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THE EXISTENCE RESULT OF RENORMALIZED SOLUTION FOR NONLINEAR PARABOLIC SYSTEM WITH VARIABLE EXPONENT AND L[1] DATA

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Abstract. In this paper, we prove the existence result of a renormalized solution to a class of nonlinear parabolic systems, which has a variable exponent Laplacian term and a Leary lions operator with data belong to L^1.

1. Introduction

Let Ω is bounded open domain of $\mathbb{R}^N$, $(N \geq 2)$ with lipschiz boundary $\partial\Omega$, T is a positive number oure aime is to study the existence of renormalized solution for a class of nonlinear parabolic systeme with variable exponent and L^1 data. More precisely, we study the asymptotic behavrior of the problem

$$\begin{cases} (b_1(u))_t - \text{div}\mathcal{A}(x,t,\nabla u) + \gamma(u) = f_1(x,t,u,v) & \text{in} \quad Q = \Omega\times]0,T[, \\ (b_2(v))_t - \Delta v + = f_2(x,t,u,v) & \text{in} \quad Q = \Omega\times]0,T[, \\ u = v = 0 & \text{on} \quad \Sigma = \partial\Omega\times]0,T[, \\ b_1(u)(t=0) = b_1(u_0) & \text{in} \quad \Omega, \\ b_2(v)(t=0) = b_2(v_0) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $\text{div}\mathcal{A}(x,t,\nabla u) = \text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is a Leary lions operator (see assumptions (3.1)-(3.3)) with $p : \overline{\Omega} \longrightarrow [1,+\infty)$ be a continuous real-valued function and let $p^- = \min_{x\in\overline{\Omega}} p(x)$ and $p^+ = \max_{x\in\overline{\Omega}} p(x)$

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with $1 < p^- \leq p^+ < N$. Let $\gamma : \mathbb{R} \to \mathbb{R}$ with $\gamma(s) = \lambda |s|^{p(x)-2} s$ is a continuous increasing function for $\lambda > 0$ and $\gamma(0) = 0$ such that γ is assumed to belong to $L^1(Q)$. The function $f_i : Q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for $i = \overline{1,2}$ be a Carathéodory function (see assumptions (3.5)-(3.7)).

Finally the function $b : \mathbb{R} \to \mathbb{R}$ is a strictly increasing C^1-function lipchizienne with $b_i(0) = 0$ (see (3.4)), the data f_i and $(b_1(u_0), b_2(v_0))$ is in $(L^1)^2$, for $i = \overline{1,2}$.

The study of differential equations and variational problems with nonstandard growth conditions arouses much interest with the development of elastic mechanics, electro-rheological fluid dynamics and image processing, etc (see [9], [19]) .

Problems of this type have been studied by serval a authors. In 2007 H. Redwane, studied the existence of solutions for a class of nonlinear parabolic systems see [18], in 2013 Youssef. B and all studied the existence of a renormalized solution for the nonlinear parabolic systems with unbounded nonlinearities see [2] agin in 2016 B . El Hamdaoui and all in [11] studied the renormalized solutions for nonlinear parabolic systems in the Lebesgue Sobolev Space with variable exponent and L^1 data. In 2016 [17] authors proved the existence and uniqueness of renormalized solution of a reaction diffusion systems which has a variable exponent Laplacian term and could be applied to image denoising for the case of parabolic equations. In 2010 T. M. Bendahmane, P. Wittbold and A.Zimmermann [7] have proved the existence and uniqueness of renormalized solution to nonlinear parabolic equations with variable exponent and L^1 data. C. Zhang and S. Zhou studied the renormalized and entropy solution for nonlinear parabolic equation with variable exponent and L^1 data. Moreover, they obtain the equivalence of renormalized solution and entropy solution(see [23]).

In the present paper we prove the existence of renormalized solution for nonlinear parabolic systems with variable exponent and L^1 data of (1.1). The notion of renormalized solution was introduced by Diperna and Lions [10] in their study of the Boltzmann equation, and this result can be seen as a generalization of the results obtained by F. Souilah and all in [12].

The paper is organized as follows: Section 2, to recall some basic notations and properties of variable exponent Lebesgue Sobolev space. Section 3, is devoted to specify the assumptions on, $\mathcal{A}(x,t,\xi)$, γ, b_1, b_2, f_1, f_2, $b_1(u_0)$ and $b_2(v_0)$ needed in the present study. Section 4, to give the definition of a renormalized solution of (1.1), and we establish (Theorem (4.1)) the existence of such a solution.

2. The Mathematical Preliminaries on Variable Exponent Sobolev Spaces

In this section, we first recall some results on generalized Lebesgue-Sobolev spaces $L^{p(.)}(\Omega)$, $W^{1,p(.)}(\Omega)$ and $W_0^{1,p(.)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. We refer to [13] for further properties of Lebesgue Sobolev spaces with variable exponents. Let $p : \overline{\Omega} \longrightarrow [1, +\infty)$ be a continuous real-valued function and let $p^- = \min_{x\in\overline{\Omega}} p(x)$, $p^+ = \max_{x\in\overline{\Omega}} p(x)$ with $1 < p(.) < N$.

2.1. **Generalized Lebesgue-Sobolev Spaces.** First, denote the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ by

$$L^{p(\cdot)}(\Omega) = \{u \text{ measurable function in } \Omega : \rho_{p(.)}(u) = \int_\Omega |u|^{p(x)}\, dx\}$$

,

which is equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\mu > 0, \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}. \tag{2.1}$$

The space $L^{p(x)}(\Omega)$ is also called a generalized Lebesgue space.

The space $(L^{p(\cdot)}(\Omega); \|.\|_{p(.)})$ is a separable Banach space. Moreover, if $1 < p^- \le p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p\prime(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, for $x \in \Omega$.

The following inequality will be used later:

$$\min\left\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right\} \le \int_\Omega |u(x)|^{p(x)}\, dx \le \max\left\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right\}. \tag{2.2}$$

Finally, the Hölder type inequality

$$\left|\int_\Omega uv dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p^+}\right) \|u\|_{p(.)} \|v\|_{p'(.)}, \tag{2.3}$$

for all $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$.

Next, define the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ by

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega), |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}, \tag{2.4}$$

which is Banach space equiped with the following norm

$$\|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}. \tag{2.5}$$

The space $(W^{1,p(\cdot)}(\Omega); \|.\|_{1,p(.)})$ is a separable and reflexive Banach space. An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(\cdot)}(\Omega)$. To have the following result:

Proposition 2.1. *If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p^+ < +\infty$, the following properties hold true.*

(i) $\|u\|_{p(.)} > 1 \Longrightarrow \|u\|^{p^+}_{p(.)} < \rho_{p(.)}(u) < \|u\|^{p^-}_{p(.)}$,
(ii) $\|u\|_{p(.)} < 1 \Longrightarrow \|u\|^{p^-}_{p(.)} < \rho_{p(.)}(u) < \|u\|^{p^+}_{p(.)}$,
(iii) $\|u\|_{p(.)} < 1$ *(respectively* $= 1; > 1$*)*$\Longleftrightarrow \rho_{p(.)}(u) < 1$ *(respectively* $= 1; > 1$*),*
(iv) $\|u_n\|_{p(.)} \longrightarrow 0$ *(respectively* $\longrightarrow +\infty$*)*$\Longleftrightarrow \rho_{p(.)}(u_n) < 1$*(respectively* $\longrightarrow +\infty$*),*

(v) $\rho_{p(.)}\left(\frac{u}{\|u\|_{p(.)}}\right)=1.$

For a measurable function $u:\Omega\longrightarrow\mathbb{R}$, *we introduce the following notation*

$$\rho_{1,p(.)}=\int_{\Omega}|u|^{p(x)}\,dx+\int_{\Omega}|\nabla u|^{p(x)}\,dx.$$

Proposition 2.2. *If* $u\in W^{1,p(.)}(\Omega)$ *and* $p^{+}<+\infty$, *the following properties hold true.*

(i) $\|u\|_{1,p(.)}>1\Longrightarrow\|u\|^{p^{+}}_{1,p(.)}<\rho_{1,p(.)}(u)<\|u\|^{p^{-}}_{1,p(.)}$,

(ii) $\|u\|_{1,p(.)}<1\Longrightarrow\|u\|^{p^{-}}_{1,p(.)}<\rho_{1,p(.)}(u)<\|u\|^{p^{+}}_{1,p(.)}$,

(iii) $\|u\|_{1,p(.)}<1$ *(respectively* $=1;>1$*)*$\Longleftrightarrow\rho_{1,p(.)}(u)<1$*(respectively* $=1;>1$*).*

Extending a variable exponent $p:\overline{\Omega}\longrightarrow[1,+\infty)$ *to* $\overline{Q}=[0,T]\times\overline{\Omega}$ *by setting* $p(x,t)=p(x)$ *for all* $(x,t)\in\overline{Q}$.

We may also consider the generalized Lebesgue space

$$L^{p(\cdot)}(Q)=\left\{u:Q\longrightarrow\mathbb{R}\text{ mesurable such that}\int_{Q}|u(x,t)|^{p(x)}\,d(x,t)<\infty\right\},$$

endowed with the norm

$$\|u\|_{L^{p(\cdot)}(Q)}=\inf\left\{\mu>0,\int_{Q}\left|\frac{u(x,t)}{\mu}\right|^{p(x)}d(x,t)\leq 1\right\},$$

which share the same properties as $L^{p(\cdot)}(\Omega)$.

3. The Assumptions on The Data

This paper, we assume that the following assumptions hold true:

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N\geq 2)$, $T>0$ is given and we set $Q=\Omega\times]0,T[$, and $\mathcal{A}:Q\times\mathbb{R}^N\to\mathbb{R}^N$ be a Carathéodory function such that for all $\xi,\eta\in\mathbb{R}^N,\xi\neq\eta$

$$\mathcal{A}(x,t,\xi).\xi\geqslant\alpha\,|\xi|^{p(x)}, \tag{3.1}$$

$$|\mathcal{A}(x,t,\xi)|\leqslant\beta\left[L(x,t)+|\xi|^{p(x)-1}\right], \tag{3.2}$$

$$(\mathcal{A}(x,t,\xi)-\mathcal{A}(x,t,\eta)).(\xi-\eta)>0, \tag{3.3}$$

where $1<p^{-}\leq p^{+}<+\infty$, α,β are positives constants and L is a nonnegative function in $L^{p'(\cdot)}(Q)$, $\gamma:\mathbb{R}\to\mathbb{R}$ is a continuous increasing function with $\gamma(0)=0$.

Let $b_i:\mathbb{R}\to\mathbb{R}$ is a strictly increasing C^1-function lipchizienne with $b_i(0)=0$ and for any ρ,τ are positives constants and for $i=\overline{1,2}$ such that

$$\rho\leq b_i'(s)\leq\tau,\quad\forall s\in\mathbb{R}, \tag{3.4}$$

$f_i : Q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for any $k > 0$, there exists $\sigma_k > 0$, $\;c_k \in L^1(Q)$ such that

$$|f_1(x,t,s_1,s_2)| \leq c_k(x,t) + \sigma_k |s_2|^2, \tag{3.5}$$

for almost every $(x,t) \in (Q)$, for every s_1 such that $|s_1| \leq k$, and for every $s_2 \in \mathbb{R}$.

For any $k > 0$, there exists $\zeta_k > 0$ and $G_k \in L^{p'(.)}(Q)$ such that

$$|f_2(x,t,s_1,s_2)| \leq G_k(x,t) + \zeta_k |s_1|^{p(x)-1}, \tag{3.6}$$

for almost every $(x,t) \in (Q)$, for every s_2 such that $|s_2| \leq k$, and for every $s_1 \in \mathbb{R}$.

$$f_1(x,t,s_1,s_2)s_1 \geq 0 \text{ and } f_2(x,t,s_1,s_2)s_2 \geq 0, \tag{3.7}$$

$$(b_1(u_0), b_2(v_0)) \in (L^1(\Omega))^2. \tag{3.8}$$

4. **The Main Results**

In this section, we study the existence of renormalized solutions to problem (1.1).

Definition 4.1. *Let* $2 - \dfrac{1}{N+1} < p^- \leq p^+ < N$ *and* $(b_1(u_0), b_2(v_0)) \in (L^1(\Omega))^2$. *A measurable functions* $(u,v) \in (C(]0,T[; L^1(\Omega)))^2$ *is a renormalized solution of the problem* (1.1) *if ,*

$$T_k(u) \in L^{p^-}(]0,T[; W_0^{1,p(.)}(\Omega)), T_k(v) \in L^2(]0,T[; H_0^1(\Omega)) \text{ for any } k > 0\;, \tag{4.1}$$

$$\gamma(u) \in L^1(Q) \text{ and } f_i(x,t,u,v) \in (L^1(Q))^2, \;\; \forall i = \overline{1,2},$$

$$b_1(u) \in L^\infty\left(]0,T[; L^1(\Omega)\right) \cap L^{q^-}(]0,T[; W_0^{1,q(.)}(\Omega)) \tag{4.2}$$

$$\text{and} \qquad b_2(v) \in L^\infty\left(]0,T[; L^1(\Omega)\right) \cap L^2(]0,T[; H_0^1(\Omega)),$$

for all continuous functions $q(x)$ *on* $\overline{\Omega}$ *satisfying* $q(x) \in \left[1, p(x) - \frac{N}{N+1}\right)$ *for all* $x \in \overline{\Omega}$,

$$\lim_{n\to\infty} \int_{\{n \leq |u| \leq n+1\}} \mathcal{A}(x,t,\nabla u)\nabla u dxdt + \lim_{n\to\infty} \int_{\{n \leq |v| \leq n+1\}} |\nabla v|^2 dxdt = 0, \tag{4.3}$$

and if, for every function $S \in W^{2,\infty}(\mathbb{R})$ *which is piecewise* C^1 *and such that* S' *has compact support on* $\mathbb{R}$, *to have,*

$$(B_S^1(u))_t - div(\mathcal{A}(x,t,\nabla u)S'(u)) + S''(u)\mathcal{A}(x,t,\nabla u)\nabla u + \gamma(u)S'(u) \tag{4.4}$$

$$= f_1(x,t,u,v)S'(u) \text{ in } \mathcal{D}'(Q),$$

$$(B_S^2(v))_t - div(\nabla v S'(v)) + S''(v)\nabla v = f_2(x,t,u,v)S'(v) \text{ in } \mathcal{D}'(Q), \tag{4.5}$$

$$B_S^1(u)(t=0) = S(b_1(u_0)) \ in \ \Omega, \tag{4.6}$$

$$B_S^2(v)(t=0) = S(b_2(v_0)) \ in \ \Omega, \tag{4.7}$$

where $B_S^i(z) = \int_0^z b_i'(r)S'(r)dr$, *for* $i = \overline{1,2}$.

The following remarks are concerned with a few comments on definition (4.1).

Remark 4.1. *Note that, all terms in* (4.4) *are well defined. Indeed, let* $k > 0$ *such that* $supp(S') \subset [K, K]$, *we have* $B_S^i(u)$ *belongs to* $L^\infty(Q)$ *for all* $i = \overline{1,2}$ *because*

$$|B_S^1(u)| \leq \int_0^u |b_1'(r)S'(r)|dr \leq \tau \|S'\|_{L^\infty(\mathbb{R})},$$

and

$$|B_S^2(v)| \leq \int_0^v |b_2'(r)S'(r)|dr \leq \tau \|S'\|_{L^\infty(\mathbb{R})},$$

and

$$S(u) = S(T_k(u)) \in L^{p-}(]0,T[; W_0^{1,p(.)}(\Omega)), S(v) = S(T_k(v)) \in L^2(]0,T[; H_0^1(\Omega))$$

and $\frac{\partial B_S^i(u)}{\partial t} \in (\mathcal{D}'(Q))^2$ *for* $i = \overline{1,2}$. *The term* $S'(u)\mathcal{A}(x,t,\nabla T_k(u))$ *identifes with* $S'(T_k(u))\mathcal{A}(x,t,\nabla(T_k(u)))$ *a.e. in* Q, *where* $u = T_k(u)$ *in* $\{|u| \leq k\}$, *assumptions* (3.2) *imply that*

$$\begin{gathered} |S'(T_k(u))\mathcal{A}(x,t,\nabla T_k(u))| \\ \leq \beta \|S'\|_{L^\infty(\mathbb{R})} \left[L(x,t) + |\nabla(T_k(u))|^{p(x)-1}\right] \ a.e \ in \ Q. \end{gathered} \tag{4.8}$$

Using (3.2) *and* (4.1), *it follows that* $S'(u)\mathcal{A}(x,t,\nabla u) \in (L^{p'(.)}(Q))^N$. *The term* $S''(u)\mathcal{A}(x,t,\nabla u)\nabla(u)$ *identifes with* $S''(u)\mathcal{A}(t,x,\nabla(T_k(u)))\nabla T_k(u)$ *and in view of* (3.2), (4.1) *and* (4.8), *to obtain* $S''(u)\mathcal{A}(x,t,\nabla u)\nabla(u) \in L^1(Q)$ *and* $S'(u)\gamma(u) \in L^1(Q)$. *Finally* $f_1(x,t,u,v)\ S'(u) = f_1(x,t,T_k(u),v)S'(u)$ *a.e in* Q . *Since* $|T_k(u)| \leq k$ *and* $S'(u) \in L^\infty(Q)$, $c_k(x,t) \in L^1(Q)$, *to obtain from* (3.5) *that* $f_1(x,t,T_k(u),v)S'(u) \in L^1(Q)$, *and* $f_2(x,t,u,v)\ S'(v) = f_2(x,t,u,T_k(v))S'(v)$ *a.e in* Q. *Since* $|T_k(v)| \leq k$ *and* $S'(v) \in L^\infty(Q)$, $G_k(x,t) \in L^{p'(.)}(Q)$, *to obtain from* (3.6) *that* $f_2(x,t,u,T_k(v))S'(v) \in L^1(Q)$. *Also* $\frac{\partial B_S^1(u)}{\partial t} \in L^{(p^-)'}(]0,T[; W^{-1,p'(.)}(\Omega)) + L^1(Q)$ *and* $B_S^1(u) \in L^{p^-}(]0,T[; W_0^{1,p(.)}(\Omega)) \cap L^\infty(Q)$, *and* $\frac{\partial B_S^2(v)}{\partial t} \in L^2(]0,T[; H^{-1}(\Omega)) + L^1(Q)$ *and* $B_S^2(v) \in L^2(]0,T[; H_0^1(\Omega)) \cap L^\infty(Q)$, *which implies that* $(B_S^1(u), B_S^2(v)) \in (C(]0,T[; L^1(\Omega)))^2$.

4.1. **The Existence Theorem.**

Theorem 4.1. *Let* $(b_1(u_0), b_2(v_0)) \in (L^1(\Omega))^2$*, assume that* (3.1)-(3.8) *hold true, then there exists at least one renormalized solution* $(u, v) \in (C(]0, T[, L^1(\Omega)))^2$ *of Problem* (1.1) *(in the sens of Definition* (4.1) *).*

Proof. of Theorem (4.1) The above theorem is to be proved in five steps.

- **Step 1: Approximate problem and a priori estimates.** Let us define the following approximation of b and f for $\varepsilon > 0$ fixed and for $i = \overline{1, 2}$

$$b^i_\varepsilon(r) = T_{\frac{1}{\varepsilon}}(b_i(r)) \text{ a.e in } \Omega \text{ for } \varepsilon > 0, \quad \forall r \in \mathbb{R}, \tag{4.9}$$

$$b^i_\varepsilon(u^\varepsilon_0) \text{ are a sequence of } (C^\infty_c(\Omega))^2 \text{ functions such that} \tag{4.10}$$
$$(b^1_\varepsilon(u^\varepsilon_0), b^2_\varepsilon(v^\varepsilon_0)) \to (b_1(u_0), b_2(v_0)) \text{ in } (L^1(\Omega))^2 \text{ as } \varepsilon \text{ tends to } 0.$$

$$\begin{aligned} f^\varepsilon_1(x, t, r_1, r_2) &= f_1(x, t, T_{\frac{1}{\varepsilon}}(r_1), r_2), \\ f^\varepsilon_2(x, t, r_1, r_2) &= f_2(x, t, r_1, T_{\frac{1}{\varepsilon}}(r_2)), \end{aligned} \tag{4.11}$$

in view of (3.5), (3.6) and (3.7), there exist $G^\varepsilon_k \in L^{p'(\cdot)}(Q)$, $c^\varepsilon_k \in L^1(Q)$ and $\sigma^\varepsilon_k, \zeta^\varepsilon_k > 0$ such that

$$|f^\varepsilon_1(x, t, s_1, s_2)| \le c^\varepsilon_k(x, t) + \sigma^\varepsilon_k |s_2|^2, \tag{4.12}$$

$$|f^\varepsilon_2(x, t, s_1, s_2)| \le G^\varepsilon_k(x, t) + \zeta^\varepsilon_k |s_1|^{p(x)-1}, \tag{4.13}$$

for almost every $(x, t) \in (Q)$, $s_1, s_2 \in \mathbb{R}$,

$$f^\varepsilon_1(x, t, s_1, s_2)s_1 \ge 0 \text{ and } f^\varepsilon_2(x, t, s_1, s_2)s_2 \ge 0. \tag{4.14}$$

Let us now consider the approximate problem:

$$\left(b^1_\varepsilon(u^\varepsilon)\right)_t - div\mathcal{A}(x, t, \nabla u^\varepsilon) + \gamma(u^\varepsilon) = f^\varepsilon_1(x, t, u^\varepsilon, v^\varepsilon) \text{ in } Q, \tag{4.15}$$

$$\left(b^2_\varepsilon(v^\varepsilon)\right)_t - \Delta v^\varepsilon = f^\varepsilon_2(x, t, u^\varepsilon, v^\varepsilon) \text{ in } Q, \tag{4.16}$$

$$u^\varepsilon = v^\varepsilon = 0 \text{ on }]0, T[\times \partial\Omega, \tag{4.17}$$

$$b^1_\varepsilon(u^\varepsilon)(t = 0) = b^1_\varepsilon(u^\varepsilon_0) \text{ in } \Omega, \tag{4.18}$$

$$b^2_\varepsilon(v^\varepsilon)(t = 0) = b^2_\varepsilon(v^\varepsilon_0) \text{ in } \Omega. \tag{4.19}$$

As a consequence, proving existence of a weak solution $u^\varepsilon \in L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega))$ and $v^\varepsilon \in L^2(]0,T[;H_0^1(\Omega))$ of (4.15)-(4.18) is an easy task (see [15]).

we choose $T_k(u^\varepsilon)\chi_{(0,t)}$ as a test function in (4.15), to get

$$\begin{aligned}\int_\Omega B_k^{1,\varepsilon}(u^\varepsilon)(t)dx \quad &+ \quad \int_0^t\int_\Omega \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon) + \int_0^t\int_\Omega \gamma\left(u^\varepsilon\right)T_k(u^\varepsilon)dxds \\ &= \int_0^t\int_\Omega f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)T_k(u^\varepsilon)dxds + \int_\Omega B_k^{1,\varepsilon}(u_0^\varepsilon)dx, \qquad (4.20)\end{aligned}$$

for almost every t in $(0,T)$, and where

$$B_k^{i,\varepsilon}(r) = \int_0^r T_k(s)\frac{\partial b_\varepsilon^i(s)}{\partial s}ds.\forall i = \overline{1,2}.$$

Under the definition of $B_k^{i,\varepsilon}(r)$ the inequality

$$0 \le \int_\Omega B_k^{1,\varepsilon}(u_0^\varepsilon)(t)dx \le k\int_\Omega |b_\varepsilon^1(u_0^\varepsilon)|dx, \quad k > 0.$$

Using (3.1), $f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)T_k(u^\varepsilon) \ge 0$, and we have $\gamma(u^\varepsilon) = \lambda|u^\varepsilon|^{p(x)-1}u^\varepsilon \ge 0$ because $1 < p^- \le p(x) \le +\infty$ and the definition of $B_k^\varepsilon(r)$ in (4.20), to obtain

$$\int_\Omega B_k^\varepsilon(u^{1,\varepsilon})(t)dx + \alpha\int_{E_k}|\nabla u^\varepsilon|^{p(x)}\,dxds \le k\left\|b_\varepsilon^1(u_0^\varepsilon)\right\|_{L^1(Q)}, \qquad (4.21)$$

where $E_k = \{(x,t) \in Q : |u^\varepsilon| \le k\}$, using $\overline{B}_k^\varepsilon(u^\varepsilon)(t) \ge 0$ and inequality (2.2) in (4.21), to get

$$\begin{aligned}\alpha\int_0^T \min\left\{\|\nabla T_k(u^\varepsilon)\|_{L^{p(x)}(\Omega)}^{p^-}, \|\nabla T_k(u^\varepsilon)\|_{L^{p(x)}(\Omega)}^{p^+}\right\} \quad &\le \quad \alpha\int_{\{(x,t)\in Q:\ |u^\varepsilon|\le k\}}|\nabla u^\varepsilon|^{p(x)}\,dxdt \\ &\le \quad C, \qquad (4.22)\end{aligned}$$

then is $T_k(u^\varepsilon)$ is bounded in $L^{p-}(]0,T[\,;W_0^{1,p(x)}(\Omega))$.

Similarly, we choose $T_k(v^\varepsilon)\chi_{(0,t)}$ as a test function in (4.16), to get

$$\int_\Omega B_k^{2,\varepsilon}(v^\varepsilon)(t)dx + \alpha\int_{F_k}|\nabla v^\varepsilon|^2\,dxds \le k\left\|b_\varepsilon^2(v_0^\varepsilon)\right\|_{L^1(Q)}, \qquad (4.23)$$

where $F_k = \{(x,t) \in Q : |v^\varepsilon| \le k\}$, then is $T_k(v^\varepsilon)$ is bounded in $L^2(]0,T[\,;H_0^1(\Omega))$. Adding (4.21) and (4.23), one gets

$$\int_\Omega B_k^{1,\varepsilon}(u^\varepsilon)(t)dx + \int_\Omega B_k^{2,\varepsilon}(v^\varepsilon)(t)dx \le k\left\|(b_\varepsilon^1(u_0^\varepsilon), b_\varepsilon^2(v_0^\varepsilon))\right\|_{L^1(Q)\times L^1(Q)}. \qquad (4.24)$$

Also, to obtain

$$k\int_{\{(t,x)\in Q:|u^\varepsilon|>k\}}|\gamma(u^\varepsilon)|\,dxdt \le k\left\|b_\varepsilon(u_0^\varepsilon)\right\|_{L^1(Q)}. \qquad (4.25)$$

Hence

$$k \int_{\{(x,t)\in Q:|u^\varepsilon|>k\}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt + k \int_{\{(x,t)\in Q:|v^\varepsilon|>k\}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \leq k \left\|(b_\varepsilon^1(u_0^\varepsilon), b_\varepsilon^2(v_0^\varepsilon))\right\|_{L^1(Q)\times L^1(Q)}. \tag{4.26}$$

Now, let $T_1(s - T_k(s)) = T_{k,1}(s)$ and take $T_{k,1}(b_\varepsilon^1(u^\varepsilon))$ as test function in (4.15). Reasoning as above, by $\nabla T_{k,1}(s) = \nabla s \chi_{\{k\leq|s|\leq k+1\}}$ and the young's inequality, to obtain

$$\begin{aligned}
\alpha \int_{\{k\leq|b_\varepsilon^1(u^\varepsilon)|\leq k+1\}} b'_{1,\varepsilon}(u^\varepsilon)\,|\nabla(u^\varepsilon)|^{p(x)}\,dxdt &\leq k \int_{\{|b_\varepsilon^1(u_0^\varepsilon)|>k\}} \left|b_\varepsilon^1(u_0^\varepsilon)\right| dx \\
&+ Ck \int_{\{|b_\varepsilon^1(u^\varepsilon)|>k\}} |\gamma(u^\varepsilon)|\,dxdt \\
&+ Ck \int_{\{|b_\varepsilon^1(u^\varepsilon)|>k\}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\leq C_1,
\end{aligned}$$

inequality (2.2) implies that

$$\begin{aligned}
&\int_0^T \alpha \chi_{\{k\leq|b_\varepsilon^1(u^\varepsilon)|\leq k+1\}} \min\left\{\left\|\nabla(b_\varepsilon^1(u^\varepsilon))\right\|_{L^{p(x)}(\Omega)}^{p-}, \left\|\nabla(b_\varepsilon^1(u^\varepsilon))\right\|_{L^{p(x)}(\Omega)}^{p+}\right\} \\
&\leq \alpha \int_{\{k\leq|b_\varepsilon^1(u^\varepsilon)|\leq k+1\}} b'_{1,\varepsilon}(u^\varepsilon)\,|\nabla(u^\varepsilon)|^{p(x)}\,dxdt \leq C_1.
\end{aligned} \tag{4.27}$$

Similarly, we choose $T_k(b_\varepsilon^2(v^\varepsilon))$ as test function in (4.16), to have

$$\begin{aligned}
\int_{\{|b_\varepsilon^2(v^\varepsilon)|\leq k\}} b'_{2,\varepsilon}(v^\varepsilon)\,|\nabla(v^\varepsilon)|^2\,dxdt &\leq k \int_{\{|b_\varepsilon^2(v_0^\varepsilon)|>k\}} \left|b_\varepsilon^2(v_0^\varepsilon)\right| dx \\
&+ Ck \int_{\{|b_\varepsilon^2(v^\varepsilon)|>k\}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \leq C_2,
\end{aligned}$$

we know that properties of $B_k^{i,\varepsilon}(u^\varepsilon)$, $(B_k^{i,\varepsilon}(r^\varepsilon) \geq 0,\ \ B_k^{i,\varepsilon}(r^\varepsilon)) \geq \rho(|r|-1)$, for all $i = \overline{1,2}$, to obtain

$$\begin{aligned}
\int_\Omega \left|B_k^{1,\varepsilon}(u^\varepsilon)(t)\right| dx + \int_\Omega \left|B_k^{2,\varepsilon}(v^\varepsilon)(t)\right| dx &\leq k \int_\Omega \left|b_\varepsilon^1(u^\varepsilon)(t)\right| dx + k \int_\Omega \left|b_\varepsilon^2(v^\varepsilon)(t)\right| dx \\
&\leq \rho\left(2meas(\Omega) + k\left\|(b_\varepsilon^1(u_0^\varepsilon), b_\varepsilon^2(v_0^\varepsilon))\right\|_{L^1(Q)\times L^1(Q)}\right).
\end{aligned} \tag{4.28}$$

From the estimation (4.22), (4.23), (4.27) , (4.28) and the properites of $B_k^{i,\varepsilon}$ and $b_\varepsilon^1(u_0^\varepsilon)$, $b_\varepsilon^2(v_0^\varepsilon)$, we deduce that

$$b_\varepsilon^1(u^\varepsilon) \text{ and } b_\varepsilon^2(v^\varepsilon) \text{ is bounded in } L^\infty\left(]0,T[;L^1(\Omega)\right), \tag{4.29}$$

$$u^\varepsilon \text{ and } v^\varepsilon \text{ is bounded in } L^\infty\left(]0,T[;L^1(\Omega)\right), \tag{4.30}$$

and

$$b_\varepsilon^1(u^\varepsilon) \text{ is bounded in } L^{p-}(]0,T[\,;W_0^{1,p(x)}(\Omega)), \tag{4.31}$$

and

$$b_\varepsilon^2(v^\varepsilon) \text{ is bounded in } L^2(]0,T[\,;H_0^1(\Omega)), \tag{4.32}$$

by (4.27), (4.28) and Lemma 2.1 in [7] by and if

$$2-\frac{1}{N+1}<p(.)<N,$$

to obtain

$$b_\varepsilon^1(u^\varepsilon) \text{ is bounded in } L^{q-}(]0,T[\,;W_0^{1,q(x)}(\Omega)), \tag{4.33}$$

for all continuous variable exponents $q \in C(\overline{\Omega})$ satisfying

$$1 \le q(x) < \frac{N(p(x)-1)+p(x)}{N+1},$$

for all $x \in \Omega$.

And

$$T_k\left(u^\varepsilon\right) \text{ is bounded in } L^{p^-}\left(]0,T[;W_0^{1,p(.)}\left(\Omega\right)\right), \tag{4.34}$$

and

$$T_k\left(v^\varepsilon\right) \text{ is bounded in } L^2\left(]0,T[;H_0^1\left(\Omega\right)\right). \tag{4.35}$$

By (4.25) and (4.26), we may conclude that

$$\gamma(u^\varepsilon) \text{ is bounded in } L^1\left(]0,T[;L^1\left(\Omega\right)\right), \tag{4.36}$$

and

$$f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon) \text{ and } f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon) \text{ is bounded in } L^1\left(]0,T[;L^1\left(\Omega\right)\right), \tag{4.37}$$

independently of ε.

Proceeding as in [3], [4] that for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' is compact (supp $S' \subset [-k,k]$),

$$S\left(u^\varepsilon\right) \text{ is bounded in } L^{p-}\left(]0,T[;W_0^{1,p(.)}\left(\Omega\right)\right), \tag{4.38}$$

and

$$S\left(v^\varepsilon\right) \text{ is bounded in } L^2\left(]0,T[;H_0^1\left(\Omega\right)\right), \tag{4.39}$$

and

$$\left(S\left(u^\varepsilon\right)\right)_t \text{ is bounded in } L^1\left(Q\right)+L^{(p-)'}\left(]0,T[;W^{-1,p'(.)}\left(\Omega\right)\right), \tag{4.40}$$

and

$$\left(S\left(v^\varepsilon\right)\right)_t \text{ is bounded in } L^1\left(Q\right)+L^2\left(]0,T[;H^{-1}\left(\Omega\right)\right). \tag{4.41}$$

In fact, as a consequence of (4.34), by Stampacchia's Theorem, we obtain (4.38). To show that (4.40) holds true, we multiply the equation (4.15) by $S'(u^\varepsilon)$ and the equation (4.16) by $S'(v^\varepsilon)$, to obtain

$$\left(B_S^1(u^\varepsilon)\right)_t = div(S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)) - \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla(S'(u^\varepsilon)) \quad (4.42)$$
$$-\gamma(u^\varepsilon)S'(u^\varepsilon) + f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S'(u^\varepsilon) \text{ in } \mathcal{D}'(Q).$$

And

$$\left(B_S^2(v^\varepsilon)\right)_t = div(S'(v^\varepsilon)\nabla v^\varepsilon) - \nabla(S'(v^\varepsilon)) \quad (4.43)$$
$$+f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S'(v^\varepsilon) \text{ in } \mathcal{D}'(Q).$$

Since supp(S') and supp(S'') are both included in $[-k;k]$; u^ε may be replaced by $T_k(u^\varepsilon)$ in $\{|u^\varepsilon| \le k\}$. To have

$$|S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)| \quad (4.44)$$
$$\le \beta \|S'\|_{L^\infty}\left[L(x,t) + |\nabla T_k(u^\varepsilon)|^{p(x)-1}\right],$$

as a consequence, each term in the right hand side of (4.42) is bounded either in $L^{(p_-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right)$ or in $L^1(Q)$, and obtain (4.40).

Now we look for an estimate on a sort of energy at infinity of the approximating solutions. For any integer $n \ge 1$, consider the Lipschitz continuous function θ_n defined through

$$\theta_n(s) = T_{n+1}(s) - T_n(s) = \begin{cases} 0 & \text{if } |s| \le n, \\ (|s|-n)\,sign(s) & \text{if } n \le |s| \le n+1, \\ sign(s) & \text{if } |s| \ge n. \end{cases}$$

Remark that $||\theta_n||_{L^\infty} \le 1$ for any $n \ge 1$ and that $\theta_n(s) \to 0$, for any s when n tends to infinity. Using the admissible test function $\theta_n(u^\varepsilon)$ in (4.15) leads to

$$\int_\Omega \widetilde{\theta_n}(u^\varepsilon)(t)\,dx + \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla(\theta_n(u^\varepsilon))\,dxdt + \int_Q \gamma(u^\varepsilon)\theta_n(u^\varepsilon)dxdt$$
$$= \int_Q f^\varepsilon(x,t,u^\varepsilon)\theta_n(u^\varepsilon)dxdt + \int_\Omega \widetilde{\theta_n}(u_0^\varepsilon)\,dx, \quad (4.45)$$

where $\widetilde{\theta_n}(r)(t) = \int_0^r \theta_n(s)\dfrac{\partial b_\varepsilon^i(s)}{\partial s}ds$, for all $i = \overline{1,2}$,

for almost any t in $]0,T[$ and where $\widetilde{\theta_n}(r) = \int_0^r \theta_n(s)ds \ge 0$. Hence, dropping a nonnegative term

$$\int_{\{n \le |u^\varepsilon| \le n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt \quad (4.46)$$

$$\leq \int_Q \gamma(u^\varepsilon)\,\theta_n(u^\varepsilon)dxdt + \int_Q f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)\theta_n(u^\varepsilon)dxdt + \int_\Omega \widetilde{\theta_n}\,(u_0^\varepsilon)\,dx$$

$$\leq \int_{\{|u^\varepsilon|\geq n\}} |\gamma(u^\varepsilon)|\,dxdt + \int_{\{|u^\varepsilon|\geq n\}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt + \int_{\{|b_\varepsilon^1(u_0^\varepsilon)|\geq n\}} \left|b_\varepsilon^1(u_0^\varepsilon)\right|dx.$$

Similarly, we take test function $\theta_n(v^\varepsilon)$ in (4.16) leads to

$$\int_{\{n\leq|v^\varepsilon|\leq n+1\}} |\nabla v^\varepsilon|^2 dxdt \tag{4.47}$$

$$\leq \int_Q f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)\theta_n(v^\varepsilon)dxdt + \int_\Omega \widetilde{\theta_n}\,(v_0^\varepsilon)\,dx \leq \int_{\{|v^\varepsilon|\geq n\}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt$$

$$+ \int_{\{|b_\varepsilon^2(v_0^\varepsilon)|\geq n\}} \left|b_\varepsilon^2(v_0^\varepsilon)\right|dx.$$

Next, we study the convergence of $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ in $C(]0,T[;L^1(\Omega))$.

Lemma 4.1. *Both $(u^{\varepsilon_n})_{n\in\mathbb{N}}$ and $(v^{\varepsilon_n})_{n\in\mathbb{N}}$ are Cauchy sequences in $C(]0,T[;L^1(\Omega))$.*

Proof. Let ε_n and ε_m two positive integers. It follows frome (4.15) and (4.16) that

$$\int_\Omega \frac{\partial b_{\varepsilon_n}^1(u^{\varepsilon_n}-u^{\varepsilon_m})}{\partial t}\varphi dx + \int_0^t\int_\Omega (\mathcal{A}(x,t,\nabla u^{\varepsilon_n}) - \mathcal{A}(x,t,\nabla u^{\varepsilon_m}))\nabla\varphi dxdt$$

$$+\int_0^t\int_\Omega \lambda\left[|u^{\varepsilon_n}|^{p(x)-2}u^{\varepsilon_n} - |u^{\varepsilon_m}|^{p(x)-2}u^{\varepsilon_m}\right]\phi dxds$$

$$= \int_0^t\int_\Omega \left[f_1^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n}) - f_1^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]\varphi dxds, \tag{4.48}$$

and

$$\int_\Omega \frac{\partial b_{\varepsilon_n}^2(v^{\varepsilon_n}-v^{\varepsilon_m})}{\partial t}\phi dx + \int_0^t\int_\Omega (\nabla v^{\varepsilon_n} - \nabla v^{\varepsilon_m})\nabla\phi dxdt \tag{4.49}$$

$$= \int_0^t\int_\Omega \left[f_2^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n}) - f_2^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]\phi dxds,$$

where $\varphi \in L^\infty(]0,T[;W^{1,p(\cdot)}(\Omega))$ and $\phi \in L^2(]0,T[;H_0^1(\Omega))$. To do this fix $\tau \in [0,T]$. Taking $\varphi = \frac{1}{k}T_k(u^{\varepsilon_n} - u^{\varepsilon_m})1_{\{[0,\tau[\}}$ in (4.48) and $\phi = \frac{1}{k}T_k(v^{\varepsilon_n} - v^{\varepsilon_m})1_{\{[0,\tau[\}}$ in (4.49), one gets

$$\begin{aligned}
&\frac{1}{k}\int_\Omega B_k^{1,\varepsilon_n}(u^{\varepsilon_n}(\tau) - u^{\varepsilon_n}(\tau))dx - \frac{1}{k}\int_\Omega B_k^{1,\varepsilon_n}(u^{\varepsilon_n}(0) - u^{\varepsilon_m}(0))dx \\
&+\int_0^\tau\int_\Omega \frac{1}{k}(\mathcal{A}(x,t,\nabla u^{\varepsilon_n}) - \mathcal{A}(x,t,\nabla u^{\varepsilon_m}))\nabla T_k(u^{\varepsilon_n} - u^{\varepsilon_m})dxdt \\
&+\int_0^\tau\int_\Omega \frac{\lambda}{k}\left[|u^{\varepsilon_n}|^{p(x)-2}u^{\varepsilon_n} - |u^{\varepsilon_m}|^{p(x)-2}u^{\varepsilon_m}\right]T_k(u^{\varepsilon_n} - u^{\varepsilon_m})dxds \\
&= \int_0^t\int_\Omega \frac{1}{k}\left[f_1^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n}) - f_1^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]T_k(u^{\varepsilon_n} - u^{\varepsilon_m})dxds,
\end{aligned} \tag{4.50}$$

and

$$\begin{aligned}
&\frac{1}{k}\int_\Omega B_k^{2,\varepsilon_n}(v^{\varepsilon_n}(\tau) - v^{\varepsilon_m}(\tau))dx - \frac{1}{k}\int_\Omega B_k^{2,\varepsilon_n}(v^{\varepsilon_n}(0) - v^{\varepsilon_m}(0))dx \\
&+\frac{1}{k}\int_0^t\int_\Omega \nabla(v^{\varepsilon_n} - v^{\varepsilon_m})\nabla T_k(v^{\varepsilon_n} - v^{\varepsilon_m})dxdt \\
&= \int_0^t\int_\Omega \frac{1}{k}\left[f_2^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n}) - f_2^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]T_k(v^{\varepsilon_n} - v^{\varepsilon_m})dxds,
\end{aligned} \tag{4.51}$$

where

$$B_k^{i,\varepsilon_n}(r) = \int_0^r T_k(s)\frac{\partial b_{\varepsilon_n}^i(s)}{\partial s}ds. \ \forall i = \overline{1,2},$$

adding (4.50) and (4.51), we get

$$\begin{aligned}
&\frac{1}{k}\int_\Omega B_k^{1,\varepsilon_n}(u^{\varepsilon_n}(\tau) - u^{\varepsilon_m}(\tau))dx + \frac{1}{k}\int_\Omega B_k^{2,\varepsilon_n}(v^{\varepsilon_n}(\tau) - v^{\varepsilon_m}(\tau))dx \\
&\leq \int_0^\tau\int_\Omega \lambda\left[|u^{\varepsilon_n}|^{p(x)-2}u^{\varepsilon_n} - |u^{\varepsilon_m}|^{p(x)-2}u^{\varepsilon_m}\right]dxdt + \\
&\int_0^\tau\int_\Omega \left[f_1^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n}) - f_1^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]dxdt + \\
&\int_0^\tau\int_\Omega \left[f_2^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n}) - f_2^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]dxdt + \\
&\int_\Omega \left|b_{\varepsilon_n}^1(u_0^{\varepsilon_n} - u_0^{\varepsilon_m})\right|dx + \int_\Omega \left|b_{\varepsilon_n}^2(v_0^{\varepsilon_n} - v_0^{\varepsilon_m})\right|dx,
\end{aligned}$$

since $B_k^{i,\varepsilon_n}(r) \geq \rho \int_0^r T_k(s)ds \geq \rho\left(|s|-1\right).\forall i=\overline{1,2}$

$$\int_\Omega |u^{\varepsilon_n}(\tau)-u^{\varepsilon_m}(\tau)|\,dx+\int_\Omega |v^{\varepsilon_n}(\tau)-v^{\varepsilon_m}(\tau)|\,dx$$
$$\leq 2k\ \ meas(\Omega)+\int_0^\tau\int_\Omega k\lambda\left[|u^{\varepsilon_n}|^{p(x)-2}u^{\varepsilon_n}-|u^{\varepsilon_m}|^{p(x)-2}u^{\varepsilon_m}\right]dxdt$$
$$+k\int_0^\tau\int_\Omega\left[f_1^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n})-f_1^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]dxdt$$
$$+k\int_0^\tau\int_\Omega\left[f_2^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n})-f_2^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]dxdt$$
$$+k\int_\Omega\left|b_{\varepsilon_n}^1(u_0^{\varepsilon_n}-u_0^{\varepsilon_m})\right|dx+k\int_\Omega\left|b_{\varepsilon_n}^2(v_0^{\varepsilon_n}-v_0^{\varepsilon_m})\right|dx,$$

letting ε_n, $\varepsilon_m\to\infty$ and them $k\to 0$, to obtain

$$\sup_{\tau\in[0,T]}\int_\Omega |u^{\varepsilon_n}(\tau)-u^{\varepsilon_m}(\tau)|\,dx+\sup_{\tau\in[0,T]}\int_\Omega |v^{\varepsilon_n}(\tau)-v^{\varepsilon_m}(\tau)|\,dx$$
$$\leq\int_0^\tau\int_\Omega k\lambda\left[|u^{\varepsilon_n}|^{p(x)-2}u^{\varepsilon_n}-|u^{\varepsilon_m}|^{p(x)-2}u^{\varepsilon_m}\right]dxdt$$
$$+k\int_0^\tau\int_\Omega\left[f_1^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n})-f_1^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]dxdt$$
$$+k\int_0^\tau\int_\Omega\left[f_2^{\varepsilon_n}(x,t,u^{\varepsilon_n},v^{\varepsilon_n})-f_2^{\varepsilon_n}(x,t,u^{\varepsilon_m},v^{\varepsilon_m})\right]dxdt$$
$$+k\int_\Omega\left|b_{\varepsilon_n}^1(u_0^{\varepsilon_n}-u_0^{\varepsilon_m})\right|dx+k\int_\Omega\left|b_{\varepsilon_n}^2(v_0^{\varepsilon_n}-v_0^{\varepsilon_m})\right|dx.$$

□

- **Step 2: The limit of the solution of the approximated problem.** Arguing again as in [[3], [4], [5]] estimates (4.38), (4.40), (4.39) and (4.41) imply that, for a subsequence still indexed by ε,

$$(u^\varepsilon,v^\varepsilon)\text{ converge almost every where to }(u,v),\tag{4.52}$$

using (4.15), (4.34), (4.35) and (4.44), to get

$$T_k(u^\varepsilon)\text{ converge weakly to }T_k(u)\text{ in }L^{p-}\left(]0,T[\,;W_0^{1,p(.)}(\Omega)\right),\tag{4.53}$$

and

$$T_k(v^\varepsilon)\text{ converge weakly to }T_k(v)\text{ in }L^2\left(]0,T[\,;H_0^1(\Omega)\right),\tag{4.54}$$

$$\chi_{\{|u^\varepsilon|\le k\}}\mathcal{A}(x,t,\nabla u^\varepsilon) \rightharpoonup \eta_k \text{ weakly in } \left(L^{p'(\cdot)}(Q)\right)^N, \tag{4.55}$$

as ε tends to 0 for any $k>0$ and any $n\ge 1$ and where for any $k>0$, η_k belongs to $\left(L^{p'(\cdot)}(Q)\right)^N$. Since $\gamma(u^\varepsilon)$ is a continuous incrassing function, from the monotone convergence theorem and (4.25) and by (4.52), to obtain that

$$\gamma(u^\varepsilon) \text{ converge weakly to } \gamma(u) \text{ in } L^1(Q). \tag{4.56}$$

We now establish that $(b_1(u),b_2(v))$ belongs to $(L^\infty(]0,T[\,;L^1(\Omega)))^2$. Indeed using (4.20) and $\left|B_k^{i,\varepsilon}(s)\right| \ge \rho(|s|-1)$, $\forall i=\overline{1,2}$, leads to

$$\begin{aligned}\int_\Omega \left|b_\varepsilon^1(u^\varepsilon)\right|(t)dx + \int_\Omega \left|b_\varepsilon^2(v^\varepsilon)\right|(t)dx \;\le\;& \rho(2meas(\Omega)\\ &+ \|(f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon), f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon))\|_{(L^1(Q))^2}\\ &+ k\,\|\gamma(u^\varepsilon)\|_{L^1(Q)}\\ &+ k\left\|(b_\varepsilon^1(u_0^\varepsilon), b_\varepsilon^2(v_0^\varepsilon))\right\|_{(L^1(\Omega))^2}).\end{aligned}$$

By lemma (4.1) and (4.46), (4.47), we conclude that there exist two subsequences of u^{ε_n} and v^{ε_n}, still denoted by themselves for convenience, such that u^{ε_n} converges to a function u in $C(]0,T[;L^1(\Omega))$, v^{ε_n} converges to a function v in $C(]0,T[;L^1(\Omega))$. Using (4.25) and (4.10),(4.26), we have $(b_1(u),b_2(v))$ belongs to $(L^\infty(]0,T[\,;L^1(\Omega)))^2$. We are now in a position to exploit (4.46) and (4.47). Since $(u^\varepsilon,v^\varepsilon)$ is bounded in $(L^\infty(]0,T[\,;L^1(\Omega)))^2$, to get

$$\lim_{n\to+\infty}\left(\sup_\varepsilon meas\{|u^\varepsilon|\ge n\}\right)=0. \tag{4.57}$$

and

$$\lim_{n\to+\infty}\left(\sup_\varepsilon meas\{|v^\varepsilon|\ge n\}\right)=0. \tag{4.58}$$

The equi-integrability of the sequence $f_i^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)$ in $(L^1(Q))^2$. We shall now prove that $f_i^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)$ converges to $f_i(x,t,u,v)$ strongly in $(L^1(Q))^2$, for all $i=\overline{1,2}$ by using Vitali's theorem. Since $f_i^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)\to f_i(x,t,u,v)$ a.e in Q it suffices to prove that $f_i^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)$ are equi-integrable in Q. Let $\delta_1>0$ and $\mathbf{A}$ be a measurable subset belonging to $\Omega\times]0,T[$, we define the following sets

$$G_{\delta_1} = \{(x,t)\in Q:\ |u_n|\le\delta_1\}, \tag{4.59}$$

$$F_{\delta_1} = \{(x,t)\in Q:\ |u_n|>\delta_1\}. \tag{4.60}$$

Using the generalized Hölder's inequality and Poincaré inequality, to have

$$\int_{\mathbf{A}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt = \int_{\mathbf{A}\cap G_{\delta_1}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt + \int_{\mathbf{A}\cap F_{\delta_1}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt,$$

therfore

$$
\begin{aligned}
\int\limits_{\mathbf{A}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt &\leq \int\limits_{\mathbf{A}\cap G_{\delta_1}} \left(c_{k,\varepsilon}(x,t)+\sigma_{k,\varepsilon}\,|v^\varepsilon|^2\right)dxdt \\
&+ \int\limits_{\mathbf{A}\cap F_{\delta_1}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\leq \int\limits_{\mathbf{A}} c_{k,\varepsilon}(x,t)dxdt + \sigma_{k,\varepsilon}\int\limits_{Q} |\nabla T_{\delta_1}(v^\varepsilon)|^2\,dxdt \\
&\quad + \int\limits_{\mathbf{A}\cap F_{\delta_1}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\leq \int\limits_{\mathbf{A}} c_{k,\varepsilon}(x,t)dxdt + \sigma_{k,\varepsilon}\left(meas(\mathbf{Q})+1\right)^{\frac{1}{2}} \\
&\quad \left(\int\limits_{Q_T} |\nabla T_{\delta_1}(v^\varepsilon)|^2\,dxdt\right)^{\frac{1}{2}} + \int\limits_{\mathbf{A}\cap F_{\delta_1}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\leq K_1 + C_2\left(\frac{k}{\alpha}\left\|b_\varepsilon^2(v_0^\varepsilon)\right\|_{L^1(\Omega)}\right)^{\frac{1}{2}} \\
&+ \int\limits_{\mathbf{A}\cap F_{\delta_1}} \frac{1}{|u^\varepsilon|}\,|u^\varepsilon f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\leq K_2 + \int\limits_{\mathbf{A}\cap F_{\delta_1}} \frac{1}{\delta_1}\,|u^\varepsilon f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\leq K_2 + \frac{1}{\delta_1}\left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\left(\int\limits_{\mathbf{A}\cap F_{\delta_1}} |u^\varepsilon|^{p(x)}\,dxdt\right)^{\frac{1}{p^-}} \\
&\quad \left(\int\limits_{\mathbf{A}\cap F_{\delta_1}} |f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|^{p'(x)(p(x)-1)}\,dxdt\right)^{\frac{1}{p'^-}} \\
&\to 0 \text{ when } meas(\mathbf{A}) \to \mathbf{0}.
\end{aligned}
$$

Which shows that $f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)$ is equi-integrable. By using Vitali's theorem, to get

$$f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon) \to f_1(x,t,u,v) \text{ strongly in } L^1(Q). \tag{4.61}$$

Now we prove that

$$f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon) \to f_2(x,t,u,v) \text{ strongly in } L^1(Q). \tag{4.62}$$

Let $\delta_2 > 0$ and $\mathbf{A}$ be a measurable subset belonging to $\Omega\times]0,T[$, we define the following sets

$$G_{\delta_2} = \{(x,t) \in Q : \ |v_n| \le \delta_2\}, \tag{4.63}$$

$$F_{\delta_2} = \{(x,t) \in Q : \ |v_n| > \delta_2\}. \tag{4.64}$$

Using the generalized Hölder's inequality and Poincaré inequality, to get

$$\int_{\mathbf{A}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt = \int_{\mathbf{A}\cap G_{\delta_2}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt + \int_{\mathbf{A}\cap F_{\delta_2}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt,$$

therfore

$$\begin{aligned}
\int_{\mathbf{A}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt &\le \int_{\mathbf{A}\cap G_{\delta_2}} \left(G_k^\varepsilon(x,t) + \xi_k^\varepsilon\,|u^\varepsilon|^{p(x)-1}\right) dxdt \\
&+ \int_{\mathbf{A}\cap F_{\delta_2}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le \int_{\mathbf{A}} G_k^\varepsilon(x,t)dxdt + \xi_k^\varepsilon \int_Q |\nabla T_{\delta_2}(u^\varepsilon)|^{p(x)-1}\,dxdt \\
&\quad + \int_{\mathbf{A}\cap F_{\delta_2}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le \int_{\mathbf{A}} G_k^\varepsilon(x,t)dxdt + \xi_k^\varepsilon \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)(meas(\mathbf{Q})+1)^{\frac{1}{p^-}} \\
&\quad \left(\int_{Q_T} |\nabla T_{\delta_2}(u^\varepsilon)|^{(p(x)-1)p'(x)}\,dxdt\right)^{\frac{1}{p'^-}} \\
&+ \int_{\mathbf{A}\cap F_{\delta_2}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le K_3 + C_4\left(\frac{k}{\alpha}\left\|b_\varepsilon^1(u_0^\varepsilon)\right\|_{L^1(\Omega)}\right)^{\frac{1}{2}} \\
&+ \int_{\mathbf{A}\cap F_{\delta_2}} \frac{1}{|v^\varepsilon|}\,|v^\varepsilon f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le K_4 + \int_{\mathbf{A}\cap F_{\delta_2}} \frac{1}{\delta_2}\,|v^\varepsilon f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|\,dxdt \\
&\le K_4 + \frac{1}{\delta_2}\left(\int_{\mathbf{A}\cap F_{\delta_2}} |v^\varepsilon|^2\,dxdt\right)^{\frac{1}{2}}\left(\int_{\mathbf{A}\cap F_{\delta_2}} |f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)|^2\,dxdt\right)^{\frac{1}{2}} \\
&\to 0 \text{ when } meas(\mathbf{A}) \to \mathbf{0}.
\end{aligned}$$

Which shows that $f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)$ is equi-integrable. By using Vitali's theorem, to get

$$f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon) \to f_2(x,t,u,v) \text{ strongly in } L^1(Q). \tag{4.65}$$

Using (4.56), (4.61) and the equi-integrability of the sequence $|b^1_\varepsilon(u^\varepsilon_0)|$ in $L^1(\Omega)$ and $|b^2_\varepsilon(v^\varepsilon_0)|$ in $L^1(\Omega)$, we deduce that

$$\lim_{n\to+\infty}\left(\sup_{\varepsilon}\left(\int\limits_{\{n\le|u^\varepsilon|\le n+1\}}\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt+\int\limits_{\{n\le|v^\varepsilon|\le n+1\}}|\nabla v^\varepsilon|^2dxdt\right)\right)=0. \tag{4.66}$$

- **Step 4: Strong convergence.** The specifie time regularization of $T_k(u)$ (for fixed $k\ge 0$) is defined as follows. Let $(v_0^\mu)_\mu$ be a sequaence in $L^\infty(\Omega)\cap W_0^{1,p(.)}(\Omega)$ such that $\|v_0^\mu\|_{L^\infty(\Omega)}\le k,\ \ \forall\mu>0$, and $v_0^\mu\to T_k(u_0)$ a.e in Ω with $\frac{1}{\mu}\|v_0^\mu\|_{L^{p(.)}(\Omega)}\to 0$ as $\mu\to+\infty$.

For fixed $k\ge 0$ and $\mu>0$, let us consider the unique solution $T_k(u)_\mu\in L^\infty(\Omega)\cap L^{p-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right)$ of the monotone problem

$$\frac{\partial T_k(u)_\mu}{\partial t}+\mu\left(T_k(u)_\mu-T_k(u)\right)=0 \text{ in } \mathcal{D}'(Q), \tag{4.67}$$

$$T_k(u)_\mu(t=0)=v_0^\mu. \tag{4.68}$$

The behavior of $T_k(u)_\mu$ as $\mu\to+\infty$ is investigated in [9] and we just recall here that (4.67)-(4.68) imply that

$$T_k(u)_\mu\to T_k(u) \text{ strongly in } L^{p-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right) \text{ a.e in } Q,\ \text{ as } \mu\to+\infty, \tag{4.69}$$

with $\|T_k(u)_\mu\|_{L^\infty(\Omega)}\le k$, for any μ, and $\frac{\partial T_k(u)_\mu}{\partial t}\in L^{(p-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right)$.

The main estimate is the following

Lemma 4.2. *Let S be an increasing $C^\infty(\mathbb{R})-$ function such that $S(r)=r$ for $r\le k$, and suppS' is compact. Then*

$$\liminf_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\left\langle\frac{\partial B^1_S(u^\varepsilon)}{\partial t},(T_k(u^\varepsilon)_\mu-T_k(u))\right\rangle dt\ge 0,$$

where here $\langle .,.\rangle$ denotes the duality pairing between $L^1(\Omega)+W^{-1,p'(.)}(\Omega)$ and $L^\infty(\Omega)\cap W_0^{1,p(.)}(\Omega)$, and where $B^1_S(z)=\int_0^z b_1'(r)S'(r)dr$.

Proof. See [5], Lemma 1. □

Now we are to prove that the weak limit η_k and we prove the weak L^1 convergence of the "truncted" energy $\mathcal{A}(x,t,\nabla T_k(u^\varepsilon))$ as ε tends to 0. In order to show this result we recall the lemma below.

Lemma 4.3. *The subsequence of u^ε defined in step 3 satisfies*

$$\limsup_{\varepsilon\to 0}\int_Q\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon)dxdt\le\int_Q\eta_k\nabla T_k(u)dxdt, \tag{4.70}$$

$$\lim_{\varepsilon\to 0}\int_Q \left[\mathcal{A}\left(x,t,\nabla u^{\varepsilon}_{\chi_{\{|u^{\varepsilon}|\leq k\}}}\right)-\mathcal{A}\left(x,t,\nabla u_{\chi_{\{|u|\leq k\}}}\right)\right]$$

$$\times\left[\nabla u^{\varepsilon}_{\chi_{\{|u^{\varepsilon}|\leq k\}}}-\nabla u_{\chi_{\{||\leq k\}}}\right]dxdt=0 \tag{4.71}$$

$\eta_k=\mathcal{A}\left(x,t,\nabla u_{\chi_{\{|u|\leq k\}}}\right)$ *a.e in* Q*, for any* $k\geq 0$*, as* ε *tends to* 0.

$$\mathcal{A}\left(x,t,\nabla u^{\varepsilon}\right)\nabla T_k(u^{\varepsilon})\to\mathcal{A}\left(x,t,\nabla u\right)\nabla T_k(u) \text{ weakly in } L^1\left(Q\right). \tag{4.72}$$

Proof. Let us introduce a sequence of increasing $C^{\infty}(\mathbb{R})$-functions S_n such that, for any $n\geq 1$

$$\begin{cases} S_n(r)=r \text{ if } |\mathrm{r}|\leq n,\\ \operatorname{supp}\left(S_n'\right)\subset\left[-(n+1),(n+1)\right],\\ \|S_n''\|_{L^{\infty}(\mathbb{R})}\leq 1. \end{cases} \tag{4.73}$$

For fixed $k\geq 0$, we consider the test function $S_n'(u^{\varepsilon})\left(T_k(u_{\varepsilon})-(T_k(u))_{\mu}\right)$ in (4.15), we use the definition (4.73) of S_n' and we definie $W^{\varepsilon}_{\mu}=T_k(u_{\varepsilon})-(T_k(u))_{\mu}$, to get

$$\int_0^T\left\langle\left(B^1_S(u^{\varepsilon})\right)_t,W^{\varepsilon}_{\mu}\right\rangle dt+\int_Q S_n'(u^{\varepsilon})\mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla W^{\varepsilon}_{\mu}dxdt \tag{4.74}$$
$$+\int_Q S_n''(u^{\varepsilon})\mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla u^{\varepsilon}W^{\varepsilon}_{\mu}dxdt+\int_Q\gamma(u^{\varepsilon})S_n'(u^{\varepsilon})W^{\varepsilon}_{\mu}dxdt$$
$$=\int_Q f_1^{\varepsilon}(x,t,u^{\varepsilon},v^{\varepsilon})S_n'(u^{\varepsilon})W^{\varepsilon}_{\mu}dxdt.$$

Now we pass to the limit in (4.74) as $\varepsilon\to 0$, $\mu\to+\infty$, $n\to+\infty$ for k real number fixed. In order to perform this task, we prove below the following results for any $k\geq 0$:

$$\liminf_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\left\langle\left(B^1_S(u^{\varepsilon})\right)_t,W^{\varepsilon}_{\mu}\right\rangle dt\geq 0 \text{ for any } n\geq k, \tag{4.75}$$

$$\lim_{n\to+\infty}\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q S_n''(u^{\varepsilon})\mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla u^{\varepsilon}W^{\varepsilon}_{\mu}dxdt=0, \tag{4.76}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q\gamma(u^{\varepsilon})S_n'(u^{\varepsilon})W^{\varepsilon}_{\mu}dxdt=0, \text{ for any } n\geq 1, \tag{4.77}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q f_1^{\varepsilon}(x,t,u^{\varepsilon},v^{\varepsilon})S_n'(u^{\varepsilon})W^{\varepsilon}_{\mu}dxdt=0, \text{ for any } n\geq 1. \tag{4.78}$$

Proof of (4.75). In view of the definition W^{ε}_{μ}, we apply lemma (4.2) with $S=S_n$ for fixed $n\geq k$. As a consequence, (4.75) hold true. □

Proof of (4.76). For any $n \geq 1$ fixed, we have $supp(S''_n) \subset [-(n+1), -n] \cup [n, n+1]$, $\left\|W^\varepsilon_\mu\right\|_{L^\infty(Q)} \leq 2k$ and $\|S''_n\|_{L^\infty(\mathbb{R})} \leq 1$, to get

$$\left|\int_Q S''_n(u^\varepsilon)\mathcal{A}\left(x,t,\nabla u^\varepsilon\right)\nabla u^\varepsilon W^\varepsilon_\mu dxdt\right| \tag{4.79}$$
$$\leq \quad 2k \int_{\{n\leq|u^\varepsilon|\leq n+1\}} \mathcal{A}\left(x,t,\nabla u^\varepsilon\right)\nabla u^\varepsilon dxdt,$$

for any $n \geq 1$, by (4.66) it possible to etablish (4.76) □

Proof of (4.77). For fixed $n \geq 1$ and in view (4.56) . Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $n \geq 1$

$$\lim_{\varepsilon\to 0}\int_Q \gamma(u^\varepsilon)S'_n(u^\varepsilon)W^\varepsilon_\mu \ dxdt = \int_Q \gamma(u)S'_n(u)(T_k(u) - T_k\left(u\right)_\mu)dxdt. \tag{4.80}$$

Appealing now to (4.69) and passing to the limit as $\mu \to +\infty$ in (4.80) allows to conclude that (4.77) holds true. □

Proof of (4.78). By (4.11), (4.61) and Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $n \geq 1$, it is possible to pass to the limit for $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0}\int_Q f^\varepsilon_1(x,t,u^\varepsilon,v^\varepsilon)S'_n(u^\varepsilon)W^\varepsilon_\mu \ dxdt = \int_Q f_1(x,t,u,v)S'_n(u)(T_k(u) - T_k\left(u\right)_\mu)dxdt,$$

using (4.69) permits to the limit as μ tends to $+\infty$ in the above equality to obtain (4.78). □

Now turn back to the proof of Lemma (4.3), due to (4.75)-(4.78), we are in a position to pass to the limit-sup when $\varepsilon \to 0$, then to the limit-sup when $\mu \to +\infty$ and then to the limit as $n \to +\infty$ in (4.74). Using the definition of W^ε_μ, we deduce that for any $k \geq 0$,

$$\lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)S'_n(u^\varepsilon)\nabla\left(T_k(u^\varepsilon) - T_k(u)_\mu\right)dxdt \leq 0.$$

Since $\mathcal{A}(x,t,\nabla u^\varepsilon)S'_n(u^\varepsilon)\nabla T_k(u^\varepsilon) = \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon)$ fo $k \leq n$, the above inequality implies that for $k \leq n$,

$$\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon)dxdt \tag{4.81}$$
$$\leq \quad \lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(t,x,\nabla u^\varepsilon)S'_n(u^\varepsilon)\nabla T_k(u)_\mu dxdt.$$

Due to (4.55), to have

$$\mathcal{A}(x,t,\nabla u^\varepsilon)S'_n(u^\varepsilon) \to \eta_{n+1}S'_n(u) \text{ weakly in } \left(L^{p'(\cdot)}\left(Q\right)\right)^N \text{ as } \varepsilon \to 0,$$

and the strong convergence of $T_k(u)_\mu$ to $T_k(u)$ in $L^{p^-}(]0,T[;W_0^{1,p}(\Omega))$ as $\mu \to +\infty$, to get

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(u^\varepsilon)\nabla T_k(u)_\mu dxdt \tag{4.82}$$
$$= \int_Q S_n'(u)\eta_{n+1}\nabla T_k(u)dxdt = \int_Q \eta_{n+1}\nabla T_k(u)dxdt,$$

as soon as $k \le n$, since $S_n'(s) = 1$ for $|s| \le n$. Now, for $k \le n$, to have

$$S_n'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)_{\chi_{\{|u^\varepsilon|\le k\}}} = \mathcal{A}(x,t,\nabla u^\varepsilon)_{\chi_{\{|u^\varepsilon|\le k\}}} \text{ a.e in } Q.$$

Letting $\varepsilon \to 0$, to obtain

$$\eta_{n+1}\chi_{\{|u|\le k\}} = \eta_k\chi_{\{|u|\le k\}} \text{ a.e in } Q - \{|u| = k\} \text{ for } k \le n.$$

Recalling (4.81) and (4.82) allows to conclude that (4.70) holds true. □

Proof of (4.71). Let $k \ge 0$ be fixed. We use the monotone character (3.3) of $\mathcal{A}(x,t,\xi)$ with respest to ξ, to obtain

$$I^\varepsilon = \int_Q \left(\mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}}) - \mathcal{A}(x,t,\nabla u\chi_{\{|u|\le k\}})\right)\left(\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}} - \nabla u\chi_{\{|u|\le k\}}\right)dxdt \ge 0. \tag{4.83}$$

Inequality (4.83) is split into $I^\varepsilon = I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon$ where

$$I_1^\varepsilon = \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}})\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}}dxdt,$$
$$I_2^\varepsilon = -\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}})\nabla u\chi_{\{|u|\le k\}}dxdt,$$
$$I_3^\varepsilon = -\int_Q \mathcal{A}(x,t,\nabla u\chi_{\{|u|\le k\}})\left(\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}} - \nabla u\chi_{\{|u|\le k\}}\right)dxdt.$$

We pass to the limit-sup as $\varepsilon \to 0$ in I_1^ε, I_2^ε and I_3^ε. Let us remark that we have $u^\varepsilon = T_k(u^\varepsilon)$ and $\nabla u^\varepsilon\chi_{\{|u^\varepsilon|\le k\}} = \nabla T_k(u^\varepsilon)$ a.e in Q, and we can assume that k is such that $\chi_{\{|u^\varepsilon|\le k\}}$ almost everywhere converges to $\chi_{\{|u|\le k\}}$(in fact this is true for almost every k, see Lemma 3.2 in [6]). Using (4.70), to obtain

$$\lim_{\varepsilon\to 0} I_1^\varepsilon = \lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon)dxdt \tag{4.84}$$
$$\le \int_Q \eta_k\nabla T_k(u)dxdt.$$

In view of (4.53) and (4.55), to have

$$\lim_{\varepsilon\to 0} I_2^\varepsilon = -\lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon \chi_{\{|u^\varepsilon|\leq k\}})\,(\nabla T_k(u))\,dxdt = -\int_Q \eta_k\,(\nabla T_k(u))\,dxdt. \tag{4.85}$$

As a consequence of (4.53), we have for all $k>0$

$$\lim_{\varepsilon\to 0} I_3^\varepsilon = -\int_Q \mathcal{A}(x,t,\nabla u\chi_{\{|u|\leq k\}})\,(\nabla T_k(u^\varepsilon)-\nabla T_k(u))\,dxdt = 0. \tag{4.86}$$

Taking the limit-sup as $\varepsilon\to 0$ in (4.83) and using (4.84), (4.85) and (4.86) show that (4.71) holds true. □

Proof of (4.72). Using (4.71) and the usual Minty argument applies it follows that (4.72) holds true.

Lemma 4.4. *$\nabla T_k(v^\varepsilon)$ converges to $\nabla T_k(v)$ in $(L^2(Q))^N$.*

Proof. Denote $V_\mu^\varepsilon = T_k(v_\varepsilon)-(T_k(v))_\mu$ and choose $S_n'(v^\varepsilon)\left(T_k(v_\varepsilon)-(T_k(v))_\mu\right)$ the test function in (4.16). One can get that

$$\int_0^T \left\langle \left(B_S^2(v^\varepsilon)\right)_t, V_\mu^\varepsilon\right\rangle dt + \int_Q S_n'(v^\varepsilon)\nabla v^\varepsilon \nabla V_\mu^\varepsilon dxdt + \int_Q S_n''(v^\varepsilon)|\nabla v^\varepsilon|^2 V_\mu^\varepsilon dxdt = \int_Q f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S_n'(v^\varepsilon)V_\mu^\varepsilon dxdt. \tag{4.87}$$

By a similar discussion, one has

$$\liminf_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T \left\langle \left(B_S^2(v^\varepsilon)\right)_t, V_\mu^\varepsilon\right\rangle dt \geq 0 \text{ for any } n\geq k, \tag{4.88}$$

$$\lim_{n\to+\infty}\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q S_n''(v^\varepsilon)|\nabla v^\varepsilon|^2 V_\mu^\varepsilon dxdt = 0, \tag{4.89}$$

and

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S_n'(v^\varepsilon)V_\mu^\varepsilon dxdt = 0, \text{ for any } n\geq 1. \tag{4.90}$$

Hence

$$\lim_{n\to+\infty}\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q S_n'(v^\varepsilon)\nabla v^\varepsilon \nabla V_\mu^\varepsilon dxdt \leq 0. \tag{4.91}$$

□

Similarly, one gets that $\nabla T_k(v^\varepsilon)$ converges to $\nabla T_k(v)$ in $(L^2(Q))^N$. □

- **Step 5:** In this step we prove that (u,v) satisfies (4.3), (4.4)-(4.7) . For any fixed $n \geq 0$ one has

$$\begin{aligned}&\int\limits_{\{n\leq|u^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt\\ =& \int\limits_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_{n+1}(u^\varepsilon)dxdt - \int\limits_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_n(u^\varepsilon)dxdt.\end{aligned}$$

According to (4.55) and (4.72) one is at liberty to pass to the limit as ε tends to 0 for fixed $n \geq 1$ and to obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int\limits_{\{n\leq|u^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt\\ =& \int\limits_Q \mathcal{A}(x,t,\nabla u)\nabla T_{n+1}(u)dxdt - \int\limits_Q \mathcal{A}(x,t,\nabla u)\nabla T_n(u)dxdt\\ =& \int\limits_{\{n\leq|u^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u)\nabla u dxdt.\end{aligned} \tag{4.92}$$

Letting n tends to $+\infty$ in (4.92), it follows from estimate (4.66), that

$$\lim_{\varepsilon\to 0}\lim \int\limits_{\{n\leq|u^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt = 0.$$

Similarly, one can prove

$$\lim_{\varepsilon\to 0}\lim \int\limits_{\{n\leq|v^\varepsilon|\leq n+1\}} |\nabla v^\varepsilon|^2 dxdt = 0.$$

Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact. Let k be a positive real number such that $\text{supp}(S') \subset [-k,k]$. Pontwise multiplication of that approximate equation (4.15) by $(S'(u^\varepsilon), S'(v^\varepsilon))$ leads to

$$\begin{aligned}&\left(B_S^1(u^\varepsilon)\right)_t - div(S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon))\\ &+S''(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla(u^\varepsilon) + \gamma(u^\varepsilon)S'(u^\varepsilon) = f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S'(u^\varepsilon) \text{ in } \mathcal{D}'(Q),\end{aligned} \tag{4.93}$$

and

$$\begin{aligned}&\left(B_S^2(v^\varepsilon)\right)_t - div(S'(v^\varepsilon)\nabla v^\varepsilon)\\ &+S''(v^\varepsilon)|\nabla(v^\varepsilon)|^2 = f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S'(v^\varepsilon) \text{ in } \mathcal{D}'(Q).\end{aligned} \tag{4.94}$$

In what follows to pass to the limit as ε tends to 0 in each term of (4.93). Since S is bounded, and $(S(u^\varepsilon), S(v^\varepsilon))$ converges to $(S(u), S(v))$ a.e in Q and in $(L^\infty(Q))^2$ *-weak, then

$\left(\left(B_S^1(u^\varepsilon)\right)_t, \left(B_S^2(v^\varepsilon)\right)_t\right)$ converges to $\left(\left(B_S^1(u)\right)_t, \left(B_S^1(v)\right)_t\right)$ in $\mathcal{D}'(Q)$ as ε tends to 0. Since supp$(S') \subset [-k, k]$,

$$S'(u^\varepsilon)\mathcal{A}(t,x,\nabla u^\varepsilon) = S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\chi_{\{|u^\varepsilon|\leq k\}} \text{ a.e in } Q.$$

The pointwise convergence of u^ε to u as ε tends to 0, the bounded character of S and (4.72) of Lemma(4.3) imply that $S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)$ converges to $S'(u)\mathcal{A}(x,t,\nabla u)$ weakly in $\left(L^{p'(\cdot)}(Q)\right)^N$ as ε tends to 0, because $S'(u) = 0$ for $|u| \geq k$ a.e in Q and $S'(v^\varepsilon)\nabla v^\varepsilon$ converges to $S'(v)\nabla v$ weakly in $L^2(Q)$ as ε tends to 0. The pointwise convergence of u^ε to u, the bounded character of S', S'' and (4.72) of Lemma (4.3) allow to conclude that

$$S''(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon) \to S''(u)\mathcal{A}(x,t,\nabla u)\nabla T_k(u) \text{ weakly in } L^1(Q)$$

as $\varepsilon \to 0$, and lemma (4.1) shows that

$$S''(v^\varepsilon)\nabla^\varepsilon v\nabla T_k(v^\varepsilon) \to S''(v)\nabla v\nabla T_k(v) \text{ weakly in } L^1(Q).$$

The use of (4.56) to obtain that $\gamma(u^\varepsilon)S'(u^\varepsilon)$ converges to $\gamma(u)S'(u)$ in $L^1(Q)$, and we use (4.11), (4.53) and we obtain that

$$f_1^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S'(u^\varepsilon) \text{ converges to } f_1(x,t,u,v)S'(u) \text{ in } L^1(Q)$$

and

$$f_2^\varepsilon(x,t,u^\varepsilon,v^\varepsilon)S'(v^\varepsilon) \text{ converges to } f_2(x,t,u,v)S'(v) \text{ in } L^1(Q).$$

As a consequence of the above convergence result, the position to pass to the limit as ε tends to 0 in equation (4.93) and (4.94), we conclude that (u,v) satisfies (4.4) and (4.5).

It remains to show that $S(u)$ satisfies the initial condition (4.6) and $S(v)$ satisfies the initial condition (4.7). To this end, firstly remark that, S being bounded, $(S(u^\varepsilon), S(v^\varepsilon))$ is bounded in $(L^\infty(Q))^2$, $(B_S^1(u^\varepsilon), B_S^2(v^\varepsilon))$ is bounded in $L^\infty(Q) \times L^\infty(Q)$. Secondly, (4.93) and (4.94), the above considerations on the behavior of the terms of this equation show that $\frac{\partial B_S^1(u^\varepsilon)}{\partial t}$ is bounded in $L^1(Q) + L^{(p_-)'}(]0,T[; W^{-1,p'(\cdot)}(\Omega))$ and $\frac{\partial B_S^2(v^\varepsilon)}{\partial t}$ is bounded in $L^1(Q) + L^2(]0,T[; H_0^1(\Omega))$. As a consequence, an Aubin's type lemma ([20], Corollary 4) implies that $(B_S^1(u^\varepsilon), B_S^2(v^\varepsilon))$ lies in a compact set of $(C(]0,T[; L^1(\Omega)))^2$. It follows that, on the one hand, $B_S^1(u^\varepsilon)(t=0)$ converges to $B_S^1(u)(t=0)$ strongly in $L^1(\Omega)$ and $B_S^2(v^\varepsilon)(t=0)$ converges to $B_S^2(v)(t=0)$ strongly in $L^1(\Omega)$. Due to (4.10), to conclude that (4.6) and (4.7) holds true. As a conclusion of **Step 3** and **Step 5**, the proof of Theorem (4.1) is complete.

□

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