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MICRO SEPARATION AXIOMS

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Abstract. In this paper, some new types of spaces are defined and studied in micro topological spaces namely, Micro T_0, Micro T_1, Micro T_2, Micro R_0 and Micro R_1 spaces. Properties and the relationships of these spaces are introduced. Finally, the relationships between these spaces and the related concepts are investigated.

1. Introduction

Topology and its branches have become hot topics not only for almost all fields of mathematics, but also for many areas of science such as chemistry [21], and information systems [23]. The notion of rough sets was introduced by Pawlak [22]. Rough set theory is an important tool for data mining. Lower and upper approximation operators are two important basic concepts in the rough set theory. The classical Pawlak rough approximation operators are based on equivalence relations and have been extended to relation-based generalized rough approximation operators. The notation of nano topology was introduced by Thivagar et al [25, 26] which was defined in terms of approximations and boundary region of a subset of an universe using an equivalence relation on it. The concept of micro topology was introduced and investigated by Chandrasekar [1]. The notion of micro $T_{\frac{1}{2}}$ space was introduced by Ibrahim [16]. In the past few years, different forms of separation axioms have been studied [2–14, 17–20].

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2. Preliminaries

The following recalls requisite ideas and preliminaries necessitated in the sequel of this work.

Definition 2.1. *[24] Let U be a nonempty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.*

(1) *The lower approximation of U with respect to R is the set of all objects, which can be for certain classified as X with respect to R and its is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by x.*
(2) *The upper approximation of U with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \cap X \neq \phi\}$.*
(3) *The boundary region of U with respect to R is the set of all objects, which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = L_R(X) - L_R(X)$.*

Definition 2.2. *[25, 26] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), L_R(X), B_R(X)\}$, where $X \subseteq U$. Then, $\tau_R(X)$ satisfies the following axioms:*

(1) *U and $\phi \in \tau_R(X)$.*
(2) *The union of the elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$.*
(3) *The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.*

That is, $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X. We call $(U, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano open sets. A subset F of U is nano closed if its complement is nano open.

Definition 2.3. *[1] Let $(U, \tau_R(X))$ be a nano topological space. Then, $\mu_R(X) = \{N \cup (N' \cap \mu) : N, N' \in \tau_R(X)$ and $\mu \notin \tau_R(X)\}$ is called the micro topology on U with respect to X. The triplet $(U, \tau_R(X), \mu_R(X))$ is called micro topological space and the elements of $\mu_R(X)$ are called Micro open sets and the complement of a Micro open set is called a Micro closed set.*

Definition 2.4. *[1] The Micro closure of a set A is denoted by $Mic\text{-}cl(A)$ and is defined as $Mic\text{-}cl(A) = \cap\{B : B$ is Micro closed and $A \subseteq B\}$.*

Definition 2.5. *[1] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Let A and B be any two subsets of U. Then:*

(1) *A is a Micro closed set if and only if $Mic\text{-}cl(A) = A$.*
(2) *If $A \subseteq B$, then $Mic\text{-}cl(A) \subseteq Mic\text{-}cl(B)$.*
(3) *$Mic\text{-}cl(Mic\text{-}cl(A)) = Mic\text{-}cl(A)$.*

Remark 2.1. *[15] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space and A be any subset of U. Then:*

1. *Mic-$cl(A)$ is Micro closed.*
2. *$A \subseteq Mic$-$cl(A)$.*
3. *$x \in Mic$-$cl(A)$ if and only if for every Micro open subset L of U containing x, $A \cap L \neq \phi$.*

Definition 2.6. *[16] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. A subset A of U is said to be a Micro generalized closed (briefly, Micro g.closed) if Mic-$cl(A) \subseteq L$ whenever $A \subseteq L$ and L is a Micro open set in U.*

Definition 2.7. *[16] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, a subset A of U is called a Micro Difference set (briefly, MD-set) if there are $L, K \in \mu_R(X)$ such that $L \neq U$ and $A = L \setminus K$.*

Definition 2.8. *[16] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space and A be a subset of U. Then, the Micro kernel of A denoted by $Mker(A)$ is defined to be the set*

$$Mker(A) = \cap\{L \in \mu_R(X) \colon A \subseteq L\}.$$

Definition 2.9. *[16] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is said to be Micro $T_{\frac{1}{2}}$ if every Micro g.closed in U is Micro closed.*

Definition 2.10. *[16] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is said to be Micro symmetric if for x and y in U such that $x \in Mic$-$cl(\{y\})$ implies $y \in Mic$-$cl(\{x\})$.*

Definition 2.11. *[16] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is said to be:*

1. *Micro D_0 if for any pair of distinct points x and y of U there exists a MD-set of U containing x but not y or a MD-set of U containing y but not x.*
2. *Micro D_1 if for any pair of distinct points x and y of U there exists a MD-set of U containing x but not y and a MD-set of U containing y but not x.*
3. *Micro D_2 if for any pair of distinct points x and y of U there exist disjoint MD-set G and E of U containing x and y, respectively.*

Theorem 2.1. *[16] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is a Micro $T_{\frac{1}{2}}$ if and only if $\{x\}$ is Micro closed or Micro open, for each $x \in U$.*

Remark 2.2. *[16] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. If U is Micro D_k, then it is Micro D_{k-1}, for $k = 1, 2$.*

Theorem 2.2. *[16] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, then the following statements are equivalent:*

(1) *U is a Micro symmetric.*

(2) *$\{x\}$ is Micro g.closed, for each $x \in U$.*

Theorem 2.3. *[16] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space and $x \in U$. Then, $y \in Mker(\{x\})$ if and only if $x \in Mic\text{-}cl(\{y\})$.*

Theorem 2.4. *[16] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space and A be a subset of U. Then, $Mker(A) = \{x \in U\colon Mic\text{-}cl(\{x\}) \cap A \neq \phi\}$.*

Theorem 2.5. *[16] Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, for any points x and y in U the following statements are equivalent:*

(1) *$Mker(\{x\}) \neq Mker(\{y\})$.*

(2) *$Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$.*

Definition 2.12. *[1] Let $(U, \tau_R(X), \mu_R(X))$ and $(V, \tau_R(Y), \mu_R(Y))$ be two micro topological spaces. Then, A function $f : U \to V$ is said to be: Micro-continuous if $f^{-1}(K)$ is Micro open in U, for every Micro open set K in V.*

3. Micro T_k $(k = 0, 1, 2)$

The following definitions are introduced via Micro open sets.

Definition 3.1. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is said to be:*

(1) *Micro T_0 if for each pair of distinct points x, y in U, there exists a Micro open set L such that either $x \in L$ and $y \notin L$ or $x \notin L$ and $y \in L$.*

(2) *Micro T_1 if for each pair of distinct points x, y in U, there exist two Micro open sets L and K such that $x \in L$ but $y \notin L$ and $y \in K$ but $x \notin K$.*

(3) *Micro T_2 if for each distinct points x, y in U, there exist two disjoint Micro open sets L and K containing x and y respectively.*

Theorem 3.1. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is Micro T_0 if and only if for each pair of distinct points x, y of U, $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$.*

Proof. Necessity. Let U be Micro T_0 and x, y be any two distinct points of U. Then, there exists a Micro open set L containing x or y, say x but not y. Then, $U \setminus L$ is a Micro closed set which does not contain x but contains y. Since $Mic\text{-}cl(\{y\})$ is the smallest Micro closed set containing y, then $Mic\text{-}cl(\{y\}) \subseteq U \setminus L$ and therefore $x \notin Mic\text{-}cl(\{y\})$. Consequently $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$.

Sufficiency. Suppose that $x, y \in U$, $x \neq y$ and $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$. Let z be a point of U such that $z \in Mic\text{-}cl(\{x\})$ but $z \notin Mic\text{-}cl(\{y\})$. We claim that $x \notin Mic\text{-}cl(\{y\})$. For, if $x \in Mic\text{-}cl(\{y\})$ then

$Mic\text{-}cl(\{x\}) \subseteq Mic\text{-}cl(\{y\})$. This contradicts the fact that $z \notin Mic\text{-}cl(\{y\})$. Consequently x belongs to the Micro open set $U \setminus Mic\text{-}cl(\{y\})$ to which y does not belong. □

Theorem 3.2. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is Micro T_1 if and only if the singletons are Micro closed sets.*

Proof. Let U be Micro T_1 and x any point of U. Suppose $y \in U \setminus \{x\}$, then $x \neq y$ and so there exists a Micro open set L such that $y \in L$ but $x \notin L$. Consequently $y \in L \subseteq U \setminus \{x\}$, that is $U \setminus \{x\} = \cup\{L : y \in U \setminus \{x\}\}$ which is Micro open.

Conversely, suppose $\{p\}$ is Micro closed for every $p \in U$. Let $x, y \in U$ with $x \neq y$. Now, $x \neq y$ implies $y \in U \setminus \{x\}$. Hence, $U \setminus \{x\}$ is a Micro open set contains y but not x. Similarly $U \setminus \{y\}$ is a Micro open set contains x but not y. Accordingly U is Micro T_1. □

Theorem 3.3. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, the following statements are equivalent:*

(1) *U is Micro T_2.*
(2) *Let $x \in U$. For each $y \neq x$, there exists a Micro open set L containing x such that $y \notin Mic\text{-}cl(L)$.*
(3) *For each $x \in U$, $\cap\{Mic\text{-}cl(L) : L \in \mu_R(X)$ and $x \in L\} = \{x\}$.*

Proof. (1) ⇒ (2): Since U is Micro T_2, then there exist disjoint Micro open sets L and K containing x and y respectively. So, $L \subseteq U \setminus K$. Therefore, $Mic\text{-}cl(L) \subseteq U \setminus K$. So, $y \notin Mic\text{-}cl(L)$.

(2) ⇒ (3): If possible for some $y \neq x$, we have $y \in Mic\text{-}cl(L)$ for every Micro open set L containing x, which then contradicts (2).

(3) ⇒ (1): Let $x, y \in U$ and $x \neq y$. Then, there exists a Micro open set L containing x such that $y \notin Mic\text{-}cl(L)$. Let $K = U \setminus Mic\text{-}cl(L)$, then $y \in K$, $x \in L$ and $L \cap K = \phi$. Thus, U is Micro T_2. □

Theorem 3.4. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, then the following statements are hold:*

(1) *Every Micro T_2 space is Micro T_1.*
(2) *Every Micro T_1 space is Micro $T_{\frac{1}{2}}$.*
(3) *Every Micro $T_{\frac{1}{2}}$ space is Micro T_0.*

Proof. (1) The proof is straightforward from the definitions.

(2) The proof is obvious by Theorem 3.2.

(3) Let x and y be any two distinct points of U. By Theorem 2.1, the singleton set $\{x\}$ is Micro closed or Micro open.
   (a) If $\{x\}$ is Micro closed, then $U \setminus \{x\}$ is Micro open. So $y \in U \setminus \{x\}$ and $x \notin U \setminus \{x\}$. Therefore, we have U is Micro T_0.

(b) If $\{x\}$ is Micro open, then $x \in \{x\}$ and $y \notin \{x\}$. Therefore, we have U is Micro T_0.

□

Remark 3.1. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then,*

(1) *if U is Micro T_1, then $\mu_R(X)$ is discrete micro topology on U.*

(2) *U is Micro T_1 if and only if it is Micro T_2.*

Example 3.1. *Consider $U = \{a, b, c\}$ with $U/R = \{\{a\}, \{b, c\}\}$ and $X = \{a\}$. Then, $\tau_R(X) = \{U, \phi, \{a\}\}$. If $\mu = \{a, c\}$, then $\mu_R(X) = \{U, \phi, \{a\}, \{a, c\}\}$. Then, U is Micro T_0 but not Micro $T_{\frac{1}{2}}$.*

Example 3.2. *Consider $U = \{a, b, c\}$ with $U/R = \{\{c\}, \{a, b\}\}$ and $X = \{a, b\}$. Then, $\tau_R(X) = \{U, \phi, \{a, b\}\}$. If $\mu = \{a, c\}$, then $\mu_R(X) = \{U, \phi, \{a\}, \{a, b\}, \{a, c\}\}$. Then, U is Micro $T_{\frac{1}{2}}$ but not Micro T_1.*

Remark 3.2. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. If U is Micro T_k, then it is Micro D_k, for $k = 0, 1, 2$.*

Proof. Obvious. □

Theorem 3.5. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is Micro D_0 if and only if it is Micro T_0.*

Proof. Suppose that U is Micro D_0. Then, for each distinct pair $x, y \in U$, at least one of x, y, say x, belongs to a MD-set G but $y \notin G$. Let $G = L_1 \setminus L_2$ where $L_1 \neq U$ and $L_1, L_2 \in \mu_R(X)$. Then, $x \in L_1$, and for $y \notin G$ we have two cases: (a) $y \notin L_1$ (b) $y \in L_1$ and $y \in L_2$.
In case (a), $x \in L_1$ but $y \notin L_1$.
In case (b), $y \in L_2$ but $x \notin L_2$.
Thus in both the cases, we obtain that U is Micro T_0.
Conversely, if U is Micro T_0, by Remark 3.2, U is Micro D_0. □

Corollary 3.1. *If U is Micro D_1, then it is Micro T_0.*

Proof. Follows from Remark 2.2 and Theorem 3.5. □

Here is an example which shows that the converse of Corollary 3.1 is not true in general.

Example 3.3. *Consider $U = \{a, b, c\}$ with $U/R = \{\{a\}, \{b, c\}\}$ and $X = \{a\}$. Then, $\tau_R(X) = \{U, \phi, \{a\}\}$. If $\mu = \{a, b\}$, then $\mu_R(X) = \{U, \phi, \{a\}, \{a, b\}\}$. Then, U is Micro T_0 but not Micro D_1 because there is no MD-set containing c but not b.*

Corollary 3.2. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. If U is Micro T_1, then it is Micro symmetric.*

Proof. In Micro T_1, every singleton is Micro closed and therefore is Micro g.closed. Then, by Theorem 2.2, U is Micro symmetric. □

Corollary 3.3. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, the following statements are equivalent:*

(1) *U is Micro symmetric and Micro T_0.*

(2) *U is Micro T_1.*

Proof. By Corollary 3.2 and Theorem 3.4, it suffices to prove only (1) $\Rightarrow$ (2).

Let $x \neq y$ and as U is Micro T_0, we may assume that $x \in L \subseteq U \setminus \{y\}$ for some $L \in \mu_R(X)$. Then, $x \notin Mic\text{-}cl(\{y\})$ and hence $y \notin Mic\text{-}cl(\{x\})$. There exists a Micro open set K such that $y \in K \subseteq U \setminus \{x\}$ and thus U is a Micro T_1 space. □

Theorem 3.6. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. If U is Micro symmetric, then the following statements are equivalent:*

(1) *U is Micro T_0.*

(2) *U is Micro $T_{\frac{1}{2}}$.*

(3) *U is Micro T_1.*

Proof. (1) $\Leftrightarrow$ (3): Obvious from Corollary 3.3.

(3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1): Directly from Theorem 3.4. □

Corollary 3.4. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. If U is Micro symmetric, then the following statements are equivalent:*

(1) *U is Micro T_0.*

(2) *U is Micro D_1.*

(3) *U is Micro T_1.*

Proof. (1) $\Rightarrow$ (3). Follows from Corollary 3.3.

(3) $\Rightarrow$ (2) $\Rightarrow$ (1). Follows from Remark 3.2 and Corollary 3.1. □

Definition 3.2. *A function $f : U \to V$ is called Micro-open if the image of every Micro open set in U is a Micro open set in V.*

Theorem 3.7. *Suppose that $f : U \to V$ is Micro-open and surjective. Then:*

(1) *If U is Micro T_0, then V is Micro T_0.*

(2) *If U is Micro T_1, then V is Micro T_1.*

(3) *If U is Micro T_2, then V is Micro T_2.*

Proof. We prove only the case for Micro T_1 the others are similarly.

Let U be Micro T_1 and $y_1, y_2 \in V$ with $y_1 \neq y_2$. Since f is surjective, so there exist distinct points x_1, x_2 of U such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since U is Micro T_1, then there exist Micro open sets G and H such that $x_1 \in G$ but $x_2 \notin G$ and $x_2 \in H$ but $x_1 \notin H$. Since f is Micro-open, then $f(G)$ and $f(H)$ are Micro open sets of V such that $y_1 = f(x_1) \in f(G)$ but $y_2 = f(x_2) \notin f(G)$, and $y_2 = f(x_2) \in f(H)$ but $y_1 = f(x_1) \notin f(H)$. Hence, V is Micro T_1. □

Theorem 3.8. *If $f : U \to V$ is a Micro-continuous injective function and V is Micro T_2, then U is Micro T_2.*

Proof. Let x and y in U be any pair of distinct points, then there exist disjoint Micro open sets A and B in V such that $f(x) \in A$ and $f(y) \in B$. Since f is Micro-continuous, then $f^{-1}(A)$ and $f^{-1}(B)$ are Micro open in U containing x and y respectively, we have $f^{-1}(A) \cap f^{-1}(B) = \phi$. Thus, U is Micro T_2. □

4. Micro R_k $(k = 0, 1)$

Definition 4.1. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is said to be Micro R_0 if L is a Micro open set and $x \in L$, then $Mic\text{-}cl(\{x\}) \subseteq L$.*

Theorem 4.1. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, the following properties are equivalent:*

(1) *U is Micro R_0.*

(2) *For any $F \in \mu_R^c(X)$, $x \notin F$ implies $F \subseteq L$ and $x \notin L$ for some $L \in \mu_R(X)$. Where $\mu_R^c(X)$ is the family of all Micro closed sets.*

(3) *For any $F \in \mu_R^c(X)$, $x \notin F$ implies $F \cap Mic\text{-}cl(\{x\}) = \phi$.*

(4) *For any distinct points x and y of U, either $Mic\text{-}cl(\{x\}) = Mic\text{-}cl(\{y\})$ or $Mic\text{-}cl(\{x\}) \cap Mic\text{-}cl(\{y\}) = \phi$.*

Proof. (1) ⇒ (2): Let $F \in \mu_R^c(X)$ and $x \notin F$. Then by (1), $Mic\text{-}cl(\{x\}) \subseteq U \setminus F$. Set $L = U \setminus Mic\text{-}cl(\{x\})$, then L is a Micro open set such that $F \subseteq L$ and $x \notin L$.

(2) ⇒ (3): Let $F \in \mu_R^c(X)$ and $x \notin F$. Then, there exists $L \in \mu_R(X)$ such that $F \subseteq L$ and $x \notin L$. Since $L \in \mu_R(X)$, then $L \cap Mic\text{-}cl(\{x\}) = \phi$ and $F \cap Mic\text{-}cl(\{x\}) = \phi$.

(3) ⇒ (4): Suppose that $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$ for distinct points $x, y \in U$. Then, there exists $z \in Mic\text{-}cl(\{x\})$ such that $z \notin Mic\text{-}cl(\{y\})$ (or $z \in Mic\text{-}cl(\{y\})$ such that $z \notin Mic\text{-}cl(\{x\})$). There exists $K \in \mu_R(X)$

such that $y \notin K$ and $z \in K$; hence $x \in K$. Therefore, we have $x \notin Mic\text{-}cl(\{y\})$. By (3), we obtain $Mic\text{-}cl(\{x\}) \cap Mic\text{-}cl(\{y\}) = \phi$.

(4) $\Rightarrow$ (1): Let $K \in \mu_R(X)$ and $x \in K$. For each $y \notin K$, $x \neq y$ and $x \notin Mic\text{-}cl(\{y\})$. This shows that $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$. By (4), $Mic\text{-}cl(\{x\}) \cap Mic\text{-}cl(\{y\}) = \phi$ for each $y \in U \setminus K$ and hence $Mic\text{-}cl(\{x\}) \cap (\bigcup_{y \in U \setminus K} Mic\text{-}cl(\{y\})) = \phi$. On other hand, since $K \in \mu_R(X)$ and $y \in U \setminus K$, we have $Mic\text{-}cl(\{y\}) \subseteq U \setminus K$ and hence $U \setminus K = \bigcup_{y \in U \setminus K} Mic\text{-}cl(\{y\})$. Therefore, we obtain $(U \setminus K) \cap Mic\text{-}cl(\{x\}) = \phi$ and $Mic\text{-}cl(\{x\}) \subseteq K$. This shows that U is Micro R_0. □

Theorem 4.2. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is Micro T_1 if and only if U is both Micro T_0 and Micro R_0.*

Proof. Necessity. Let L be any Micro open subset of U and $x \in L$. Then by Theorem 3.2, we have $Mic\text{-}cl(\{x\}) \subseteq L$ and so by Theorem 3.4, it is clear that U is Micro T_0 and Micro R_0.

Sufficiency. Let x and y be any distinct points of U. Since U is Micro T_0, then there exists a Micro open set L such that $x \in L$ and $y \notin L$. As $x \in L$ implies that $Mic\text{-}cl(\{x\}) \subseteq L$. Since $y \notin L$, so $y \notin Mic\text{-}cl(\{x\})$. Hence, $y \in K = U \setminus Mic\text{-}cl(\{x\})$ and it is clear that $x \notin K$. Thus, it follows that there exist Micro open sets L and K containing x and y respectively, such that $y \notin L$ and $x \notin K$. This implies that U is Micro T_1. □

Theorem 4.3. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, the following properties are equivalent:*

(1) *U is Micro R_0.*
(2) *$x \in Mic\text{-}cl(\{y\})$ if and only if $y \in Mic\text{-}cl(\{x\})$, for any points x and y in U.*

Proof. (1) $\Rightarrow$ (2): Assume that U is Micro R_0. Let $x \in Mic\text{-}cl(\{y\})$ and K be any Micro open set such that $y \in K$. Now by hypothesis, $x \in K$. Therefore, every Micro open set which contain y contains x. Hence, $y \in Mic\text{-}cl(\{x\})$.

(2) $\Rightarrow$ (1): Let L be a Micro open set and $x \in L$. If $y \notin L$, then $x \notin Mic\text{-}cl(\{y\})$ and hence $y \notin Mic\text{-}cl(\{x\})$. This implies that $Mic\text{-}cl(\{x\}) \subseteq L$. Hence, U is Micro R_0. □

Remark 4.1. *From Definition 2.10 and Theorem 4.3, the notions of Micro symmetric and Micro R_0 are equivalent.*

Theorem 4.4. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is Micro R_0 if and only if for every x and y in U, $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$ implies $Mic\text{-}cl(\{x\}) \cap Mic\text{-}cl(\{y\}) = \phi$.*

Proof. Necessity. Suppose that U is Micro R_0 and $x, y \in U$ such that $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$. Then, there exists $z \in Mic\text{-}cl(\{x\})$ such that $z \notin Mic\text{-}cl(\{y\})$ (or $z \in Mic\text{-}cl(\{y\})$ such that $z \notin Mic\text{-}cl(\{x\})$) and there exists $K \in \mu_R(X)$ such that $y \notin K$ and $z \in K$, hence $x \in K$. Therefore, we have $x \notin Mic\text{-}cl(\{y\})$.

Thus, $x \in U \setminus Mic\text{-}cl(\{y\}) \in \mu_R(X)$, which implies $Mic\text{-}cl(\{x\}) \subseteq U \setminus Mic\text{-}cl(\{y\})$ and $Mic\text{-}cl(\{x\}) \cap Mic\text{-}cl(\{y\}) = \phi$.

Sufficiency. Let $K \in \mu_R(X)$ and $x \in K$. We still show that $Mic\text{-}cl(\{x\}) \subseteq K$. Let $y \notin K$, that is $y \in U \setminus K$. Then, $x \neq y$ and $x \notin Mic\text{-}cl(\{y\})$. This shows that $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$. By assumption, $Mic\text{-}cl(\{x\}) \cap Mic\text{-}cl(\{y\}) = \phi$. Hence, $y \notin Mic\text{-}cl(\{x\})$ and therefore $Mic\text{-}cl(\{x\}) \subseteq K$. Thus, U is Micro R_0 □

Theorem 4.5. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is Micro R_0 if and only if for any points x and y in U, $Mker(\{x\}) \neq Mker(\{y\})$ implies $Mker(\{x\}) \cap Mker(\{y\}) = \phi$.*

Proof. Suppose that U is Micro R_0. Thus, by Theorem 2.5, for any points x and y in U if $Mker(\{x\}) \neq Mker(\{y\})$ then $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$. Now we prove that $Mker(\{x\}) \cap Mker(\{y\}) = \phi$. Assume that $z \in Mker(\{x\}) \cap Mker(\{y\})$. By $z \in Mker(\{x\})$ and Theorem 2.3, it follows that $x \in Mic\text{-}cl(\{z\})$. Since $x \in Mic\text{-}cl(\{x\})$, by Theorem 4.1, $Mic\text{-}cl(\{x\}) = Mic\text{-}cl(\{z\})$. Similarly, we have $Mic\text{-}cl(\{y\}) = Mic\text{-}cl(\{z\}) = Mic\text{-}cl(\{x\})$. This is a contradiction. Therefore, we have $Mker(\{x\}) \cap Mker(\{y\}) = \phi$.

Conversely, suppose that for any points x and y in U, $Mker(\{x\}) \neq Mker(\{y\})$ implies $Mker(\{x\}) \cap Mker(\{y\}) = \phi$. If $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$, then by Theorem 2.5, $Mker(\{x\}) \neq Mker(\{y\})$. Hence, $Mker(\{x\}) \cap Mker(\{y\}) = \phi$ which implies $Mic\text{-}cl(\{x\}) \cap Mic\text{-}cl(\{y\}) = \phi$. Because $z \in Mic\text{-}cl(\{x\})$ implies that $x \in Mker(\{z\})$ and therefore $Mker(\{x\}) \cap Mker(\{z\}) \neq \phi$. By hypothesis, we have $Mker(\{x\}) = Mker(\{z\})$. Then $z \in Mic\text{-}cl(\{x\}) \cap Mic\text{-}cl(\{y\})$ implies that $Mker(\{x\}) = Mker(\{z\}) = Mker(\{y\})$. This is a contradiction. Therefore, $Mic\text{-}cl(\{x\}) \cap Mic\text{-}cl(\{y\}) = \phi$ and by Theorem 4.1, U is Micro R_0. □

Theorem 4.6. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, the following properties are equivalent:*

(1) *U is Micro R_0.*

(2) *For any non-empty set A and $G \in \mu_R(X)$ such that $A \cap G \neq \phi$, there exists $F \in \mu_R^c(X)$ such that $A \cap F \neq \phi$ and $F \subseteq G$.*

(3) *For any $G \in \mu_R(X)$, we have $G = \cup\{F \in \mu_R^c(X)\colon F \subseteq G\}$.*

(4) *For any $F \in \mu_R^c(X)$, we have $F = \cap\{G \in \mu_R(X)\colon F \subseteq G\}$.*

(5) *For every $x \in U$, $Mic\text{-}cl(\{x\}) \subseteq Mker(\{x\})$.*

Proof. (1) ⇒ (2): Let A be a non-empty subset of U and $G \in \mu_R(X)$ such that $A \cap G \neq \phi$. Then, there exists $x \in A \cap G$. Since $x \in G \in \mu_R(X)$, so $Mic\text{-}cl(\{x\}) \subseteq G$. Set $F = Mic\text{-}cl(\{x\})$, then $F \in \mu_R^c(X)$, $F \subseteq G$ and $A \cap F \neq \phi$.

(2) ⇒ (3): Let $G \in \mu_R(X)$, then $G \supseteq \cup\{F \in \mu_R^c(X)\colon F \subseteq G\}$. Let x be any point of G. Then, there exists $F \in \mu_R^c(X)$ such that $x \in F$ and $F \subseteq G$. Therefore, we have $x \in F \subseteq \cup\{F \in \mu_R^c(X)\colon F \subseteq G\}$ and hence

$G = \cup\{F \in \mu_R^c(X)\colon F \subseteq G\}$.

(3) $\Rightarrow$ (4): Obvious.

(4) $\Rightarrow$ (5): Let x be any point of U and $y \notin Mker(\{x\})$. Then, there exists $K \in \mu_R(X)$ such that $x \in K$ and $y \notin K$, hence $Mic\text{-}cl(\{y\}) \cap K = \phi$. By (4), $(\cap\{G \in \mu_R(X)\colon Mic\text{-}cl(\{y\}) \subseteq G\}) \cap K = \phi$ and there exists $G \in \mu_R(X)$ such that $x \notin G$ and $Mic\text{-}cl(\{y\}) \subseteq G$. Therefore, $Mic\text{-}cl(\{x\}) \cap G = \phi$ and $y \notin Mic\text{-}cl(\{x\})$. Consequently, we obtain $Mic\text{-}cl(\{x\}) \subseteq Mker(\{x\})$.

(5) $\Rightarrow$ (1): Let $G \in \mu_R(X)$ and $x \in G$. Let $y \in Mker(\{x\})$, then $x \in Mic\text{-}cl(\{y\})$ and $y \in G$. This implies that $Mker(\{x\}) \subseteq G$. Therefore, we obtain $x \in Mic\text{-}cl(\{x\}) \subseteq Mker(\{x\}) \subseteq G$. This shows that U is Micro R_0. □

Corollary 4.1. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, the following properties are equivalent:*

(1) *U is Micro R_0.*

(2) *$Mic\text{-}cl(\{x\}) = Mker(\{x\})$ for all $x \in U$.*

Proof. (1) $\Rightarrow$ (2): Suppose that U is Micro R_0. By Theorem 4.6, $Mic\text{-}cl(\{x\}) \subseteq Mker(\{x\})$ for each $x \in U$. Let $y \in Mker(\{x\})$, then $x \in Mic\text{-}cl(\{y\})$ and by Theorem 4.1, $Mic\text{-}cl(\{x\}) = Mic\text{-}cl(\{y\})$. Therefore, $y \in Mic\text{-}cl(\{x\})$ and hence $Mker(\{x\}) \subseteq Mic\text{-}cl(\{x\})$. This shows that $Mic\text{-}cl(\{x\}) = Mker(\{x\})$.

(2) $\Rightarrow$ (1): Follows from Theorem 4.6. □

Theorem 4.7. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, the following properties are equivalent:*

(1) *U is Micro R_0.*

(2) *If F is Micro closed, then $F = Mker(F)$.*

(3) *If F is Micro closed and $x \in F$, then $Mker(\{x\}) \subseteq F$.*

(4) *If $x \in U$, then $Mker(\{x\}) \subseteq Mic\text{-}cl(\{x\})$.*

Proof. (1) $\Rightarrow$ (2): Let F be Micro closed and $x \notin F$. Thus, $(U \setminus F)$ is a Micro open set containing x. Since U is Micro R_0, $Mic\text{-}cl(\{x\}) \subseteq (U \setminus F)$. Thus, $Mic\text{-}cl(\{x\}) \cap F = \phi$ and by Theorem 2.4, $x \notin Mker(F)$. Therefore, $Mker(F) = F$.

(2) $\Rightarrow$ (3): In general, $A \subseteq B$ implies $Mker(A) \subseteq Mker(B)$. Therefore, it follows from (2), that $Mker(\{x\}) \subseteq Mker(F) = F$.

(3) $\Rightarrow$ (4): Since $x \in Mic\text{-}cl(\{x\})$ and $Mic\text{-}cl(\{x\})$ is Micro closed, by (3), $Mker(\{x\}) \subseteq Mic\text{-}cl(\{x\})$.

(4) $\Rightarrow$ (1): We show the implication by using Theorem 4.3. Let $x \in Mic\text{-}cl(\{y\})$. Then by Theorem 2.3, $y \in Mker(\{x\})$ and by (4), we obtain $y \in Mker(\{x\}) \subseteq Mic\text{-}cl(\{x\})$. Therefore, $x \in Mic\text{-}cl(\{y\})$ implies $y \in Mic\text{-}cl(\{x\})$. The converse is obvious and U hence is Micro R_0. □

Definition 4.2. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is said to be Micro R_1 if for x, y in U with $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$, there exist disjoint Micro open sets L and K such that $Mic\text{-}cl(\{x\}) \subseteq L$ and $Mic\text{-}cl(\{y\}) \subseteq K$.*

Theorem 4.8. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is Micro R_1 if it is Micro T_2.*

Proof. Let x and y be any points of U such that $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$. By Theorem 3.4 (1), U is Micro T_1. Therefore, by Theorem 3.2, $Mic\text{-}cl(\{x\}) = \{x\}$, $Mic\text{-}cl(\{y\}) = \{y\}$ and hence $\{x\} \neq \{y\}$. Since U is Micro T_2, then there exist disjoint Micro open sets L and K such that $Mic\text{-}cl(\{x\}) = \{x\} \subseteq L$ and $Mic\text{-}cl(\{y\}) = \{y\} \subseteq K$. This shows that U is Micro R_1. □

Theorem 4.9. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, the following statements are equivalent:*

(1) *U is Micro R_1.*
(2) *If $x, y \in U$ such that $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$, then there exist Micro closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $U = F_1 \cup F_2$.*

Proof. Obvious. □

Theorem 4.10. *If U is Micro R_1, then U is Micro R_0.*

Proof. Let L be Micro open such that $x \in L$. If $y \notin L$, then $x \notin Mic\text{-}cl(\{y\})$ and $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$. So, there exists a Micro open set K such that $Mic\text{-}cl(\{y\}) \subseteq K$ and $x \notin K$, which implies $y \notin Mic\text{-}cl(\{x\})$. Hence, $Mic\text{-}cl(\{x\}) \subseteq L$. Therefore, U is Micro R_0. □

Corollary 4.2. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is Micro R_1 if and only if for $x, y \in U$, $Mker(\{x\}) \neq Mker(\{y\})$, there exist disjoint Micro open sets L and K such that $Mic\text{-}cl(\{x\}) \subseteq L$ and $Mic\text{-}cl(\{y\}) \subseteq K$.*

Proof. Follows from Theorem 2.5. □

Theorem 4.11. *Let $(U, \tau_R(X), \mu_R(X))$ be a micro topological space. Then, U is Micro R_1 if and only if $x \in U \setminus Mic\text{-}cl(\{y\})$ implies that x and y have disjoint Micro open neighbourhoods.*

Proof. Necessity. Let $x \in U \setminus Mic\text{-}cl(\{y\})$. Then, $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$. Thus, x and y have disjoint Micro open neighbourhoods.

Sufficiency. First, we show that U is Micro R_0. Let L be a Micro open set and $x \in L$. Suppose that $y \notin L$. Then, $Mic\text{-}cl(\{y\}) \cap L = \phi$ and $x \notin Mic\text{-}cl(\{y\})$. So, there exist Micro open sets L_x and L_y such that $x \in L_x$, $y \in L_y$ and $L_x \cap L_y = \phi$. Hence, $Mic\text{-}cl(\{x\}) \subseteq Mic\text{-}cl(L_x)$ and $Mic\text{-}cl(\{x\}) \cap L_y \subseteq Mic\text{-}cl(L_x) \cap L_y = \phi$.

Therefore, $y \notin Mic\text{-}cl(\{x\})$. Consequently, $Mic\text{-}cl(\{x\}) \subseteq L$ and hence U is Micro R_0. Next, we show that U is Micro R_1. Suppose that $Mic\text{-}cl(\{x\}) \neq Mic\text{-}cl(\{y\})$. Then, we can assume that there exists $z \in Mic\text{-}cl(\{x\})$ such that $z \notin Mic\text{-}cl(\{y\})$. Then, there exist Micro open sets K_z and K_y such that $z \in K_z$, $y \in K_y$ and $K_z \cap K_y = \phi$. Since $z \in Mic\text{-}cl(\{x\})$, then $x \in K_z$. Since U is Micro R_0, we obtain $Mic\text{-}cl(\{x\}) \subseteq K_z$, $Mic\text{-}cl(\{y\}) \subseteq K_y$ and $K_z \cap K_y = \phi$. This shows that U is Micro R_1. $\square$

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