



ON THE NORMALITY OF THE PRODUCT OF TWO OPERATORS IN HILBERT SPACE

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ABSTRACT. In this paper we present the results of the maximality of operators not necessarily bounded. For that, we will see the results obtained by operators in situation of extension. Regarding the normal product of normal operators we seem to be the key to maximality.

1. INTRODUCTION

First, we assume that all operators operators are non necessarily bounded on a complex Hilbert space H , Let us, however, recall some notations that will be met below. If A and B are two operators with dense domains $D(A)$ and $D(B)$ respectively, then B is called an extension of A , and we write $A \subset B$, if $D(A) \subset D(B)$ and if A and B coincide on $D(A)$. The product AB of two operators is defined by

$$AB(x) = A(Bx) \quad \text{for } x \in D(AB)$$

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where

$$D(AB) = \{x \in D(B) : Bx \in D(A)\}.$$

Recall too that the unbounded operator A , defined on $D(A)$, is said to be invertible if there exists an everywhere defined (i.e. on the whole of H) bounded operator B , which then will be designated by A^{-1} , such that

$$A^{-1}A \subset AA^{-1} = I$$

where I is the identity operator on H . An operator A is said to be closed if its graph is closed in $H \oplus H$. The closing of the domain $D(A)$ of A implies the closing of A if A is bounded on $D(A)$. It is known that the product operators AB is closed if for instance A is closed and $B \in B(H)$, or $A^{-1} \in B(H)$ and B is closed. We also recall that an operator S is said to be densely defined if its domain $D(S)$ is dense in H . It is known that in such case its adjoint S^* exists and is unique. If $T \subset S$, then $S^* \subset T^*$. Notice that if S , T and ST are all densely defined, then we are only sure of

$$T^*S^* \subset (ST)^*,$$

and a full equality occurring if e.g. $T^{-1} \in B(H)$ or $S \in B(H)$. The bounded operator $T \in B(H)$ is said to be unitary if $TT^* = T^*T = I$. A densely defined operator S is said to be symmetric if $S \subset S^*$. It is called self-adjoint if $S = S^*$. S is called essential self-adjoint if the closure of S is self-adjoint (i.e. $(\overline{S})^* = \overline{S}$). We say that S is normal if S is densely defined, and $\|Sx\| = \|S^*x\|$ for all $x \in D(S) = D(S^*)$ (hence from known facts normal operators are automatically closed). Recall that the previous is equivalent to S is closed and $SS^* = S^*S$. Other classes of operators are defined in the usual fashion. Let us also agree that any operator is linear and non necessarily bounded unless we specify that it belongs to $B(H)$. We also assume the basic theory of operators (see e.g. [1] or [20]). We do recall the celebrated Fuglede-Putnam Theorem though:

Theorem 1.1. *(for a proof, see e.g. [1]) Let $T \in B(H)$ and let M, N be two normal non necessarily bounded operators. Then*

$$TN \subset MT \implies TN^* \subset M^*T.$$

One of the main objectives of this work is to impose conditions to obtain other results, starting from an extension. The following theorem and corollary result are a powerful tool to prove results on unbounded operators. For instance, Statement (3) of the next theorem is used in the proof of the "unbounded" version of the spectral theorem of normal operators (see e.g. [15]). For other uses, see e.g. [6] or [10].

Let us now list some known (see e.g. [15] or [16]) maximality results:

Theorem 1.2. *Let S, T be two operators with (dense when necessary) domains $D(S)$ and $D(T)$ respectively such that $S \subset T$. Then $S = T$ when one of the following occurs:*

- (1) S is surjective and T is injective.
- (2) T is symmetric and S is self-adjoint (resp. normal). We then say that self-adjoint (resp. normal) operators are maximally symmetric.
- (3) T and S are normal (we say that normal operators are maximally normal). Hence, self-adjoint (resp. normal) operators are maximally normal (resp. self-adjoint).

Commutativity of operators must be handled with care. First, recall the definition of two strongly commuting (normal) operators (see e.g. [16]):

Definition 1.1. *Let A and B be two normal operators. We say that A and B strongly commute if all the projections in their associated projection-valued measures commute.*

Now, let us recall results obtained by Devinatz-Nussbaum (and von Neumann) on strong commutativity:

Theorem 1.3. *(Devinatz-Nussbaum-von Neumann, [2] and cf. [13]). If there exists a self-adjoint operator A such that $A \subseteq BC$, where B and C are self-adjoint, then B and C strongly commute.*

Corollary 1.1. *Let A , B and C be self-adjoint operators. Then*

$$A \subseteq BC \implies A = BC$$

2. MAIN RESULTS

The normality of unbounded products of normal operators has been studied recently. See e.g. [5] and the references therein. We recall

Theorem 2.1. *(for a proof, see e.g. [11]) Let A, B be normal operators with $B \in B(H)$. If $BA \subset AB$, then AB and \overline{BA} are both normal (and so $AB = \overline{BA}$).*

Theorem 2.2. *Let T, A, B be non necessarily bounded operators such that T and B are self-adjoint with $B \in B(H)$ and A is normal. Assume further that $BA \subset T$. Then*

$$\overline{BA} = T.$$

Proof. We have:

$$\begin{aligned} BA \subset T &\implies BA \subset T \subset A^*B \\ &\implies BA^* \subset AB \text{ (by Fuglede-Putnam Theorem)}. \end{aligned}$$

It is clear that BA is closable and densely defined. Let's show now that \overline{BA} is normal. Indeed

$$(\overline{BA})^* \overline{BA} = (BA)^*(BA)^{**} = A^*B(A^*B)^* \supset A^*BBA \supset BABA \supset B^2A^*A.$$

Since the operators $B^2, A^*A, (\overline{BA})^*\overline{BA}$ are self-adjoint with $B^2 \in B(H)$, then

$$(\overline{BA})^*\overline{BA} \subset A^*AB^2,$$

by corollary 1.1, we obtain

$$(\overline{BA})^*\overline{BA} = A^*AB^2.$$

Similarly, we obtain

$$\overline{BA}(\overline{BA})^* = A^*AB^2,$$

establishing the normality of \overline{BA} . Theorem 1.2 gives us

$$\overline{BA} = T.$$

□

Corollary 2.1. *Let T, B, A be non necessarily bounded operators such that T is normal and B symmetric and invertible (hence B is self-adjoint) and that A is self-adjoint, then*

$$T \subset BA \implies A = \overline{B^{-1}T}.$$

Proof. Clearly,

$$T \subset BA \implies B^{-1}T \subset A,$$

by theorem 2.2, we obtain

$$A = \overline{B^{-1}T}.$$

□

Proposition 2.1. *Let A, B and T be operators where $B \in B(H)$. Assume that T^* is symmetric, B is self-adjoint and A is normal. If $T \subset AB$, then BA is essential self-adjoint.*

Proof. Since AB is closed, we have

$$T \subset AB \implies \overline{T} \subset AB,$$

and

$$\begin{aligned} T \subset AB &\implies BA^* \subset T^* \subset T^{**} = \overline{T} \subset AB \\ &\implies BA \subset A^*B \text{ (by Fuglede-Putnam Theorem)} \\ &\implies \overline{BA} \subset A^*B \text{ (because } A^*B \text{ is closed)}. \end{aligned}$$

We can show the normality of \overline{BA} . We have

$$(\overline{BA})^*\overline{BA} = A^*B(A^*B)^* \supset A^*BBA \supset B^2A^*A.$$

Since B^2, A^*A are self-adjoint with $B^2 \in B(H)$, then

$$(\overline{BA})^* \overline{BA} \subset A^*AB^2,$$

by corollary 1.1, we obtain

$$(\overline{BA})^* \overline{BA} = A^*AB^2.$$

Similarly,

$$\overline{BA}(\overline{BA})^* = A^*AB^2,$$

i.e. \overline{BA} is normal. Since $(\overline{BA})^*$ too is normal and normal operators are maximally normal, we get

$$\overline{BA} = (\overline{BA})^* = A^*B,$$

i.e. BA is essentially self-adjoint. □

Proposition 2.2. *Let T, B, A be non necessarily bounded operators such that T is self-adjoint and B symmetric and invertible (hence B is self-adjoint) and that A^* is symmetric. Then*

$$AB \subset T \implies TB^{-1} = \overline{B^{-1}T} = \overline{A}.$$

Proof. Clearly,

$$\begin{aligned} AB \subset T &\implies A \subset TB^{-1} \\ &\implies B^{-1}T \subset A^* \subset A^{**} = \overline{A} \subset \overline{TB^{-1}} = TB^{-1}. \end{aligned}$$

From theorem 2.1, we have TB^{-1} and $B^{-1}T$ are normal. Hence

$$TB^{-1} = \overline{B^{-1}T} = \overline{A}.$$

□

Proposition 2.3. *Let T, B, A be non necessarily bounded operators such that T and B are self-adjoint with $B \in B(H)$ and invertible and A^* is symmetric. If $BA \subset T$, then $\overline{BA} = \overline{AB} = T$.*

Proof. We have:

$$\begin{aligned} BA \subset T &\implies A \subset B^{-1}T \\ &\implies TB^{-1} \subset A^* \subset A^{**} = \overline{A} \subset \overline{B^{-1}T} = B^{-1}T. \end{aligned}$$

Left and right multiplying by B give

$$BT \subset B\overline{AB} \subset TB.$$

By theorem 2.1, we obtain TB, BT are normal and

$$TB = BT = B\overline{AB},$$

hence

$$B\bar{A} = \overline{AB} = T.$$

□

Theorem 2.3. *Let T, A, B be non necessarily bounded operators such that A and B are normal with $B \in B(H)$ and T is unitary. Assume further that TA est normal. If $BA \subset TAB$ and , then AB and \overline{BA} are normal. Also, if A and T commute, then TAB is normal.*

Proof. Obviously,

$$\begin{aligned} BA \subset TAB &\implies B^*A^*T^* \subset A^*B^* \\ &\implies B^*TA \subset AB^* \text{ (by Fuglede-Putnam Theorem).} \\ &\implies BA^* \subset A^*T^*B. \end{aligned}$$

It is clear that AB is closed and we have:

$$(AB)^*AB \supset B^*A^*AB \supset B^*A^*BA \supset B^*BA^*A.$$

Since $(AB)^*AB, B^*B, A^*A$ are self-adjoint with $B^*B \in B(H)$, then

$$(AB)^*AB \subset A^*AB^*B,$$

and by corollary 1.1, we obtain

$$(AB)^*AB = A^*AB^*B.$$

We also have,

$$AB(AB)^* \supset ABB^*A^* \supset AB^*BA^* \supset B^*TABA^* \supset B^*TT^*BAA^* = B^*BAA^*.$$

Similarly, we obtain

$$AB(AB)^* = A^*AB^*B \text{ (because } A \text{ and } B \text{ are normal).}$$

and this marks the end of the proof of the normality of AB . Let's show now that \overline{BA} is normal. Indeed

$$(\overline{BA})^*\overline{BA} = A^*B^*(A^*B^*)^* \supset A^*B^*BA \supset B^*A^*T^*BA \supset B^*BA^*A,$$

i.e.

$$(\overline{BA})^*\overline{BA} = A^*AB^*B$$

Similarly,

$$\overline{BA}(\overline{BA})^* = A^*AB^*B,$$

that is, \overline{BA} is normal.

Let's show now that TAB is normal. We have TAB is closed because T is invertible and AB is closed. we Also have

$$\begin{aligned} TA \subset AT &\implies TA^* \subset A^*T \text{ (by Fuglede-Putnam Theorem).} \\ &\implies T^*A \subset AT^* \\ &\implies T^*A^* \subset A^*T^* \text{ (by Fuglede-Putnam Theorem).} \end{aligned}$$

Indeed

$$TAB(TAB)^* \supset B^*A^*T^*TAB = B^*A^*AB \supset B^*A^*T^*BA \supset B^*BA^*A,$$

since $TAB(TAB)^*, B^*B, A^*A$ are self-adjoint with $B^*B \in B(H)$, we get

$$TAB(TAB)^* \subset A^*AB^*B.$$

By corollary 1.1, we obtain

$$TAB(TAB)^* = A^*AB^*B.$$

Similarly,

$$(TAB)^*TAB = TAB(TAB)^* = A^*AB^*B,$$

and this marks the end of the proof of the normality of TAB . □

The following result is already seen in ([12]), we can consider it as a consequence of the preceding theorem. Also for $T = I$ (where I is the identity operator on H) we will get the theorem 2.1.

Corollary 2.2. *Let A, B be normal operators with $B \in B(H)$. Assume that $BA \subset \lambda AB$ where $\lambda \in \mathbb{C}$. Then AB and \overline{BA} are both normal if $|\lambda| = 1$ (and so $AB = \lambda \overline{BA}$).*

Proof. For $T = \lambda I$ where I is the identity operator on H , we obtain $T^* = \overline{\lambda}I$, i.e. T is unitary (because $|\lambda| = 1$). Theorem 2.3 yields the normality of AB, \overline{BA} and λAB . Since AB is closed, we may also write

$$BA \subset \lambda AB \implies \overline{BA} \subset \lambda AB$$

But, normal operators are maximally normal, therefore, we finally infer that

$$\overline{BA} = \lambda AB.$$

□

Closely related to the previous results, we have another proof for the closure of bounded operators on a domain.

Proposition 2.4. *Let T is a bounded operator on $D(T)$. Then T is closed if $D(T)$ is closed on H .*

The proof requires the following lemma whose proof is very akin to the one in [15].

Lemma 2.1. *Let $f : X \rightarrow Y$ is continuous such that Y is Hausdorff space. Then the graph of f is closed on $X \times Y$.*

Now we prove proposition 2.4

Proof. We denote the graph of T by $G_r(T)$. By lemma 2.1, we obtain $G_r(T)$ is closed on $D(T) \times H$, i.e. $(G_r(T))^C$ is open. We may write

$$(G_r(T))^C = \bigcup_{(i,j) \in I \times J} U_i^1 \times U_j^2$$

where I, J are arbitrary and U_i^1, U_j^2 are open on $D(T), H$ respectively. Hence

$$G_r(T) = \bigcap_{(i,j) \in I \times J} (F_i^1 \cap D(T)) \times F_j^2,$$

with F_i^1, F_j^2 closed sets on H . Therefore $G_r(T)$ becomes closed on $H \times H$ when $D(T)$ is closed on H (It's the induced topology). \square

Remark 2.1. *In the previous proof, we did not use the linearity of T , we used only topological notions.*

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