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ON SOME NEW CONTRACTIVE CONDITIONS FOR ASYMPTOTICALLY REGULAR SET-VALUED MAPPINGS

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Abstract. In this paper, we introduce two new contractive conditions of Proinov-type for asymptotically regular set-valued mappings and prove the existence of their fixed points. Our results extend some results due to Nadler and Boyd and Wong.

1. Preliminaries

The notion of asymptotic regularity was introduced by Browder and Petryshyn [6]. This notion has been exploited by several authors to obtain fixed points of the concerned map. In this context, fixed points of dissipative multivalued maps were studied by Aubin and Siegel [1]. Weak convergence of asymptotically regular (in short, AR) sequences for nonexpansive mappings was studied by Engl [11] and its relation with Chebychef-centers was established. Fixed points of AR mappings and their applications were studied by Guay and Singh [12], Rhoades et al. [17], Beg and Azam [2], Singh et. al. [19], Debnath and de La Sen [9].

The connection between fixed points and asymptotic regularity may easily be understood from the fact that contraction maps are AR. Nadler [14] introduced the set-valued version of Banach's contraction principle with

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the help of Hausdorff metric. Aubin and Siegel [1] has discussed about the applications of AR multivalued maps in control theory, optimisation and system theory. Sequences of AR multivalued maps have been used by Itoh and Takahashi [13] and Rhoades et al. [18]. In [16], Rhoades compared several contractive conditions and found that most of the conditions imply asymptotic regularity. As such, study of AR maps is a well motivated area of research in fixed point theory.

We recall the definition of Pompeiu-Hausdorff metric which plays a crucial role in set-valued analysis.

Let $\Gamma(X)$ denote the class of all nonempty closed and bounded subsets of a non-empty set X and $(\Gamma(X), \mathcal{PH})$ denote the Pompeiu-Hausdorff metric in a MS (X, δ). The metric function $\mathcal{PH} : \Gamma(X) \times \Gamma(X) \to [0, \infty)$ is defined by

$$\mathcal{PH}(U, V) = \max\{\sup_{\xi \in V} \Delta(\xi, U), \sup_{\eta \in U} \Delta(\eta, V)\}, \text{ for all } U, V \in \Gamma(X),$$

where $\Delta(\eta, V) = \inf_{\xi \in V} \delta(\eta, \xi)$.

Definition 1.1. *[14] Let $R : X \to \Gamma(X)$ be a set-valued map. $\mu \in X$ is called a fixed point of R if $\mu \in R\mu$.*

Following results are important in the present context.

Lemma 1.1. *[4, 7] Let (X, δ) be a MS and $U, V, W \in \Gamma(X)$. Then*

(1) $\Delta(\mu, V) \leq \delta(\mu, \gamma)$ *for any* $\gamma \in V$ *and* $\mu \in X$*;*
(2) $\Delta(\mu, V) \leq \mathcal{PH}(U, V)$ *for any* $\mu \in U$*;*
(3) $\Delta(\mu, U) \leq \delta(\mu, \nu) + \Delta(\nu, U)$ *for all* $\mu, \nu \in X$*.*

Lemma 1.2. *[14] Let $\{U_n\}$ be a sequence in $\Gamma(X)$ and $\lim_{n\to\infty} \mathcal{PH}(U_n, U) = 0$ for some $U \in \Gamma(X)$. If $\mu_n \in U_n$ and $\lim_{n\to\infty} \delta(\mu_n, \mu) = 0$ for some $\mu \in X$, then $\mu \in U$.*

Definition 1.2. *[14] Let $R : X \to \Gamma(X)$ be a set-valued map. R is said to be a set-valued contraction if $\mathcal{PH}(R\mu, R\nu) \leq \lambda\delta(\mu, \nu)$ for all $\mu, \nu \in X$, where $\lambda \in [0, 1)$.*

Orbital sequence is one of the important components in the investigation of fixed points for set-valued maps (see [3, 8]).

Definition 1.3. *[10] Let (X, δ) be a MS and $R : X \to \Gamma(X)$ a set-valued map. An R-orbital (or, simply orbital) sequence of R at a point $\mu \in X$ is a set $O(\mu, R)$ of points in X defined by $O(\mu, R) = \{\mu_0 = \mu, \mu_{n+1} \in R\mu_n, n = 0, 1, 2, \ldots\}$.*

Inspired by the recent work of Proinov [15], in the present paper we introduce some new contractive definitions for set-valued maps and show that our results extend some results due to Nadler [14] and Boyd and Wong [5].

The rest of the paper is organised as follows. Section 2 contains three lemmas on the asymptotic regularity of the set-valued map under consideration, while in Section 3 we present our main results with the help of those lemmas. Section 4 contains conclusions and future work.

2. Some lemmas on asymptotic regularity

In this section we present three lemmas. The first lemma provides us with conditions on the two auxiliary control functions g and h those ensure that the set-valued map under consideration is AR.

The recent proofs due to Proinov [15] will be taken as a framework and his proofs will be extended to their set-valued analogues using the function Δ and the Pompeiu-Hausdorff metric $\mathcal{PH}$.

Definition 2.1. *[9] Let (X, δ) be a MS. A set-valued map $R : X \to \Gamma(X)$ is said to be asymptotically regular (in short, AR) at a point $\mu_0 \in X$, if for any orbital sequence $\{\mu_n\} = O(\mu_0, R)$, we have*

$$\lim_{n\to\infty} \delta(\mu_n, \mu_{n+1}) = 0.$$

If R is AR at all points of its domain, then it is called AR.

Lemma 2.1. *Let (X, δ) be a MS and $R : X \to \Gamma(X)$ be a set-valued map satisfying $h(\mathcal{PH}(R\mu, R\nu)) \leq g(\delta(\mu, \nu))$ for all $\mu, \nu \in X$, where $g, h : (0, \infty) \to \mathbb{R}$ are functions such that*

(1) *$g(t) < h(t)$ for all $t > 0$;*
(2) *$\inf_{t>\epsilon} h(t) > -\infty$ for any $\epsilon > 0$.*

Further, suppose that at least one of the following is true:

(1) *h is nondecreasing and $\limsup_{t\to\epsilon+} g(t) < h(\epsilon+)$ for any $\epsilon > 0$;*
(2) *if $\{h(t_n)\}$ and $\{g(t_n)\}$ are convergent sequences with the same limit and $\{h(t_n)\}$ is strictly decreasing, then $\lim_{n\to\infty} t_n = 0$.*

Then R is AR.

Proof. Fix $\mu_0 \in X$ and consider the orbital sequence $\{\mu_n\} = O(\mu_0, R)$. Let $E_n = \delta(\mu_n, \mu_{n+1})$. To prove that $E_n \to 0$.

Rest of the proof follows verbatim from the proof of Lemma 3.2 in [15]. □

In our second lemma, we find conditions on g and h which show that if the map R is AR, then the orbital sequence $\{\mu_n\} = O(\mu_0, R)$ is Cauchy.

Lemma 2.2. *Let (X, δ) be a MS and $R : X \to \Gamma(X)$ be a set-valued map satisfying $h(\mathcal{PH}(R\mu, R\nu)) \leq g(\delta(\mu, \nu))$ for all $\mu, \nu \in X$, where the functions g, h satisfy at least one of the following conditions:*

(1) *h is nondecreasing, $g(t) < h(t)$ for all $t > 0$ and $\limsup_{t\to\epsilon+} g(t) < h(\epsilon+)$ for any $\epsilon > 0$;*
(2) *$\limsup_{t\to\epsilon} g(t) < \liminf_{t\to\epsilon+} h(t)$ for any $\epsilon > 0$;*
(3) *$\limsup_{t\to\epsilon+} g(t) < \liminf_{t\to\epsilon} h(t)$ for any $\epsilon > 0$.*

If R is AR at a point $\mu_0 \in X$, then the orbital sequence $\{\mu_n\} = O(\mu_0, R)$ is a Cauchy sequence.

Proof. Let R be AR at the point $\mu_0 \in X$. Construct the orbital sequence $\{\mu_n\} = O(\mu_0, R)$. Suppose that the sequence $\{\mu_n\}$ is not Cauchy.

Next, using exactly similar arguments as in the proof of Lemma 3.3 in [15], we arrive at a contradiction. Therefore, $\{\mu_n\}$ is a Cauchy sequence. $\square$

In the third lemma, we establish conditions on the functions g and h which guarantee the existence of a fixed point if the orbital sequence $\{\mu_n\} = O(\mu_0, R)$ is convergent. In fact, the limit of the orbital sequence happens to be a fixed point of R.

We recall the definition of closed graph for a set-valued map.

Definition 2.2. *Let (X, δ) be a MS and $R : X \to \Gamma(X)$ be a set-valued map. R is said to have a closed graph $G(R)$ if $G(R) = \{(\mu, \nu) : \nu \in R\mu, \mu \in X\}$ is a closed subset of $X \times X$ with the product topology.*

Lemma 2.3. *Let (X, δ) be a MS and $R : X \to \Gamma(X)$ be a set-valued map satisfying*

$$h(\mathcal{PH}(R\mu, R\nu)) \leq g(\delta(\mu, \nu)), \tag{2.1}$$

for all $\mu, \nu \in X$, where $g, h : (0, \infty) \to \mathbb{R}$ are functions satisfying at least one of the following conditions:

(1) *R has a closed graph;*

(2) *h is nondecreasing, $g(t) < h(t)$ for all $t > 0$;*

(3) $\limsup_{t\to 0+} g(t) < \liminf_{t\to\epsilon} h(t)$ *for any $\epsilon > 0$.*

If $\{\mu_n\} = O(\mu_0, R)$ is an orbital sequence for some $\mu_0 \in X$ and $\lim_{n\to\infty} \mu_n = \theta$, then θ is a fixed point of R.

Proof. We split the proof into three parts.

Part I: Suppose condition 1 above is true, i.e., for all sequences $\{\mu_n\}$ and $\{\nu_n\}$, where $\nu_n \in R\mu_n$ for each $n = 0, 1, 2, \ldots$, such that $\mu_n \to \mu$ and $\nu_n \to \nu$ as $n \to \infty$, we have $\nu \in R\mu$.

Since $\{\mu_n\} = O(\mu_0, R)$ is an orbital sequence and $\lim_{n\to\infty} \mu_n = \theta$, we have $\mu_{n+1} \in R\mu_n$ for each $n = 0, 1, 2, \ldots$ and $\lim_{n\to\infty} \mu_{n+1} = \theta$. Thus by condition 1, we have $\theta \in R\theta$.

Part II: Consider the orbital sequence $\{\mu_n\} = O(\mu_0, R)$. If $\mathcal{PH}(R\mu_n, R\theta) = 0$ for all but finitely many values of n, then

$$\begin{aligned}\Delta(\theta, R\theta) &\leq \delta(\theta, \mu_{n+1}) + \Delta(\mu_{n+1}, R\theta)\\ &\leq \delta(\theta, \mu_{n+1}) + \mathcal{PH}(R\mu_n, R\theta), \text{(using property 3 of Lemma 1.1)}\\ &= \delta(\theta, \mu_{n+1}),\end{aligned}$$

for those values of n. Taking limit in the above inequality, as $n \to \infty$, we have $\Delta(\theta, R\theta) = 0$, i.e., $\theta \in R\theta$.

Further suppose that $\mathcal{PH}(R\mu_n, R\theta) > 0$ for infinitely many n and let condition 2 above be true. Then using (2.1) from hypothesis with $\mu = \mu_n$ and $\nu = \theta$, we have

$$h(\mathcal{PH}(R\mu_n, R\theta)) \leq g(\delta(\mu_n, \theta)) \tag{2.2}$$

$$\implies h(\Delta(\mu_{n+1}, R\theta)) \leq g(\delta(\mu_n, \theta)), (\text{ since } \Delta(\mu_{n+1}, R\theta) \leq \mathcal{PH}(R\mu_n, R\theta) \text{ and } h \text{ is nondecreasing}). \tag{2.3}$$

Since condition 2 is true, we have $h(\Delta(\mu_{n+1}, R\theta)) \leq g(\delta(\mu_n, \theta)) < h(\delta(\mu_n, \theta))$. Thus we have $\Delta(\mu_{n+1}, R\theta) < \delta(\mu_n, \theta)$ and further, taking limit as $n \to \infty$, we obtain, $\Delta(\theta, R\theta) \leq 0$, i.e, $\theta \in R\theta$.

Part III: Suppose condition 3 in the hypothesis is true. Let $\alpha_n = \Delta(\mu_{n+1}, R\theta)$ and $\beta_n = \delta(\mu_n, \theta)$. Then from (2.2), we have

$$h(\alpha_n) \leq g(\beta_n) \tag{2.4}$$

for infinitely many values of n.

Clearly, $\alpha_n \to \epsilon$ and $\beta_n \to 0$ as $n \to \infty$, where $\epsilon = \Delta(\theta, R\theta) > 0$.

It follows from (2.4) that

$$\begin{aligned}\liminf_{t\to\epsilon} h(t) &\leq \liminf_{n\to\infty} h(\alpha_n)\\ &\leq \limsup_{n\to\infty} g(\beta_n), \text{ using (2.4)}\\ &\leq \limsup_{t\to 0} g(t). \end{aligned} \tag{2.5}$$

But (2.5) is a contradiction to condition 3 in the hypothesis if $\epsilon > 0$. Hence we must have $\epsilon = \Delta(\theta, R\theta) = 0$, i.e., $\theta \in R\theta$. □

3. Main Results

In this section we present our two main results with the help of the lemmas established in the previous section.

Theorem 3.1. *Let (X, δ) be a MS and $R : X \to \Gamma(X)$ be a set-valued map satisfying*

$$h(\mathcal{PH}(R\mu, R\nu)) \leq g(\delta(\mu, \nu)), \tag{3.1}$$

for all $\mu, \nu \in X$, where the functions $g, h : (0, \infty) \to \mathbb{R}$ are functions satisfying the following conditions:

(1) *h is nondecreasing;*

(2) *$g(t) < h(t)$ for all $t > 0$;*

(3) *$\limsup_{t\to\epsilon+} g(t) < h(\epsilon+)$ for any $\epsilon > 0$;*

(4) *$\inf_{t>\epsilon} h(t) > -\infty$ for any $\epsilon > 0$.*

Then R has a fixed point.

Proof. Using condition (1)-(4) and Lemma 2.1, we have that R is AR. Fix $\mu_0 \in X$ and construct the orbital sequence $\{\mu_n\} = O(\mu_0, R)$. From conditions (1)-(3) and Lemma 2.2, it follows that the orbital sequence $\{\mu_n\}$ is Cauchy.

Since (X, δ) is complete, we have $\mu_n \to \theta$ (for some $\theta \in X$) as $n \to \infty$. Again, using conditions (1) and (2) and Lemma 2.3, we can conclude that θ is a fixed point of R. □

Theorem 3.2. *Let (X, δ) be a MS and $R : X \to \Gamma(X)$ be a set-valued map satisfying*

$$h(\mathcal{PH}(R\mu, R\nu)) \leq g(\delta(\mu, \nu)), \tag{3.2}$$

for all $\mu, \nu \in X$, where the functions $g, h : (0, \infty) \to \mathbb{R}$ satisfy the following conditions:

(1) *$g(t) < h(t)$ for all $t > 0$;*
(2) *$\inf_{t>\epsilon} h(t) > -\infty$ for any $\epsilon > 0$;*
(3) *if $\{h(t_n)\}$ and $\{g(t_n)\}$ are convergent sequences with the same limit and $\{h(t_n)\}$ is strictly decreasing, then $t_n \to 0$ as $n \to \infty$;*
(4) *$\limsup_{t\to\epsilon+} g(t) < \liminf_{t\to\epsilon} h(t)$ or $\limsup_{t\to\epsilon} g(t) < \liminf_{t\to\epsilon+} h(t)$ for any $\epsilon > 0$;*
(5) *R has a closed graph or $\limsup_{t\to 0+} g(t) < \liminf_{t\to\epsilon} h(t)$ for any $\epsilon > 0$.*

Then R has a fixed point.

Proof. Fix $\mu_0 \in X$ and construct the orbital sequence $\{\mu_n\} = O(\mu_0, R)$. From conditions (1)-(3) and Lemma 2.1, we have that R is AR at μ_0. From condition (4) and Lemma 2.2, we can prove that the orbital sequence $\{\mu_n\}$ is Cauchy.

Since (X, δ) is complete, we have $\mu_n \to \theta$ (for some $\theta \in X$) as $n \to \infty$. Next by condition (5) and Lemma 2.3, we conclude that $\theta \in R\theta$. □

The remarks below show that our results extend some results due to Nadler and Boyd-Wong.

Remark 3.1. (1) *If $h(t) = t$ and $g(t) = \lambda t$, where $0 \leq \lambda < 1$, then Theorem 3.1 and Theorem 3.2 both reduce to set-valued version of Banach contraction principle due to Nadler [14].*
(2) *If $h(t) = t$, then both the theorems reduce to set-valued versions of Boyd-Wong's fixed point theorem [5].*

Conclusion. In the present paper, we extended some recent results due to Proinov to their set-valued or multi-valued counterparts. Inspired by the work of Proinov, we introduced some new contractive definitions on the map under consideration with the help of two control functions. Asymptotic regularity of the map has been established and a scheme to obtain a fixed point has been discussed. Common fixed points and coincidence points may be investigated in future in this context.

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