L}^\infty \cap \mathbb{H}^2(\mathbb{H})$. For ϕ in $L^\infty(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial\mathbb{D}$, the Toeplitz operator T_ϕ with symbol ϕ on the Hardy space $\mathbb{H}^2(\mathbb{T})$ is given by $T_\phi f := P(\phi f)$, $(f \in \mathbb{H}^2(\mathbb{T}))$, where P denotes the orthogonal projection of $L^2(\mathbb{T})$ onto $\mathbb{H}^2(T)$.

For $\phi \in L^\infty(\mathbb{T})$, a Toeplitz operator T_ϕ is m-expansive if and only if

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\left\|T_\phi^j k\right\|^2 \leq 0, \;\; \forall\, k \in \mathbb{H}^2(\mathbb{T}),$$

is m-contractive if and only if

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\left\|T_\phi^j k\right\|^2 \geq 0, \;\; \forall\, k \in \mathbb{H}^2(\mathbb{T}),$$

and m-isometric if and only if

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\left\|T_\phi^j k\right\|^2 = 0, \;\; \forall\, k \in \mathbb{H}^2(\mathbb{T}).$$

A generalization of m-isometries to operators on general Banach spaces has been presented by several authors in the last years. F. Bayart introduces in [7] the notion of (m, p)-isometries on general (real or complex) Banach spaces. An operator T on a Banach space $\mathcal{X}$ is called an (m, p)-isometry if there exists an integer $m \geq 1$ and a $p \in [1, \infty)$, with

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^k x\|^p = 0 \quad (x \in \mathcal{H}). \tag{1.2}$$

Bayart showed that all basic properties of m-isometries on Hilbert spaces (which we should now refer to as $(m,2)$-isometries) carry over to (m,p)-isometries on Banach spaces and, further, that (m,p)-isometries are never N-supercyclic if $\mathcal{X}$ is of infinite dimension and complex.

In [18] the authors took of the restriction $p \geq 1$. They considered the equation (1.2) for $p > 0$ real and studied the role of the second parameter p and also discussed the case $p = \infty$. Most results in the literature remain valid, with their existing proofs, for p in this extended range.

Let X and Y be metric spaces. A mapping $S : X \longrightarrow Y$ is called an isometry if it satisfies $d_Y(Sx, Sy) = d_X(x,y)$, for all $x, y \in X$, where $d_X(.,.)$ and $d_Y(.,.)$ denote the metrics in the spaces X and Y, respectively.

In the paper [9], the authors introduced the concept of (m,q)-isometry for maps on a metric space (X, d_X) as : a mapping $S : X \to X$ is called an (m,q)-isometry for ($m \geq 1$, integer, $q > 0$ real) if it satisfies

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}d_X\left(S^{m-k}x, S^{m-k}y\right)^q = 0, \;\; \forall \;\; x,y \in X.$$

In [12] the author consider $A(m,p)$-isometries, where, for an operator $A \in \mathcal{B}(\mathcal{X})$, $T \in \mathcal{B}(\mathcal{X})$ (the algebra of bounded linear operators) is $A(m,p)$-isometric if

$$\beta_m^{(p)}(T,A,x) := \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|AT^kx\|^p = 0 \quad (x \in \mathcal{X}). \tag{1.3}$$

Evidently, an $I(m,p)$-isometry is an (m,p)-isometry; if $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|AT^kx\|^p = 0 \Longleftrightarrow \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\||A|T^kx\|^p = 0 \quad (x \in \mathcal{X}).$$

If $\beta_m^{(p)}(T,A,x) \leq 0$ $\big($resp. $\beta_m^{(p)}(T,A,x) \geq 0\big), \forall\, x \in \mathcal{X}$, T is said to be (A,m,p)-expansive $\big($resp. (A,m,p)-contractive$\big)$. We refer the interested reader to [13, 19] for complete details.

A mapping S (not necessarily linear) on a normed space $\mathcal{X}$ ([16]) is an (m,p)-isometry ($m \geq 1$ integer and $p > 0$ real) if, for all $x, y \in \mathcal{X}$,

$$\Delta_m^p(S;\; x,y) := \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\big\|S^kx - S^ky\big\|^p = 0. \tag{1.4}$$

When $m = 1$, (1.4) is equivalent to $\|Sx - Sy\| = \|x - y\|$, $\forall\, x, y \in \mathcal{X}$, and when $m = 2$, (1.4) is equivalent to

$$\|S^2x - S^2y\|^p - 2\|Sx - Sy\|^p + \|x - y\|^p = 0, \; \forall\, x, y \in \mathcal{X}.$$

In [15] it was observed that a sequence $\big(a_n\big)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is concave if

$$a_{n+2} - 2a_{n+1} + a_n \leq 0, \;\; \forall n \in \mathbb{N} \tag{1.5}$$

or equivalently if

$$a_n \geq \frac{a_{n-1} + a_{n+1}}{2}, \quad \forall n \in \mathbb{N}. \tag{1.6}$$

A sequence $\big(a_n\big)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is log concave if

$$a_n^2 \geq a_{n-1}a_{n+1}, \quad \forall n \in \mathbb{N} \tag{1.7}$$

Remark 1.1. (1) *From $\big(1.6\big)$, we get that any concave sequence of non negative numbers is log concave.*

(2) *If $(a_n)_{n\in\mathbb{N}}$ is a log concave sequence of non negative numbers then*

$$\frac{a_{n+1}}{a_n} \leq \frac{a_n}{a_{n-1}}, \quad \forall n \in \mathbb{N}. \tag{1.8}$$

Therefore the sequence $\frac{a_{n+1}}{a_n}$ is decreasing sequence of non negative numbers.

After a short introduction and some connections with known results in this context, the main results of the paper are presented as follows. In Section 2, we will introduce and study some properties of $(2,p)$-concave mappings. The main results of this section are Proposition 2.1, Proposition 2.3, Proposition 2.4 and Corollary 2.2. In Section,3, a parallel study of the classes of nonlinear (m,p)-isometric mappings are presented. Exactly we will give conditions under which:a self mapping S is (m,p)-isometry it becomes (k,p))-isometry for $1 \leq k \leq m-1$ (Proposition 3.1, Proposition 3.2 and Corollary 3.1. The product of a (m,p)-isometry and a (k,p)-isometry is a $(m+k-1,p)$-isometry for $k = 1,2,3$. A power of $(2,p)$-isometry is again an $(2,p)$-isometry..

2. $(2,p)$-Concave mappings

In this section, let $(\mathcal{X}, \|.\|)$ be a complex normad space, $S : \mathcal{X} \longrightarrow \mathcal{X}$ is a map.

Definition 2.1. *Let S be a self map (not necessary linear) on complex normed space §. S is said to be a $(2,p)$-concave if S satisfy*

$$\big\|S^2x - S^2y\|^p - 2\big\|Sx - Sy\big\|^p + \big\|x - y\big\|^p \leq 0,$$

for all $x, y \in \mathcal{X}$.

Remark 2.1. *It is easy to check that a self map S on complex normed space $\mathcal{X}$ is $(2,p)$-concave if and only if the sequence $a_n = \big\|S^n x - S^n y\big\|^p$ is concave.*

Note the following proposition, which lists some general properties of $(2,p)$-concave mappings.

Proposition 2.1. *Let S be a self map on complex normed space $\mathcal{X}$. If S is an $(2,p)$-concave. Then the following statements hold:*

(1) $\|S^n x - S^n y\|^p + (n-1)\|x-y\|^p \leq n.\|Sx - Sy\|^p,\ x,y \in \mathcal{X},\ n = 0,1,2,...$

(2) $\|Sx - Sy\|^p \geq \dfrac{n-1}{n}\|x-y\|^p,\ \ n \geq 1,\ x,y \in \mathcal{X}.$

(3) $\|Sx - Sy\|^p \geq \|x-y\|^p$ *for all* $x,y \in \mathcal{X}$.

(4) $\|Sx - Sy\| \leq 2^{\frac{1}{p}}\|x-y\|\ \ \forall\, x,y \in S(\mathcal{X})$ *(the range of S).*

(5) S *is injective.*

Proof. (1) From the assumption that S is a $(2,p)$-concave, we get

$$\|S^2x - S^2y\|^p - \|Sx - Sy\|^p \leq \|Sx - Sy\|^p - \|x-y\|^p.$$

Replacing x by $S^k x$ and y by $S^k y$ leads to

$$\|S^{k+2}x - S^{k+2}y|^p - \|S^{k+1}x - S^{k+1}y\|^p \leq \|S^{k+1}x - S^{k+1}y\|^p - \|S^k x - S^k y\|^p,$$

for $k \geq 0$.

Thus means that

$$\begin{aligned} \|S^n x - S^n y\|^p &= \sum_{1\leq k\leq n} \|S^k x - S^k y\|^p - \left(\|S^{k-1}x - S^{k-1}y\|^p\right) + \|x-y\|^p \\ &\leq n\left(\|Sx - Sy\|^p - \|x-y\|^p\right) + \|x-y\|^p \\ &\leq n\|Sx - Sy\|^p + (1-n)\|x-y\|^p. \end{aligned}$$

Hence,

$$\|S^n x - S^n y\|^p + (n-1))\|x-y\|^p \leq \|Sx - Sy\|^p.$$

(2) Form the inequality in (1), it follows

$$\|Sx - Sy\|^p \geq \frac{n-1}{n}\|x-y\|^p.$$

(3) By taking $n \to \infty$ we get

$$\|Sx - Sy\|^p \geq \|x-y\|^p.$$

(4) From the fact that S is a $(2,p)$-concave we get

$$\|S^2x - S^2y\|^p \leq 2\|Sx - Sy\|^p - \|x-y\|^p \leq 2\|Sx - Sy\|^p,\ \ \forall\, x,y \in \mathcal{X}.$$

This means that

$$\left\|S^2x - S^2y\right\|^p \leq 2^{\frac{1}{p}} \left\|Sx - Sy\right\|^p,$$

or equivalently

$$\left\|Sx - Sy\right\|^p \leq 2^{\frac{1}{p}} \left\|x - y\right\|^p, \ \forall \, x, y \in S(\mathcal{X}).$$

(5) The injectivity of S follows immediately from the statement (3). □

Remark 2.2. *From the injectivity of a $(2,p)$-concave map S, we get, by the assertion (2) of the Remark 1.1, that for two elements of $\mathcal{X}$ $x \neq y$ the sequence* $\left(\dfrac{\left\|S^{n+1}x - S^{n+1}y\right\|^p}{\left\|S^n x - S^n y\right\|^p}\right)_n$ *is decreasing.*

Corollary 2.1. *Let S be $(2,p)$-concave on the normed space $\mathcal{X}$ and let S^{-1} be the inverse map of S as a bijection from $\mathcal{X}$ on $S(\mathcal{X})$ (the range of S). Then, both S and S^{-1} are continuous on $S(\mathcal{X})$.*

Proof. The continuity of S and of S^{-1} on $S(\mathcal{X})$ follow, respectively, from the assertions (4) and (3) in Proposition 2.1. □

Proposition 2.2. *Let S be $(2,\ p)$-concave on the normed space $\mathcal{X}$ and R be an isometric mapping of $\mathcal{X}$ such that S and R commute. Then, RS is a $(2,\ p)$-concave mapping on $\mathcal{X}$.*

Proof. The statement follow immediately from the fact

$$\left\|(RS)^2x - (RS)^2y\right\| = \left\|S^2x - S^2y\right\| \text{ and } \left\|(RS)x - (RS)y\right\| = \left\|Sx - Sy\right\|, \ \forall \, x, y \in \mathcal{X}.$$

□

The proof of the following proposition is taken from the statements (1) and (2) of Remark 1.1. For the reader 's convenience. here we give a proof.

Proposition 2.3. *Let S be a self map on complex normed space $\mathcal{X}$ such is a $(2,p)$-concave..Then the following assertions holds:*

(1) $\left\|Sx - Sy\right\|^{2p} \geq \left\|x - y\right\|^p \left\|S^2x - S^2y\right\|^p$ *for all $x, y \in \mathcal{X}$.*

(2) *For each n and $x, y \in X$ such that $x \neq y$,the sequence*

$$\left(\frac{\left\|S^{n+1}x - S^{n+1}y\right\|^p}{\left\|T^n x - T^n y\right\|^p}\right)_{n \geq 0}$$

is monotonically decreasing to 1.

Proof. (1) Since S is a $(2,p)$-concave map, it follows

$$\begin{aligned} d\|Sx-Sy\|^{2p} &\geq \left(\frac{\|x-y\|^p+\|S^2x-S^2y\|^p}{2}\right)^2 \\ &\geq \left(\|x-y\|^{\frac{p}{2}}\|S^2x-S^2y\|^{\frac{p}{2}}\right)^2 \\ &\geq \|x-y\|^p\|S^2x-S^2y\|^p, \ \ \forall\, x,y\in\mathcal{X}. \end{aligned}$$

(2) By Observing that the $(2,p)$-concavity of S implies that

$$\|S^{n+1}x-S^{n+1}y\|^p-2\|S^nx-S^ny\|^p+\|S^{n-1}x-S^{n-1}y\|^p\leq 0. \tag{2.1}$$

On the other hand, since

$$\left(\|S^{n-1}x-S^{n-1}y\|^{\frac{p}{2}}-\|S^{n+1}x-S^{n+1}y\|^{\frac{p}{2}}\right)^2\geq 0,$$

we obtain

$$\begin{aligned} &\|S^{n-1}x-S^{n-1}y\|^{\frac{p}{2}}\|S^{n+1}x-S^{n+1}y\|^{\frac{p}{2}} \\ \leq\ & \frac{\|S^{n+1}x-S^{n+1}y\|^p+d\|S^{n-1}x-S^{n-1}y\|^p}{2} \\ \leq\ & \|S^nx-S^ny\|^p \qquad \text{(by} \quad (2.1)). \end{aligned}$$

Consequently,

$$\|S^{n-1}x-S^{n-1}y\|^p\|S^{n+1}x-S^{n+1}y\|^p\leq\|S^nx-S^ny\|^{2p}.$$

Hence,

$$\frac{\|S^{n+1}x-S^{n+1}y\|^p}{\|S^nx-S^ny\|^p}\leq\frac{\|S^nx-S^ny\|^p}{\|S^{n-1}x-S^{n-1}y\|^p},$$

so the sequence is monotonically decreasing. To calculate its limit in view of the statement (3) of Proposition 2.1, divide (2.1) by $\|S^{n-1}x-S^{n-1}y\|^p$ to get

$$1-2\frac{\|S^nx-S^ny)^p}{\|S^{n-1}x-S^{n-1}y\|^p}+\frac{\|S^{n+1}x-S^{n+1}y\|^p}{\|S^nx-T^ny\|^p}\frac{\|S^nx-S^ny\|^p}{\|S^{n-1}x-S^{n-1}y\|^p}\leq 0.$$

By tanking n tend to infinity we obtain that

$$\frac{\|S^nx-S^ny\|^p}{\|S^{n-1}x-S^{n-1}y\|^p}\longrightarrow 1 \quad \text{as} \ \ n\to\infty.$$

□

Proposition 2.4. *Let S be a self map on a normed space $\mathcal{X}$. If S is an bijective $(2,p)$-concave, then , S^{-1} is an $(2,p)$-concave.*

Proof. Since S is an $(2,p)$-concave, we have

$$\|S^2x - S^2y\|^p - 2\|Sx - Sy\|^p + \|x - y\|^p \leq 0, \ \ \forall\, x, y \in \mathcal{X}.$$

Under the assumption that S is bijective, it follows by replacing x by $S^{-2}x$ and y by $S^{-2}y$ that

$$\|x - y\|^p - 2\|S^{-1}x - S^{-1}y\|^p + \|S^{-2}x - S^{-2}y\|^p \leq 0, \ \ \forall\, x, y \in \mathcal{X}.$$

Therefore S^{-1} is a $(2,p)$-concave. □

Corollary 2.2. *Let S be a self map on a normed space $\mathcal{X}$. If S is an bijective $(2,p)$-concave, then , S is isometric mapping.*

Proof. Since S is an $(2,p)$-concave , we have in view of the statement (2) of Proposition 2.1 that

$$\|Sx - Sy\|^p \geq \|x - y\|^p.$$

Moreover, since S is a bijective $(2,p)$-concave, we have S^{-1} is $(2,p)$-concave, hence

$$\|S^{-1}u - S^{-1}w\|^p \geq \|u - w\|^p \ \ \forall \ \ u, w \in \mathcal{X}.$$

Letting $u = Sx$ and $w = Sy$, this implies

$$\|Sx - Sy\|^p = \|x - y\|^p,$$

for all $x, y \in \mathcal{X}$. This means that S is isometric. □

3. Some properties of nonlinear (m,p)-isometric mapping

In this section, let $(\mathcal{X}, \|.\|)$ be a complex normad space, $S : \mathcal{X} \longrightarrow \mathcal{X}$ is a map, $m \in \mathbb{N}$ and $p \in (0, \infty)$ is a real number. We define the quantity

$$\Delta_m^p(S; x, y) := \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \|S^k x - S^k y\|^p, \tag{3.1}$$

for all $x, y \in \mathcal{X}$.

Definition 3.1. *([16]) Let $S : \mathcal{X} \longrightarrow \mathcal{X}$ be a map (not necessary linear). S is said to be an (m,p)-isometric mapping for some positive integer m and $p \in (0, \infty)$ if S satisfying*

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \|S^k x - S^k y\|^p = 0,$$

for all $x, y \in \mathcal{X}$.

Remark 3.1. (1) *A self mapping S on $\mathcal{X}$ is an $(1,p)$-isometry if*

$$\|x-y\| = \|Sx - Sy\| \ \ \forall\, x,y \in \mathcal{X}.$$

(2) *A self mapping S on $\mathcal{X}$ is an $(2,p)$-isometry if*

$$\|S^2x - S^2y\|^p - 2\|Sx - Sy\|^p + \|x-y\|^p = 0, \ \ \forall \ \ x,y \in \mathcal{X}.$$

(3) *A self mapping S on $\mathcal{X}$ is an $(3,p)$-isometry if for all $x,y \in \mathcal{X}$,*

$$\|S^3x - S^3y\|^p - 3\|S^2x - S^2y\|^p + 3\|Sx - Sy\|^p - \|x-y\|^p = 0, \ \ \forall \ \ x,y \in \mathcal{X}.$$

Remark 3.2. *The following remarks are obvious consequence of Definition 2.1.*

(1) *A $(1,p)$-isometry is an isometry and vice versa.*

(2) *Every isometric mapping is (m,p)-isometric mapping for all $m \geq 1$ and $p \in (0,\infty)$..*

Theorem 3.1. *Let S be a self map on complex normed space $\mathcal{X}$. Then following statements hold:*

(1)

$$\Delta_m^p(S;x,y) = \Delta_{m-1}^p(S;x,y) - \Delta_{m-1}^p(S;Sx,Sy), \tag{3.2}$$

for all $x,y \in \mathcal{X}$ and $m \in \mathbb{N}$.

(2) *If S is an (m,p)-isometry, then S is an (k,p)-isometry for all integer k with $k \geq m$.*

Proof. (1) In view of the standard formula $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$ for binomial coefficients, it follows that

$$\begin{aligned}
& \Delta_m^p(S;\ x,y) \\
= \ & \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \big\|S^kx - S^ky\|^p \\
= \ & \big\|x-y\big\|^p + \sum_{1\leq k\leq m-1} (-1)^k \binom{m}{k} \big\|S^kx - S^k\big\|^p + (-1)^m \big\|S^mx - S^my\big\|^p \\
= \ & \big\|x-y\big\|^p + \sum_{1\leq k\leq m-1} (-1)^k \Big(\binom{m-1}{k} + \binom{m-1}{k-1} \Big) \big\|\big(S^kx - S^ky\big\|^p + \\
& +(-1)^m \big\|S^mx - S^my\big\|^p \\
= \ & \Delta_{m-1}^p(S;\ x,y) - \Delta_{m-1}^p(S;Sx,Sy).
\end{aligned}$$

(2) The statement (2) follows from the statement (1). ∎

Lemma 3.1. *Let S be a self map on a complex normed space $\mathcal{X}$, then S is a $(2,p)$-isometry if and only if*

$$\|S^k x - S^k y\|^p - k\|Sx - Sy\|^p + (k-1)\|x-y\|^p = 0,\ \forall\, k = 0,1,\cdots, x,y \in \mathcal{X}.$$

Proof. The proof follows by using a mathematical induction, so we omit it. □

Following [10], we say that an m-isometry S is strict if $m = 1$, or $m \geq 2$ and S is not an $(m-1)$-isometry. Examples of strict m-isometries for any $m \geq 2$ are provided in [6, Proposition 8] (see also [23, Example 2.3]).

Proposition 3.1. *Let S be a self map on a complex normed $\mathcal{X}$ that is a (m,p)-isometric and a $(2,p)$-concave mapping. Then S is a $(2,p)$-isometric mapping.*

Proof. Since S is an (m,p)-isometry, $\Delta_m^p(S;\ x,y) = 0$ for all $x,y \in \mathcal{X}$. By (3.2), we have

$$\Delta_{m-1}^p(S;\ x,y) = \Delta_{m-1}^{p)}(S;\ Sx,Sy).$$

Therefore

$$\Delta_{m-1}^p(S;\ x,y) = \Delta_{m-1}^{p)}(S;\ Sx,Sy) = \Delta_{m-1}^p(S; S^n x, S^n y);$$

for $n = 1,2,....$ From the assumption that S is a $(2,p)$-concave it follows in view of the statement (2) of Proposition 2.1 that

$$\begin{cases} \|Sx - Sy\|^p - \|x-y\|^p \geq 0 \\ \\ \|x-y\|^p - 2\|Sx - Sy)^p + \|S^2x - T^2y\|^p \leq 0 \end{cases}$$

This means that the sequence $\Big(\|S^{n+1}x - S^{n+1}y\|^p - \|S^n x - S^n y\|^p\Big)_{n\geq 0}$ is monotonically non-increasing and therefore bounded so, is converges. Hence there exists a constant β such that

$$\|S^{n+1}x - S^{n+1}y\|^p - \|S^n x - S^n y\|^p \longrightarrow \beta \ \text{ as } \ n \longrightarrow \infty.$$

Note that

$$\begin{aligned} &\Delta_{m-1}^p(S; S^n x, S^n y) \\ = &\sum_{0\leq k\leq m--1} (-1)^k \binom{m-2}{k} \big[\|S^{n+k}x - S^{n+k}\|^p - \|S^{n+1+k}x - S^{n+1+k}y\|^p\big]. \end{aligned}$$

Letting $n \to \infty$ in the preceding equality leads to

$$\Delta_{m-1}^p(S; S^n x, S^n y) \to \sum_{0\leq k\leq m-2} (-1)^k \binom{m-2}{k} \beta = 0.$$

We obtain that $\Delta_{m-1}^p(S;x,y) = 0$. Applying the corresponding results of the $(m-1,p)$-isometry and $(2.p)$-concave mapping, we obtain that $\Delta_{m-2}^p(S;x,y) = 0$. Continue these processes we get $\Delta_2^p(S;x,y) = 0$. Consequently, S is a $(2,p)$-isometric mapping. □

Proposition 3.2. *Let S be a self mapping on a complex normed space $\mathcal{X}$ that is a contractive $\big(\|Sx - Sy\| \leq \|x - y\|,\ \forall\ x, y \in \mathcal{X}\big)$. In addition If S is an (m,p)-isometry then S is an $(m-1,p)$-isometry for $m \geq 2$.*

Proof. Under the assumption that S is a contractive mapping, it follows that,

$$\|S^{n+1}x - S^{n+1}y\|^p \leq \|S^n x - S^n y\|^p,\ \ \forall\ x, y \in \mathcal{X}\ \text{ and }\ n \in \mathbb{N}.$$

This means that $\big(\|S^n x - S^n y\|^p\big)_{n\in\mathbb{N}}$ is deceasing sequence, so convergent.

From the fact that S is an (m,p)-isometry and together (**??**), we get

$$\Delta^p_{m-1}(S; x, y) = \Delta^p_{m-1}(S; Sx, Sy) = ... = \Delta^p_{m-1}(S; S^n x, S^n y).$$

On the other hand, we have

$$\Delta^p_{m-1}(S; S^n x, S^n y) = \Delta^P_{m-2}(S; S^n x, S^n y) - \Delta^p_{m-2}(S; S^{n+1}x, S^{n+1}y),$$

so that

$$\begin{aligned} &\Delta^p_{m-1}(S; S^n x, S^n y) \\ = &\sum_{j=0}^{m-2} (-1)^j \binom{m-2}{j} \Big[\|S^{n+j}x - S^{n+j}y\|^p - \|S^{n+1+j}x - S^{n+1+j}y\|^p\Big]. \end{aligned}$$

By taking the limit $n \to \infty$ in the preceding equality leads to

$$\Delta^p_{m-1}(S; S^n x, S^n y) to 0.$$

Consequently,, $\Delta^p_{m-1}(S; x, y) = 0$ and hence, T is an $(m-1,p)$-isometry. □

As a consequence of this proposition, we have the following corollary.

Corollary 3.1. *Let S be a self mapping on a complex normed space $\mathcal{X}$ which is a contractive. Then S is an (m,p)-isometry if and only if S is an isometry.*

Lemma 3.2. *Let S be a self mapping on a complex normed space $\mathcal{X}$. Then S is an $(2,\ p)$-isometry if and only if*

$$\|S^n x - S^n y\|^p - \|S^{n-1}x - S^{n-1}y\|^p = \|Sx - Sy\|^p - \|x - y\|^p,$$

for all integer $n \geq 2$ and $x, y \in \mathcal{X}$.

Proof. The proof is obvious by mathematical induction. □

The author in [21] show that if S is a 2-isometric operator on a Hilbert space, then S^n is a 2-isometric operator.

Theorem 3.2. *Let S be a self map on complex normed space $\mathcal{X}$. if S is an $(2,p)$-isometry, then so is T^n for all $n \in \mathbb{N}$.*

Proof. We will induct on n, the result obviously holds for $n = 1$. Suppose then the assertion holds for $n \geq 2$, i.e

$$\|S^{2n}x - S^{2n}y\|^p - 2\|S^nx - S^ny\|^p + \|x-y\|^p = 0,\ \forall\, x, y \in \mathcal{X}.$$

Then

$$\begin{aligned}
& \|S^{2n+2}x - S^{2n+2}y\|^p - 2\|S^{n+1}x - S^{n+1}y|^p + \|x-y\|^p \\
= \ & \|S^2S^{2n}x - S^2S^{2n}y\|^p - 2\|S^{n+1}x - S^{n+1}y\|^p + \|x-y\|^p \\
= \ & 2\|S^{2n+1}x - S^{2n+1}y\|^p - \|S^{2n}x - S^{2n}y\|^p \\
& -2\|S^{n+1}x - S^{n+1}y\|^p + \|x-y\|^p \\
= \ & 2\big(2\|S^{n+1}x - S^{n+1}y\|^p - \|Sx - Sy\|^p\big) - \|S^{2n}x - S^{2n}y\|^p \\
& -2\big(\|S^{n+1}x - S^{n+1}y\|^p + \|x-y\|^p \\
= \ & 2\|S^{n+1}x - S^{n+1}y\|^p - \|S^{2n}x - S^{2n}y\|^p \\
& -2\|Sx - Sy\|^p + \|x-y\|^p \\
= \ & 2\|S^{n+1}x - S^{n+1}y\|^p - (2\|S^nx - S^ny\|^p - \|x-y\|^p) \\
& -2\|Sx - Sy\|^p + \|x-y\|^p \\
\leq \ & 2\|S^{n+1}x - S^{n+1}y\|^2 - 2\|S^nx - S^ny\|^p - 2\|Sx - Sy\|^p + 2\|\|x-y\|^p \\
= \ & 2\big(\|Sx - Sy\|^p - \|x-y\|^p\big) - 2\|Sx - Sy\|^p + 2\|x-y\|^p \quad \text{(by Lemma3.2)} \\
= \ & 0.
\end{aligned}$$

Thus means that S^n is $(2,p)$-isometric mapping. $\square$

The following theorem gives a characterization of $(3,p)$-isometric mappings.

Theorem 3.3. *Let S be a self mapping for a normed space $\mathcal{X}$. Then S is an $(3,p)$-isometric mapping if and only if S satisfies*

$$\|S^nx - S^ny\|^p = \|x-y\|^p + n\Psi_1(S,x,y) + n^2\Psi_2(S,x,y) \tag{3.3}$$

where

$$\Psi_2(S,x,y) = \frac{1}{2}\Big(\|S^2x - S^2y\|^p - 2\|Sx - Sy\|^p + \|x-y\|^p\Big) \tag{3.4}$$

and

$$\Psi_1(S,x,y) = \frac{1}{2}\Big(-\|S^2x - S^2y\|^p + 4\|Sx - Sy\|^p - 3\|x-y\|^p\Big) \tag{3.5}$$

Proof. We prove the if part of the theorem. Assume that S satisfies (3.3). For $n=3$ we obtain

$$\begin{aligned}
&\big\|S^3x-S^3y\big\|^p\\
=\;&\big\|x-y\big\|^p+3\Psi_1(S,x,y)+9\Psi_2(S,x,y)\\
=\;&\big\|x-y\big\|^p+\frac{3}{2}\Big(-\big\|S^2x-S^2y\big\|^p+4\big\|Sx-Sy\big\|^p-3\big\|x-y\big\|^p\Big)\\
&+\frac{9}{2}\Big(\big\|S^2x-S^2y\|^p-2\big\|Sx-Sy\big\|^p+\big\|x-y\big\|^p\Big)\\
=\;&\big\|x-y\|^p-3\big\|S^2x-S^2y\big\|^p-3\big\|Sx-Sy\big\|^p.
\end{aligned}$$

It follows that

$$\big\|S^3x-S^3y\|^p-3\big\|S^2x-S^2y\big\|^p+3\big\|Sx-Sy\big\|^p-\big\|x-y\big\|^p=0,$$

so that, S is an $(3,p)$–isometry.

We prove the only if part. Assume that S is an $(3,p)$-isometry. We prove (3.3) by mathematical induction. For $n=1$ it is true. Assume that (3.3) is true for n and prove it for $n+1$. Indeed, for all $x,y\in\mathcal{X}$ we have

$$\begin{aligned}
&\big\|S^{n+1}x-S^{n+1}y\big\|^p\\
=\;&\big\|S^nSx-S^nSy\big\|^p\\
=\;&\big\|Sx-Sy\|^p+n\Psi_1(S,Sx,Sy)+n^2\Psi_2(S,Sx,Sy)\\
=\;&\big\|Sx-Sy\big\|^p+\frac{n}{2}\Big(-\big\|S^3x-S^3y\big\|^p+4\big\|S^2x-S^2y\big\|^p-3\big\|Sx-Sy\big\|^p\Big)\\
&+\frac{n^2}{2}\Big(\big\|S^3x-S^3y\big\|^p-2\big\|S^2x-S^2y\big\|^p+\big\|Sx-Sy\big\|^p\Big)\\
=\;&\Big(\frac{n^2-n}{2}\Big)\big\|S^3x-S^3y\big\|^p-\big(n^2-2n\big)\big\|S^2-S^2y\big\|^p\\
&+\Big(\frac{n^2-3n+2}{2}\Big)\big\|Sx-Sy\big\|^p.
\end{aligned}$$

Now, using the fact that T is an $(3,p)$-isometry we can obtained

$$\begin{aligned}
&\big\|S^{n+1}x-S^{n+1}y\big\|^p\\
=\;&\Big(\frac{n^2-n}{2}\Big)\Big(\big\|x-y\big\|^p+3\big\|S^2x-S^2y\big\|^p-3\big\|Sx-Sy\big\|^p\Big)\\
&+-\big(n^2-2n\big)\big\|S^2x-S^2y\big\|^p\\
&+\Big(\frac{n^2-3n+2}{2}\Big)\big\|Sx-Sy\big\|^p\\
=\;&\Big(\frac{n^2+n}{2}\Big)\big\|S^2x-s^2y\big\|^p+\Big(\frac{-2n^2+2}{2}\Big)\big\|Sx-Sy\big\|^p\\
&+\Big(\frac{n^2-n}{2}\Big)\big\|x-y\big\|^p
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{n^2+n}{2}\right)\Big(\big\|x-y\big\|^p + 2\Psi_1(S,x,y) + 4\Psi_2(S,x,y)\Big) \\
&+ \left(\frac{-2n^2+2}{2}\right)\Big(\big\|x-y\big\|^p + \Psi_1(S,x,y) + \Psi_2(S,x,y)\Big) + \left(\frac{n^2-n}{2}\right)\big\|x-y\big\|^p \\
= \quad & \big\|x-y\big\|^p + (n+1)\Psi_1(S,x,y) + (n+1)^2\Psi_2(S,x,y).
\end{aligned}
$$

□

Theorem 3.4. *Let T,S be a self mappings on a complex normed space $\mathcal{X}$ such that $TS = ST$. The following properties hold for all $x,y \in \mathcal{X}$.*

(1) *If S is an $(1,p)$-isometry, then*

$$
\Delta_m^p(TS;x,y) = \Delta_m^p(T;x,y). \tag{3.6}
$$

(2) *If S is a $(2.p)$-isometry, then*

$$
\begin{aligned}
\Delta_{m+1}^p(TS;\ x,y) \quad &= \quad (m+1)\Delta_m^p(T;\ TSx,TSy) - (m+1)\Delta_m^p(T,\ .Tx,Ty) \\
&\quad +\Delta_{m+1}^p(T;\ x,y).
\end{aligned} \tag{3.7}
$$

(3) *If S is an $((3,p)$-isometry, then*

$$
\begin{aligned}
\Delta_{m+2}(TS;\ x,y) \quad &= \quad \Delta_{m+2}(T;\ x,y) + \frac{(m+2)(m+1)}{2}\Big[\Delta_m^p(T;\ T^2S^2x,T^2S^2y) \\
&\quad +\Delta_m^p(T;\ T^2Sx,T^2Sy) + \Delta_m^p(T;\ T^2x,T^2y)\Big] \\
&\quad +\frac{5(m+2)}{2}\Delta_{m+1}(T;\ TSx,TSy) - \frac{1}{2}\Delta_m(T;\ Tx,Ty).
\end{aligned} \tag{3.8}
$$

Proof. (1) From the fact that S is an $(1,p)$-isometry,i.e.;

$$
\big\|S^kx - S^ky\big\|^p = \big\|x-y\big\|^p \ \ \forall \ \ x,u \in \mathcal{X},
$$

and the fact $TS = ST$ it follows by elementary calculation that

$$
\begin{aligned}
\Delta_m^p(TS;\ x,y) \quad &= \quad \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}\big\|(TS)^kx - (TS)^ky\big\|^p \\
&= \quad \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}\big\|S^kT^k - S^kT^ky\big\|^p \\
&= \quad \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}\big\|T^kx - T^ky\big\|^p \\
&= \quad \Delta_m^p(T;\ x,y).
\end{aligned}
$$

(2) Assume that S is an $(2,p)$-isometry. Then we have by using [19, Lemma3.4] that

$$
\big\|S^kx - S^ky\big\|^p = k\big\|Sx - Sy\big\|^p + (1-k)\big\|x-y\big\|^p, \ \ \forall\, x,y \in \mathcal{X},\ k = 0,1,\cdots.
$$

A simple calculation shows that

$$\begin{aligned}
& \beta^A_{m+1}(TS;\ x,y) \\
= & \sum_{0\le k\le m+1} (-1)^{m+1-k}\binom{m+1}{k}\big\|(TS)^k x-(TS)^k y\big\|^p \\
= & \sum_{0\le k\le m+1} (-1)^{m+1-k}\binom{m+1}{k}\big\|S^kT^k-S^kT^k y\big\|^p \\
= & \sum_{0\le k\le m+1} (-1)^{m+1-k}\binom{m+1}{k}\Big[k\big\|ST^kx-ST^ky\big\|^p+(1-k)\big\|T^kx-T^ky\big\|^p\Big] \\
= & \sum_{1\le k\le m+1} (-1)^{m+1-k}k\binom{m+1}{k}\big\|T^kSx-T^kSy\big\|^p \\
& +\sum_{0\le k\le m+1} (-1)^{m+1-k}\binom{m+1}{k}(-k+1)\big\|T^kx-T^ky\big\|^p \\
= & (m+1)\Bigg(\sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k}\big\|T^{k+1}Sx-T^{k+1}Sy\big\|^p\Bigg)-(m+1)\Delta^p_m(T;\ Tx,Ty) \\
& +\Delta^p_{m+1}(T;\ x,y) \\
= & (m+1)\Delta^A_m(T;\ TSx,TSy)-(m+1)\Delta^p_m(T;\ Tx,Ty)+\Delta^p_{m+1}(T;\ x,y).
\end{aligned}$$

(3) Assume that S is an $(A,3)$-isometry and $TS=ST$. In view of Theorem 3.3 we have that

$$\begin{aligned}
\beta^A_{m+2}(TS) &= \sum_{0\le k\le m+2}(-1)^{m+2-k}\binom{m+2}{k}\big\|(ST)^kx-(ST)^ky\big\|^p \\
&= \sum_{0\le k\le m+2}(-1)^{m+2-k}\binom{m+2}{k}\big\|S^kT^kx-S^kT^ky\big\|^p \\
&= \sum_{0\le k\le m+2}(-1)^{m+2-k}\binom{m+2}{k}\Big[\big\|T^kx-T^ky\big\|^p+k\Psi_1(S,T^kx,T^y)+k^2\Psi_2(S,T^kx,T^y)\Big] \\
&= \underbrace{\Bigg\{\sum_{0\le k\le m+2}(-1)^{m+2-k}\binom{m+2}{k}\big\|T^kx-T^ky\big\|^p+}_{I} \\
&\quad \underbrace{\sum_{0\le k\le m+2}(-1)^{m+2-k}\binom{m+2}{k}k\Psi_1(S,T^kx,T^ky)+}_{J} \\
&\quad \underbrace{\sum_{0\le k\le m+2}(-1)^{m+2-k}\binom{m+2}{k}k^2\Psi_2(S,T^kx,T^ky)\Bigg\}}_{K}.
\end{aligned}$$

Clearly $I = \Delta^p_{m+2}(T;\ x,y)$.

$$\begin{aligned}
J &= \sum_{0\le k\le m+2} (-1)^{m+2-k}\binom{m+2}{k} k\Psi_1(S,T^kx,T^ky)\\
&= \sum_{0\le k\le m+2} (-1)^{m+2-k}\binom{m+2}{k} k\frac{1}{2}\Big(-\big\|S^2T^kx - S^2T^ky\big\|^p + 4\big\|ST^kx - ST^ky\big\|^p - 3\big\|T^kx - T^ky\|^p\Big)\\
&= -\frac{1}{2}(m+2)\Delta_{m+1}(T;\ TS^2x, TS^2y) + 2(m+2)\Delta_{m+1}(T;\ TSx,TSy) - \frac{3}{2}\Delta_{m+1}(T;\ Tx,Ty).
\end{aligned}$$

$$\begin{aligned}
K &= \sum_{0\le k\le m+2} (-1)^{m+2-k}\binom{m+2}{k} k^2\Psi_2(S,T^kx,T^ky)\\
&= \sum_{0\le k\le m+2} (-1)^{m+2-k}\binom{m+2}{k} k^2\frac{1}{2}\Big(\big\|S^2T^kx - S^2T^ky\big\|^p - 2\big\|ST^kx - ST^ky\|^p + \big\|T^kx - T^ky\|^p\Big)\\
&= \sum_{0\le k\le m+2} (-1)^{m+2-k}\binom{m+2}{k} k^2\frac{1}{2}\Big(\big\|T^kS^2x - T^kS^2y\big\|^p - 2\big\|T^kSx - T^kSy\|^p + \big\|T^kx - T^ky\|^p\Big)\\
&= (-1)^{m+1}(m+2)\Big(\big\|T^kS^2x - T^kS^2y\big\|^p - 2\big\|T^kSx - T^kSy\|^p + \big\|T^kx - T^ky\|^p\Big)\\
&\quad + \sum_{2\le k\le m+2} (-1)^{m+2-k}\binom{m+2}{k} k^2\frac{1}{2}\Big(\big\|T^kS^2x - T^kS^2y\big\|^p - 2\big\|T^kSx - T^kSy\|^p + \big\|T^kx - T^ky\|^p\Big)
\end{aligned}$$

By observing that $k^2 = k(k-1)+k$ and

$$k^2\binom{m+2}{k} = (m+2)(m+1)\binom{m}{k-2} + (m+2)\binom{m+1}{k-1},\ ;k\ge 2.$$

$$\begin{aligned}
K &= \frac{(m+2)(m+1)}{2}\Big[\Delta^p_m(T;\ T^2S^2x,T^2S^2y) + \Delta^p_m(T;\ T^2Sx,T^2Sy) + \Delta^p_m(T;\ T^2x,T^2y)\Big]\\
&\quad + \frac{(m+2)}{2}\Big[\Delta^p_{m+1}(T;\ TS^2x,TS^2y) + \Delta^p_{m+1}(T;\ TSx,TSy) + \Delta^p_{m+1}(T;\ Tx,Ty)\Big].
\end{aligned}$$

By combining I, J and K we obtain

$$\begin{aligned}
\Delta_{m+2}(TS;\ x,y) &= \Delta_{m+2}(T;\ x,y) + \frac{(m+2)(m+1)}{2}\Big[\Delta^p_m(T;\ T^2S^2x,T^2S^2y) + \Delta^p_m(T;\ T^2Sx,T^2Sy)\\
&\quad + \Delta^p_m(T;\ T^2x,T^2y)\Big] + \frac{5(m+2)}{2}\Delta_{m+1}(T;\ TSx,TSy) - \frac{1}{2}\Delta_m(T;\ Tx,Ty).
\end{aligned}$$

This completes the proof of the theorem. □

The proof of the following corollary follows by Combing Theorem 3.4 and Theorem 3.1.

Corollary 3.2. *Let T,S be a self mappings on a complex normed space $\mathcal{X}$ such that $TS = ST$. If T is an (m,p)-isometry and S is an (n,p)-isometry for $n\in\{1,2,3\}$, then TS is an $(m+n-1,p)$-isometry for $n\in\{1,2,3\}$.*

Theorem 3.5. *Let S be a self map on a complex normed space $\mathcal{X}$. If S is bijective (m,p)-isometry, then S^{-1} is an (m,p)-isometry.*

Proof. Since S is an (m,p)-isometry it follows that $\Delta_m^p(S;\ x,y) = 0;\ \ \forall\, x,y \in \mathcal{X}$. Taking into account the fact that S is a bijective map we get by direct calculation

$$\begin{aligned} 0 &= \Delta_m^p(S;\ S^{-m}x, S^{-m}y) \\ &= \sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k}\left\|S^{k-m}x - S^{k-m}y\right\|^p \\ &= (-1)^m \Delta_m^p(S^{-1};\ x,y). \end{aligned}$$

Therefore S^{-1} is an (m,p)-isometry. □

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