



ANALYSIS OF QUASISTATIC FRICTIONAL CONTACT PROBLEM WITH SUBDIFFERENTIAL FORM, UNILATERAL CONDITION AND LONG-TERM MEMORY

A. OURAHMOUN^{1,*}, B. BOUDERAH¹, T. SERRAR²

¹University of M'sila 28000, M'sila LMPA Laboratory, Algeria

²University of Setif 1, 19000, Algeria

*Corresponding author: ourahmounabbes@yahoo.fr

ABSTRACT. We consider a quasistatic problem which models the contact between a deformable body and an obstacle called foundation. The material is assumed to have a viscoelastic behavior that we model with a constitutive law with long-term memory, thus at each moment of time, the stress tensor depends not only on the present strain tensor, but also on its whole history. In Contact Mechanics, history-dependent operators could arise both in the constitutive law of the material and in the frictional contact conditions. The mathematical analysis of contact models leads to the study of variational and hemivariational inequalities. For this reason a large number of contact problems lead to inequalities which involve history dependent operators, called history dependent inequalities. Such inequalities could be variational or hemivariational and variational hemivariational.

In this paper we derive a weak formulation of the problem and, under appropriate regularity hypotheses, we establish an existence and uniqueness result. The proof of the result is based on arguments of variational inequalities monotone operators and Banach fixed point theorem.

1. INTRODUCTION

Contact mechanics still remain a rich domain of research, and the literature devoted to various aspects of the subject is growing. An early attempt at the study of contact problems for elastic viscoelastic materials

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within the mathematical analysis framework was introduced in the pioneering reference works [6, 7, 15]. Further extensions to non convex contact conditions with non-monotone and possible multi-valued constitutive laws led to the active domain of non-smooth mechanics within the framework of the so-called hemivariational inequalities, for a mathematical as well as mechanical treatment we refer to [10]. There is a growing interest in the study of history-dependent inequalities. For instance, a class of variational inequalities with history dependent operators was considered in [15], where abstract existence, uniqueness and regularity results were proved. These results were extended in [18] to a more general class of variational inequalities and were completed in [6] with error estimate and convergence results. Various results on hemivariational and variational-hemivariational inequalities with history dependent operators, formulated in Sobolev-type spaces, could be found in [7, 9].

We introduce a new model of frictional contact for viscoelastic materials and to illustrate the use of history dependent variational hemivariational inequality in its variational analysis. Thus, in Section 2 we introduce the contact problem, in which the material's behavior is modeled by a nonlinear viscoelastic constitutive law with long memory, the process is quasistatic, the contact is frictional and the contact conditions are in a subdifferential form with unilateral conditions for the displacement. Then, in Section 3 we list the assumptions on the data and derive the variational formulation of the problem. It is in a form of a history-dependent variational-hemivariational inequality in which the unknown is the displacement field. Next in Section 4 we state our main existence and uniqueness result, Theorem (4.2) the proof of the theorem is obtained by using arguments of elliptic variational-hemivariational inequalities and a fixed point result for history dependent operators.

2. The Contact Model

The physical setting we consider is the following. A deformable body occupies a domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$ in applications) with outer Lipschitz surface Γ that is divided into three disjoint measurable parts Γ_i ($i = 1, 2, 3$) such that $meas(\Gamma_1) > 0$. Let $[0, T]$ be the time interval of interest, where $T > 0$. The body is clamped on $\Gamma_1 \times (0, T)$ and therefore the displacement field vanishes there. A volume force of density f_0 acts in $\Omega \times (0, T)$ and surface tractions of density f_1 act on $\Gamma_2 \times (0, T)$.

The body is in contact on $\Gamma_3 \times (0, T)$ with a rigid obstacle, the so-called foundation is in frictional contact. We assume that the process is quasistatic with long term memory and we use (1) as constitutive law. We denote by \mathbf{u} , $\boldsymbol{\sigma}$ and $\varepsilon(\mathbf{u})$ the displacement field, the stress field and the linearized strain tensor, respectively, and let \mathbf{v} be the unit outward normal vector to Γ . Here and below, we sometimes do not indicate explicitly the dependence of various functions on the spatial variable $\mathbf{x} \in \Omega \cup \Gamma$. For a vector field \mathbf{u} , we use notation $\mathbf{u}_v = \mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u}_\tau = \mathbf{u} - \mathbf{u}_v \mathbf{v}$ for the normal and tangential components of \mathbf{u} on Γ . Similarly, for the stress

field σ , its normal and tangential components on the boundary are defined by equalities $\sigma_\nu = (\sigma\nu)\cdot\nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$, respectively.

Finally, we use \mathcal{S}^d for the space of second order symmetric tensors on \mathbb{R}^d and “.” will represent the canonical inner product and the Euclidean norm on the spaces \mathbb{R}^d and \mathcal{S}^d , respectively. We also use the following notation:

$$H = (L^2(\Omega))^d, \mathcal{H} = \{\sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega), 1 \leq i \leq j \leq d\}$$

$$H_1 = \{u \in H : \varepsilon(u) \in \mathcal{H}\} \quad ; \quad \mathcal{H}_1 = \{\sigma \in \mathcal{H} \mid Div\sigma \in H\}$$

Here $\varepsilon : H_1 \rightarrow H$ and $Div : \mathcal{H}_1 \rightarrow H$ are the deformation and the divergence operators, respectively, defined by:

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(u)) \quad ; \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad ; \quad Div\sigma = (\sigma_{ij,j})$$

The spaces H, \mathcal{H}, H_1 and \mathcal{H}_1 are real Hilbert spaces endowed with the canonical inner products given by:

$$(u, v)_H = \int_{\Omega} u_i \cdot v_i dx \quad , \quad (\sigma, \tau)_{\mathcal{H}} = \int_{\Omega} \sigma_{ij} \cdot \tau_{ij} dx$$

$$(u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}$$

$$(\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_H$$

We recall that C denotes the class of continuous functions; and $C^m, m \in \mathbb{N}^*$ the set of m times continuously differentiable functions.

Finally $D(\Omega)$ denotes the set of infinitely differentiable real functions with compact support in Ω ; and $W^{m,p}, m \in \mathbb{N}, 1 \leq p \leq +\infty$ for the classical Sobolev spaces; and

$$H_0^m(\Omega) := \{w \in W^{m,2}(\Omega), w = 0 \text{ on } \Gamma\}, m \geq 1.$$

With these assumptions, the classical formulation or mathematical model which describes the equilibrium of the body in the physical setting above is the following.

Problem P. Find a displacement field $\mathbf{u} : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$, a stress field $\sigma : \Omega \times \mathbb{R}_+ \rightarrow \mathcal{S}^d$ and two interface forces $\eta_\nu : \Gamma_3 \times \mathbb{R}_+ \rightarrow \mathbb{R}$ and $\xi_\nu : \Gamma_3 \times \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$\sigma(t) = \mathcal{A}\varepsilon(u(t)) + \int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds \quad \text{in } \Omega \times (0, T) \tag{2.1}$$

$$Div(\sigma(t)) + f_0(t) = 0 \quad \text{in } \Omega \times (0, T) \tag{2.2}$$

$$u(t) = 0 \quad \text{on } \Gamma_1 \times (0, T) \tag{2.3}$$

$$\sigma(t)v = f_2(t) \quad \text{on } \Gamma_2 \times (0, T) \tag{2.4}$$

$$\left\{ \begin{array}{l} u_v(t) \leq g, \sigma_v(t) + \xi_v(t) + \eta_v(t) \leq 0 \\ (u_v(t) - g)(\sigma_v(t) + \xi_v(t) + \eta_v(t)) = 0 \\ |u_v(t)| \leq F_m \left(\int_0^t u_v^+(s) ds \right) \\ \eta_v(t) = \begin{cases} 0 & \text{if } u_v(t) < 0 \\ F_m \left(\int_0^t u_v^+(s) ds \right) & \text{if } u_v(t) \geq 0 \end{cases} \\ \xi_v(t) \in \partial j_v(u_v(t)) \quad \text{on } \Gamma_3 \times (0, T) \end{array} \right. \quad \text{on } \Gamma_3 \times (0, T) \tag{2.5}$$

$$\left\{ \begin{array}{l} \|\sigma_\tau(t)\| \leq F_b(u_v(t)) \\ -\sigma_\tau(t) = F_b(u_v(t)) \frac{u_\tau(t)}{\|u_\tau(t)\|} \quad \text{if } u_\tau(t) \neq 0 \end{array} \right. \quad \text{on } \Gamma_3 \times (0, T) \tag{2.6}$$

First, Eq.(2.1) is the constitutive law for viscoelastic materials in which \mathcal{A} represent the elasticity operator and \mathcal{B} represents the relaxation tensor. Various comments and mechanical interpretation related to such kind of equations could be found in [8, 16]. Equation (2.2) is the equilibrium equation that we use here since we assume that the process is quasistatic. Conditions (2.3) and (2.4) represent the displacement and traction conditions, respectively. Condition (2.5) represents the contact condition in which $g > 0$, j_v and F_m are given functions and ∂j_v represents the Clarke subdifferential of j_v . Finally, relations (2.7) represent the static version of Coulomb’s law of dry friction. Here F_b denotes a positive function, the friction bound assumed to depend on the normal displacement u_v . The contact condition (2.5) represents the trait of novelty of our model. Note that this condition models the contact with a foundation made of a rigid body covered by a layer made of soft material and a thin crust with memory effects.

3. Variational analysis

To derive a variational formulation of the problem we use the spaces for the displacement field we use the space

$$V = \{v = (v_i) \in H^1(\Omega) \mid v = 0 \quad \text{on } \Gamma_1\} \tag{3.1}$$

which is a real Hilbert space with inner product $(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}$ where

$$(u, v)_V = \int_{\Omega} \varepsilon(u) \cdot \varepsilon(v) dx$$

and associated norm $\|\cdot\|_V$.

we consider the space of fourth-order tensor fields

$$Q_{\infty} = \left\{ \mathcal{E} = (\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^{\infty}(\Omega) \right\}$$

which is a real Banach space with norm

$$\|\mathcal{E}\|_{Q_{\infty}} = \max_{0 \leq i, j, k, l \leq d} \|\mathcal{E}_{ijkl}\|_{L^{\infty}(\Omega)}$$

Finally, we use \mathbb{N} for the set of positive integers and \mathbb{R}_+ for the set of nonnegative real numbers. For a normed space X , we use the notation $C(\mathbb{R}_+; X)$ for the space of continuous functions defined on \mathbb{R}_+ with values in X .

By the Sobolev trace theorem, we have

$$\|v\|_{L^2(\Gamma_3, \mathbb{R}^d)} \leq \|\gamma\| \|v\|_V, \quad \forall v \in V \tag{3.2}$$

$\|\gamma\|$ being the norm of the trace operator $\gamma : V \rightarrow L^2(\Gamma_3, \mathbb{R}^d)$

We now list the assumptions on the data and we assume that

1. the elasticity operator $\mathcal{A} : \Omega \times \mathcal{S}^d \rightarrow \mathcal{S}^d$ satisfies the following properties

$$\left\{ \begin{array}{l} \text{(a) There exists } L_{\mathcal{A}} > 0 \text{ such that for all } \varepsilon_1, \varepsilon_2 \in \mathcal{S}^d, \text{ a.e. } x \in \Omega, \\ \qquad \qquad \qquad \|\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)\| \leq L_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2| \\ \text{(b) There exists } m > 0 \text{ such that for all } \varepsilon_1, \varepsilon_2 \in \mathcal{S}^d, \text{ a.e. } x \in \Omega, \\ \qquad \qquad \qquad (\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m \|\varepsilon_1 - \varepsilon_2\|^2 \\ \qquad \qquad \qquad \mathcal{A}(\cdot, \varepsilon_1) \text{ is measurable on } \Omega \text{ for all } \varepsilon \in \mathcal{S}^d \\ \qquad \qquad \qquad \mathcal{A}(x, 0) = 0 \text{ for a.e. } x \in \Omega \end{array} \right. \tag{3.3}$$

2. The relaxation tensor \mathcal{B} is such that

$$\mathcal{B} \in C(\mathbb{R}_+, Q_{\infty}) \tag{3.4}$$

3. the potential function $j_v : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}$, assumed to satisfy the following conditions

$$\left\{ \begin{array}{l} \text{(a) } j_v(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there} \\ \text{exists } \bar{e} \in L^2(\Gamma_3) \text{ such that } j_v(\cdot, \bar{e}) \in L^1(\Gamma_3) \\ \text{(b) } j_v(x, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } x \in \Gamma_3 \\ \text{(c) } |j_v(x, r)| \leq \bar{c}_0 + \bar{c}_1 |r| \text{ for a.e. } x \in \Gamma_3 \text{ and} \\ \text{for all } r \in \mathbb{R} \text{ with } \bar{c}_0, \bar{c}_1 \geq 0 \end{array} \right. \quad (3.5)$$

Next, we assume that the penetration bound $g : \Gamma_3 \rightarrow \mathbb{R}$, the memory function $F_m : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+$ and the friction bound $F_b : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}$ are assumed to satisfy the following conditions.

$$g \in L^2(\Gamma_3) , \quad g(x) \geq 0 \text{ a.e. on } \Gamma_3. \quad (3.6)$$

And

$$\left\{ \begin{array}{l} \text{(a) There exists } L_{F_m} > 0 \text{ such that } |F_m(x, r_1) - F_m(x, r_2)| \leq L_{F_m} |r_1 - r_2| \\ \text{for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3 \\ \text{(b) } F_m(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \\ \text{(c) } x \rightarrow F_m(x, 0) \in L^2(\Gamma_3) \end{array} \right. \quad (3.7)$$

$$\left\{ \begin{array}{l} \text{(a) There exists } L_{F_b} > 0 \text{ such that } |F_b(x, r_1) - F_b(x, r_2)| \leq L_{F_b} |r_1 - r_2| \\ \text{for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3 \\ \text{(b) } F_b(\cdot, r) \text{ is measurable on } \Gamma_3, \text{ for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3 \\ \text{(c) } F_b(x, r) = 0 \text{ for } r \leq 0, F_b(x, r) > 0 \text{ for all } r > 0 \text{ a.e. } x \in \Gamma_3 \end{array} \right. \quad (3.8)$$

We also assume that the densities of body forces and surface tractions have the regularity

$$f_0 \in C(\mathbb{R}_+; L^2(\Omega; \mathbb{R}^d)) , \quad f_2 \in C(\mathbb{R}_+; L^2(\Gamma_2; \mathbb{R}^d)) \quad (3.9)$$

and, finally, we assume the smallness condition

$$L_{F_b} \|\gamma\| + \alpha j_v < m_F \quad (3.10)$$

We now introduce the set of the admissible displacement fields $U \subset V$ and the function $f : \mathbb{R}_+ \rightarrow V'$ defined by

$$\left\{ \begin{array}{l} U = \{ \mathbf{v} \in V \mid \mathbf{v}_v \leq g \text{ on } \Gamma_3 \} \\ \langle f(t), \mathbf{v} \rangle_{V' \times V} = (f_0(t), \mathbf{v})_{(L^2(\Omega), \mathbb{R}^d)} + (f_2(t), \mathbf{v})_{(L^2(\Gamma_3), \mathbb{R}^d)}, \text{ for all } \mathbf{v} \in \mathbf{V}, t \in \mathbb{R}_+ \end{array} \right. \quad (3.11)$$

Assume now that $(\mathbf{u}, \boldsymbol{\sigma})$ represents a couple of regular functions which satisfy (2.1) – (2.6) and let $t \in \mathbb{R}_+$, $\mathbf{v} \in U$. We perform an integration by parts, split the surface integral on three integrals on Γ_1 , Γ_2 and Γ_3 , and use the equalities (2.2) – (2.4) to deduce that

$$\begin{aligned} \int_{\Omega} \boldsymbol{\sigma}(t) \cdot (\varepsilon(\mathbf{v}) - \varepsilon(u(t))) dx &= \int_{\Omega} f_0(t) \cdot (\mathbf{v} - u(t)) dx \\ &+ \int_{\Gamma_2} f_2(t) \cdot (\mathbf{v} - u(t)) d\Gamma + \int_{\Gamma_3} \sigma_v(t) (\mathbf{v}_v - u_v(t)) d\Gamma + \int_{\Gamma_3} \sigma_\tau(t) (\mathbf{v}_\tau - u_\tau(t)) d\Gamma \end{aligned} \tag{3.12}$$

Next, we use the contact boundary condition (2.5), the definition (3.12) and the definition of the Clarke subdifferential to obtain that

$$\begin{aligned} \int_{\Gamma_3} \sigma_v(t) (\mathbf{v}_v - u_v(t)) d\Gamma + \int_{\Gamma_3} F_m \left(\int_0^t u_v^+(s) ds \right) (\mathbf{v}_v^+ - u_v^+(t)) d\Gamma \\ + \int_{\Gamma_3} j_v^0(u_v(t), \mathbf{v}_v - u_v(t)) d\Gamma \geq 0 \end{aligned} \tag{3.13}$$

Note that here and below we use notation $j_v^0(r_1; r_2)$ for the generalized directional derivative of j_v at r_1 in the direction r_2 , see [1, 2] for details.

On the other hand, the friction law (2.6) yields

$$\int_{\Gamma_3} \sigma_\tau(t) (\mathbf{v}_\tau - u_\tau(t)) d\Gamma + \int_{\Gamma_3} F_b(u_v(t) (\|v_\tau\| - \|u_\tau(t)\|)) d\Gamma \geq 0 \tag{3.14}$$

We now combine equality (3.13) with inequalities (3.14), (3.15) to deduce that

$$\begin{aligned} \int_{\Omega} \boldsymbol{\sigma}(t) \cdot (\varepsilon(\mathbf{v}) - \varepsilon(u(t))) dx + \int_{\Gamma_3} F_b(u_v(t) (\|v_\tau\| - \|u_\tau(t)\|)) d\Gamma + \\ \int_{\Gamma_3} F_m \left(\int_0^t u_v^+(s) ds \right) (\mathbf{v}_v^+ - u_v^+(t)) d\Gamma + \int_{\Gamma_3} j_v^0(u_v(t), \mathbf{v}_v - u_v(t)) d\Gamma \\ \geq \int_{\Omega} f_0(t) \cdot (\mathbf{v} - u(t)) dx + \int_{\Gamma_2} f_2(t) \cdot (\mathbf{v} - u(t)) d\Gamma \end{aligned} \tag{3.15}$$

Finally, we substitute the constitutive law (2.1) in (3.15) and use notation (3.12) to obtain the following variational formulation of Problem \mathcal{P} , in terms of displacement.

Problem \mathcal{PV} Find a displacement field $u : \mathbb{R}_+ \rightarrow U$ such that

The unique solvability of Problem \mathcal{PV} is given by the following existence and uniqueness result, that we state here and prove in the next section.

Theorem 3.1 Assume that (3.7)–(3.11) hold. Then, Problem \mathcal{PV} has a unique solution $u \in C(\mathbb{R}_+; U)$.

We end this section with some remarks on the weak solvability of the contact problem **P**.

First, a couple of functions (u, σ) defined on the positive real line \mathbb{R}_+ with values on the product space $V \times Q$ is called a weak solution to Problem **P** if u is a solution of the variational problem \mathcal{PV} and σ satisfies the constitutive law (2.1).

We conclude that, under the assumption of Theorem 8.1, Problem **P** has a unique weak solution. Moreover, the solution has the regularity $u \in C(\mathbb{R}_+; U)$ and $\sigma \in C(\mathbb{R}_+; Q)$.

Next, recall that Theorem 8.1 provides the weak solvability of the contact problem **P** under the smallness assumption (24) involving the friction bound F_b , and the normal compliance potential j_v . Finally, note that the unknowns η_v and ξ_v of Problem **P** cannot be recovered since they cannot be computed when the solution u of Problem **P** is known. Actually, these unknowns represent interface forces and, as usual in solving contact problems with unilateral constraints, we do not have information neither on the uniqueness of these functions and on their regularity.

4. An Existence and Uniqueness Result

We present in this section an abstract result on history-dependent variational-hemivariational inequalities that we shall use to prove the unique solvability of Problem \mathcal{PV} . For more details on the material presented in this section, we send the reader to [1, 2].

Theorem 4.1. Let X be a reflexive Banach space and Y be a normed space. We denote by X' the dual of X and by $\langle \cdot, \cdot \rangle_{X' \times X}$ the duality pairing of X and X' . Let K be a subset of X and $A : X \rightarrow X'$, $\Psi : C(\mathbb{R}_+; X) \rightarrow C(\mathbb{R}_+; Y)$ be given operators, consider also a function $\phi : Y \times K \times K \rightarrow \mathbb{R}$, a locally Lipschitz function $j : X \rightarrow \mathbb{R}$ and a function $f : \mathbb{R}_+ \rightarrow X'$. With these data we consider the problem of finding a function $u : \mathbb{R}_+ \rightarrow U$ such that, for each $t \in \mathbb{R}_+$, the following inequality holds:

$$\langle Au(t), v - u(t) \rangle + \phi((\Psi u)(t), u(t), v) - \phi((\Psi u)(t), u(t), u(t)) \tag{4.1}$$

$$+ j^0(u(t), v - u(t)) \geq \langle f(t), v - u(t) \rangle, \quad \text{for all } v \in K$$

In the study of (4.1), we assume the following hypotheses.

K is a nonempty, closed and convex subset of X .

$A : X \rightarrow X'$ is an operator such that

$$\left\{ \begin{array}{l} (a) A \text{ is pseudomonotone and there exist} \\ \alpha_A > 0, \beta_A, \gamma_A \in \mathbb{R} \text{ and } u_0 \in K \text{ such that:} \\ \langle Av, v - u_0 \rangle \geq \alpha_A \|v\|_X^2 - \beta_A \|v\|_X^2 - \gamma_A \text{ for all } v \in X. \\ (b) A \text{ is strongly monotone, i.e., there exists } m_A > 0 \text{ such that} \\ \langle Av_1 - Av_2, v_1 - v_2 \rangle \geq m_A \|v_1 - v_2\|_X^2, \quad \forall v_1, v_2 \in X \end{array} \right. \tag{4.2}$$

$\phi : Y \times K \times K \rightarrow \mathbb{R}$ is a function such that

$$\left\{ \begin{array}{l} (a) \phi(y, u, \cdot) \text{ is convex and } l.s.c. \text{ on } K, \text{ for all } y \in Y, u \in K \\ (b) \text{ there exist } \alpha_\phi, \beta_\phi > 0 \text{ such that} \\ \phi(y_1, u_1, v_2) - \phi(y_1, u_1, v_1) + \phi(y_2, u_2, v_1) - \phi(y_2, u_2, v_2) \leq \\ \alpha_\phi \|u_1 - u_2\|_X \|v_1 - v_2\|_X + \beta_\phi \|y_1 - y_2\|_Y \|v_1 - v_2\|_X \\ \text{for all } y_1, y_2 \in Y, u_1, u_2, v_1, v_2 \in K \end{array} \right. \quad (4.3)$$

$j : X \rightarrow \mathbb{R}$ is a function such that

$$\left\{ \begin{array}{l} (a) j \text{ is locally Lipschitz} \\ (b) \|\partial j(v)\|_{X'} \leq c_0 + c_1 \|v\|_X, \text{ for all } v \in V, c_0, c_1 \geq 0 \\ (c) \text{ there exists } \alpha_j > 0 \text{ such that} \\ j^0(v_1, v_2 - v_1) - j^0(v_2, v_1 - v_2) \leq \alpha_j \|v_1 - v_2\|_X^2, \text{ for all } v_1, v_2 \in X \end{array} \right. \quad (4.4)$$

$$\left\{ \begin{array}{l} \text{For any } n \in \mathbb{N}, \text{ there exists } s_n > 0 \text{ such that} \\ \|(\Psi u_1)(t) - (\Psi u_2)(t)\|_Y \leq s_n \int_0^t \|u_1(s) - u_2(s)\| ds \\ \text{for all } u_1, u_2 \in C(\mathbb{R}_+; X), \text{ for all } t \in [0, n]. \end{array} \right. \quad (4.5)$$

$$\alpha_\varphi + \alpha_j < m_A \quad ; \quad \alpha_\varphi < \alpha_j \quad (4.6)$$

$$f \in C(\mathbb{R}_+; X^*) \quad (4.7)$$

Note that an operator Ψ which satisfies condition (4.5) is called a history dependent operator. Inequality (4.1) is governed both by the function ϕ which is assumed to be convex with respect its second argument and by the function j which is locally Lipschitz and could be nonconvex. Therefore, this inequality is a variational-hemivariational inequality. In addition, the function ϕ in (4.1) depends on the operator Ψ , assumed to be history-dependent. For this problem we have the following existence and uniqueness result.

Theorem 4.2. Let X be a reflexive Banach space, Y a normed space, and assume that (4.2)–(4.7) hold. Then, inequality (4.1) has a unique solution $u \in C(\mathbb{R}_+; K)$.

The proof of is obtained by using arguments of elliptic variational-hemivariational inequalities and a fixed point result for history dependent operators.

Proof (Theorem 4.1) We start by defining the operators $A : V \rightarrow V', \mathcal{F} : C(\mathbb{R}_+; V) \rightarrow C(\mathbb{R}_+; Q \times L^2(\Gamma_3))$ and the functions $\phi : L^2(\Gamma_3) \times V \times V \rightarrow \mathbb{R}$ and $j : V \rightarrow \mathbb{R}$ by

$$(A\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathcal{F}\varepsilon(u) \cdot \varepsilon(v) dx \text{ for all } \mathbf{u}, \mathbf{v} \in V \quad (4.8)$$

$$(\mathcal{F}\mathbf{u})(t) = \left(\int_{\Omega} \mathcal{B}(t-s)\varepsilon(\mathbf{u}(s))ds, F_m \left(\int_0^t u_v^+(s)ds \right) \right) \tag{4.9}$$

for all $\mathbf{u} \in C(\mathbb{R}_+; V), t \in \mathbb{R}_+$

$$\varphi(\xi, \mathbf{u}, \mathbf{v}) = (\xi_1, \varepsilon(v))_Q + (\xi_2, v_v^+)_{L^2(\Gamma_3)} + (F_b(u_v), \|v_\tau\|)_{L^2(\Gamma_3)} \tag{4.10}$$

for all $\xi = (\xi_1, \xi_2) \in Q \times L^2(\Gamma_3), \mathbf{u}, \mathbf{v} \in V$

$$j(v) = \int_{\Gamma_3} j_v(\mathbf{v}_v) d\Gamma ; \text{ for all } \mathbf{v} \in \mathbf{V}. \tag{4.11}$$

Then, it is easy to see that Problem \mathcal{PV} is equivalent to the problem of finding a function $u : \mathbb{R}_+ \rightarrow U$ such that for each $t \in \mathbb{R}_+$, the following inequality holds:

$$\langle A\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t) \rangle + \varphi((\mathcal{F}\mathbf{u})(t), \mathbf{u}(t), \mathbf{v}) - \varphi((\mathcal{F}\mathbf{u})(t), \mathbf{u}(t), \mathbf{u})) \tag{4.12}$$

$$j^0(u(t), \mathbf{v} - \mathbf{u}(t)) \geq \langle f(t), \mathbf{v} - \mathbf{u}(t) \rangle \quad , \text{ for all } \mathbf{v} \in V$$

To solve this problem, we use Theorem 4.1 with $X = V, Y = L^2(\Gamma_3)$ and $K = U$ and, to this end, we check in what follows that assumptions (4.2)–(4.7) hold. We use arguments similar to those used in our previous works [8, 9] and, for this reason, we skip the details and we resume the proof as follows. First, we note that assumption (3.7) and definition (3.12) imply (4.2). Next, a simple calculation based on the definition (4.5) of the operator A and the properties (3.4) of the elasticity operator show that (4.2) holds with $m_A = \alpha_A = m_{\mathcal{F}}$. Moreover, using assumption (3.9) and the trace inequality (3.2), it is easy to see that the function ϕ defined by (4.5) satisfies condition (4.3) with $\alpha_\phi = L_{F_b} \|\gamma\|$. On the other hand, assumption (3.6) on the function j_v and definition (4.8) show that condition (4.4) holds with $\alpha_j = \alpha_{j_v}$. And, a simple calculation based on assumptions (3.5), (3.9) imply that the operator (4.9) is a history-dependent operator, i.e., it satisfies condition (4.5). Now, keeping in mind that $m_A = \alpha_A = m_{\mathcal{F}}, \alpha_\phi = L_{F_b} \|\gamma\|$ and $\alpha_j = \alpha_{j_v}$, we easily deduce that the smallness assumption (3.11) shows that conditions (4.6) hold, too. Finally, we note that regularity (3.9) on the densites of the body forces and tractions combined with definition (3.12) show that condition (4.7) is satisfied. We are now in a position to use Theorem 4.2 to deduce the existence of a unique function $u \in C(\mathbb{R}_+; U)$ such that (4.12) holds, for each $t \in \mathbb{R}_+$. And, using notation (4.8)–(4.12), we deduce that u is the unique solution to Problem \mathcal{PV} which concludes the proof.

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