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CONTINUOUS AND DISCRETE WAVELET TRANSFORMS ASSOCIATED WITH HERMITE TRANSFORM

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Abstract. In this paper, we accomplished the concept of continuous and discrete Hermite wavelet transforms. We also discussed some basic properties of Hermite wavelet transform. Inversion formula and Parsevals formula for continuous Hermite wavelet transform is established. Moreover the discrete version of wavelet transform is discussed.

1. Introduction

Many authors have defined wavelet transforms associated with different integral transforms. In ([6], [5]) Pathak and Dixit, Pathak and Pandey defined the wavelet transform which are associated with the Hankel and Laguree transform respectively. In [7] Upadhyay and Tripathi defined continuous wavelet transform corresponding to Watson transform. In 2017 Prasad and Mandal [4] studied the Kontorovich-Lebedev wavelet transform and derived many important properties related to the KL-wavelet transform. In [1] Pathak and Abhishek studied the continuous and discrete wavelet transform associated with index Whittaker transform. Hans-Jurgen Glaeske [3] defined the translation and convolution operator associated with Hermite transform and proved so many important results related to these operator. Now, however to best our knowledge wavelet

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associated with the Hermite transform is not defined. So we are interested to define the wavelet associated to Hermite transform and study the continuous as well as discrete wavelet transforms associated with this.

The wavelet transform [8] of the function $f \in L_2(\mathbf{R})$ with respect to the wavelet $\phi \in L_2(\mathbf{R})$ is defined by

$$\left(W_\phi f\right)(\rho,\sigma) = \int_{-\infty}^{\infty} f(t)\overline{\phi_{\rho,\sigma}(t)}dt, \rho \in \mathbf{R}, \sigma > 0, \tag{1.1}$$

where

$$\phi_{\rho,\sigma}(t) = \sigma^{-\frac{1}{2}}\phi\left(\frac{t-\rho}{\sigma}\right). \tag{1.2}$$

In terms of translation τ_ρ defined by

$$\tau_\rho \phi(t) = \phi(t-\rho), \rho \in \mathbf{R},$$

and dilation D_σ is defined by

$$D_\sigma \phi(t) = \sigma^{-\frac{1}{2}}\phi\left(\frac{t}{\sigma}\right), \sigma > 0,$$

we can write

$$\phi_{\rho,\sigma}(t) = \tau_\rho D_\sigma \phi(t). \tag{1.3}$$

From equation 1.1 and 1.3 it is clear that wavelet transform of the function f on $\mathbf{R}$ is an integral transform for which the kernel is the dilated translate of ϕ.

We can also express equation 1.1 as the convolution

$$\left(W_\phi f\right)(\rho,\sigma) = \left(f * g_{0,\sigma}\right)(\rho), \tag{1.4}$$

where

$$g(t) = \overline{\phi(-t)}.$$

Since associated with each integral transform there exists a special kind of convolution, one can construct wavelet transform corresponding to an integral transform using the associated convolution.

We construct wavelet and wavelet transform on the interval $(-\infty,\infty)$ by using the theory of Hermite transforms [2] and associated convolution involving the function

$$H_n^{(\mu)}(x) = exp\left(\frac{-x^2}{2}\right)\tilde{H}_n^{(\mu)}(x), x \in \mathbf{R},$$

where $\tilde{H}_n^{(\mu)}(x)$ is the normalized Hermite polynomial, where $\alpha > -1$, is given by

$$\tilde{H}_n^{(\mu)}(x) = \begin{cases} \frac{H_{2k}^{(\mu)}(x)}{H_{2k}^{(\mu)}(0)} = R_k^{\left(\mu-\frac{1}{2}\right)}(x^2), n = 2k \\ \frac{H_{2k+1}^{(\mu)}(x)}{\left(H_{2k+1}^{(\mu)}(x)\right)\prime(0)} = xR_k^{\left(\mu+\frac{1}{2}\right)}(x^2), n = 2k+1 \end{cases},$$

and

$$H_n^{(\mu)}(x) = \begin{cases} (-1)^k 2^{2k} k! L_k^{\left(\mu-\frac{1}{2}\right)}(x^2), n = 2k \\ (-1)^k 2^{2k+1} k! x L_k^{\left(\mu+\frac{1}{2}\right)}(x^2), n = 2k+1 \end{cases} .$$

Set

$$d\mu(x) = e^{-x^2}|x|^{2\mu}dx. \tag{1.5}$$

Let us consider the measurable function $f(x)$ on the interval $(-\infty, \infty)$. Then the Hermite transform is defined by

$$H[f](n) = \hat{f}(n) = \int_{-\infty}^{\infty} f(x)\tilde{H}_n(x)d\mu(x), n \in \mathbf{N}. \tag{1.6}$$

The inverse Hermite transform defined by

$$f(x) = \sum_{n=0}^{\infty} \hat{f}(n)\tilde{H}_n^{(\mu)}(x)\left[\tilde{h}_n^{(\mu)}\right]^{-1}. \tag{1.7}$$

where

$$h_n^{(\mu)} = 2^{2n}\Gamma\left(\left[\frac{n}{2}\right]+1\right)\Gamma\left(\left[\frac{(n+1)}{2}\right]+\mu+\frac{1}{2}\right).$$

Let the space of those real measurable functions f on $(-\infty, \infty)$ be $L_{p,\mu}(-\infty, \infty), 1 \leq p < \infty$, for which

$$\|f\|_{p,\mu} = \{\int_{-\infty}^{\infty} \mid f(x) \mid^p d\mu(x)\}^{\frac{1}{p}}, p < \infty. \tag{1.8}$$

$$\|f\|_{p,\mu} = esssup_{x\in\mathbf{R}} \mid f(x) \mid, p = \infty. \tag{1.9}$$

An inner product on $L_{2,\mu}$, is defined by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}d\mu(x). \tag{1.10}$$

2. Hermite translation and convolution

In this section, Hermite translation and associated convolution will be discussed. To define the Hermite convolution '*' we have to introduce Hermite translation. For this purpose we need the basic function

$$K_H^{(\mu)}(x, y, z) \sim \sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1}\tilde{H}_n^{(\mu)}(x)\tilde{H}_n^{(\mu)}(y)\tilde{H}_n^{(\mu)}(z). \tag{2.1}$$

Hence by equation 1.6 and 1.7, we have

$$\int_{-\infty}^{\infty} K_H^{(\mu)}(x, y, z)\tilde{H}_n^{(\mu)}(z)d\mu(z) = \tilde{H}_n^{(\mu)}(x)\tilde{H}_n^{(\mu)}(y). \tag{2.2}$$

Clearly $K_H^{(\mu)}(x, y, z)$ is symmetric in x, y and z.

Setting $n = 0$ in equation 2.2, we have

$$\int_{-\infty}^{\infty} K_H^{(\mu)}(x, y, z)d\mu(z) = 1. \tag{2.3}$$

The Hermite translation τ_y of $f \in L_{p,\mu}(-\infty,\infty), 1 \le p < \infty$ is defined

$$\tau_y f(x) = f(x,y) = \int_{-\infty}^{\infty} f(z) K_H^{(\mu)}(x,y,z) d\mu(z), 1 \le p < \infty. \tag{2.4}$$

Lemma 2.1. *For $f \in L_{p,\mu}$ and $1 \le p < \infty$,*

$$\|\tau_y^{(\mu)} f\|_{p,\mu} \le \|f\|_{p,\mu}, \tag{2.5}$$

and the map: $f \to \tau_y f$ is continuous and linear in $L_{p,\mu}$.

Proof. Proof is referred from [3]. □

Let $p,q,r \in (-\infty,\infty)$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Then the Hermite convolution [3] of $f \in L_{p,\mu}(-\infty,\infty)$ and $g \in L_{q,\mu}(-\infty,\infty)$ is defined by following equation

$$(f * g)(y) = \int_{-\infty}^{\infty} \tau_y^{(\mu)}(f;x)\, g(x) d\mu(x). \tag{2.6}$$

By using the relation defined in equation 2.4, convolution $(f*g)$ can be defined as

$$\begin{aligned}(f*g)(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z) g(x) dm_{x,y}^{(\mu)}(z) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z) g(x) K_H^{(\mu)}(x,y,z) d\mu(x) d\mu(z). \end{aligned} \tag{2.7}$$

Also recall the following Lemma from [3].

Lemma 2.2. *Let $p,q,r \in (-\infty,\infty)$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1, f \in L_{p,\mu}(-\infty,\infty)$ and $g \in L_{q,\mu}(-\infty,\infty)$. Then the convolution $(f*g)$ defined by equation 2.7 satisfies the following norm inequality:*

$$(i) \|f*g\|_{r,\mu} \le \|f\|_{p,\mu} \|g\|_{q,\mu}. \tag{2.8}$$

Moreover $f,g \in L_{2,\mu}$, we get

$$(ii)\, (f*g)^{\wedge}(n) = \hat{f}(n)\hat{g}(n). \tag{2.9}$$

Lemma 2.3. *For any $f \in L_{2,\mu}$ the following Parseval Identity holds for Hermite transform:*

$$\sum_n \left[\tilde{h}_n^{(\mu)}\right]^{-1} \mid \hat{f}(n) \mid^2 = \|f\|_{2,\mu}^2. \tag{2.10}$$

Proof. Proof is referred from theorem 1 in ref. [2]. □

For any $f_1, f_2 \in L_{2,\mu}(-\infty,\infty)$ the below Parseval Identity holds for Hermite transform. See ref. [2].

$$\sum_n \left[\tilde{h}_n^{(\mu)}\right]^{-1} f_1(n) f_2(n) = \int_{-\infty}^{\infty} f_1(x) f_2(x) d\mu(x)$$

and

$$\sum_n \left[\tilde{h}_n^{(\mu)}\right]^{-1} f_1(n) f_2(n) = \int_{-\infty}^{\infty} H^{-1}[f_1(n)][f_2(n)] d\mu(x).$$

In this paper, following the technique of Pathak and Dixit [6] and Trimeche [9], Hermite wavelet transform is defined. The continuity and boundedness properties of Hermite wavelet transform is derived. A semi discrete Hermite wavelet transform is defined. Furthermore discrete Hermite wavelet transform is investigated. Using discrete Hermite wavelet, frame and Riesz basis [8] are also studied.

3. Continuous Hermite Wavelet Transform

For a function $\phi \in L_{p,\mu}(-\infty,\infty)$, defined the dilation D_σ by

$$D_\sigma \phi(t) = \phi(\sigma t), \sigma > 0. \tag{3.1}$$

Using the Hermite translation 2.4 and above dilation, the Hermite wavelet $\phi_{\rho,\sigma}(t)$ is defined as follows:

$$\begin{aligned} \phi_{\rho,\sigma}(t) &= \tau_\rho D_\sigma \phi(t) \\ &= \tau_\rho \phi(\sigma t) & (3.2) \\ &= \int_0^\infty \phi(\sigma z) K_H^{(\mu)}(\rho, t, z) d\mu(z). & (3.3) \end{aligned}$$

where $\rho \geq 0$ and $\sigma > 0$. The integral is convergent by virtue of inequality 2.5.

Definition 3.1. ***Admissible Hermite wavelet***

The function $\phi(\cdot) \in L_{p,\mu}(-\infty,\infty)$ is said to be admissible Hermite wavelet if $\phi(\cdot)$ satisfies the following admissibility condition

$$C_\phi = \sum_{n=0}^{\infty} \frac{|\ \hat{\phi}(n)\ |^2}{|\ n\ |} < \infty,$$

where $\hat{\phi}(n)$ is the Hermite transform of ϕ.

Continuous Hermite wavelet transform

Using the wavelet $\phi_{\rho,\sigma}$ we now define the continuous Hermite wavelet transform.

$$\begin{aligned} \left(\tilde{H}_\phi^{(\mu)} f\right)(\rho,\sigma) &= \langle f(t), \phi_{\rho,\sigma}(t) \rangle \\ &= \int_{-\infty}^{\infty} f(t) \overline{\phi_{\rho,\sigma}(t)} d\mu(t) & (3.4) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \overline{\phi(\sigma z)} K_H^{(\mu)}(\rho, t, z) d\mu(z) d\mu(t) & (3.5) \end{aligned}$$

provided the integral is convergent. Since by inequality 2.5 and definition $\phi_{\rho,\sigma} \in L_{p,\mu}$ whenever $\phi \in L_{p,\mu}$. By virtue of Lemma 2.2, the integral 3.5 is convergent for $f \in L_{q,\mu}$, $\frac{1}{p} + \frac{1}{q} = 1$.

The Hermite wavelet transform can be expressed in the form of Hermite transform as follows.

$$H\left[\left(\tilde{H}_\phi^{(\mu)} f\right)(\rho,\sigma)\right] = \hat{f}(n)\hat{\phi}(\sigma, n).$$

Also the Hermite wavelet transform can be written as

$$\left(\tilde{H}_{\phi}^{(\mu)} f\right)(\rho, \sigma) = (f * \phi(\sigma, \cdot))(\rho).$$

The continuity and boundedness results follow from the following theorem.

Theorem 3.1. *Let $f(\cdot) \in L_{p,\mu}$ and $\phi(\cdot) \in L_{q,\mu}, \sigma > 0$ with $1 \leq p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. and $\left(\tilde{H}_{\phi}^{(\mu)} f\right)(\rho, \sigma)$ be continuous Hermite wavelet transform 3.5. Then*

(i) $\| \left(\tilde{H}_{\phi}^{(\mu)} f\right)(\rho, \sigma)\|_{r,\mu} \leq \|f\|_{p,\mu}\|\phi(\sigma, \cdot)\|_{q,\mu}, \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1, 1 \leq p, q, r < \infty.$
(ii) $\| \left(\tilde{H}_{\phi}^{(\mu)} f\right)(\rho, \sigma)\|_{\infty,\mu} \leq \|f\|_{p,\mu}\|\phi(\sigma, \cdot)\|_{q,\mu}, \frac{1}{p} + \frac{1}{q} = 1.$

Proof. (ii) Using representation 3.5, we have

$$\begin{aligned}
\left(\tilde{H}_{\phi}^{(\mu)} f\right)(\rho, \sigma) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\overline{\phi(\sigma z)} K_H^{(\mu)}(\rho, t, z) d\mu(z) d\mu(t) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\overline{\phi(\sigma z)} K_H^{(\mu)^{\frac{1}{p}}}(\rho, t, z) K_H^{(\mu)^{\frac{1}{q}}}(\rho, t, z) d\mu(z) d\mu(t)
\end{aligned}$$

using Holder's inequality, we get

$$\begin{aligned}
|\left(\tilde{H}_{\phi}^{(\mu)} f\right)(\rho, \sigma)| &\leq \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\, f(t)\, |^p\, K_H^{(\mu)}(\rho, t, z) d\mu(z) d\mu(t)\right)^{\frac{1}{p}} \times \\
&\quad \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\, \phi(\sigma z)\, |^q\, K_H^{(\mu)}(\rho, t, z) d\mu(z) d\mu(t)\right)^{\frac{1}{q}} \\
&= \left(\int_{-\infty}^{\infty} |\, f(t)\, |^p\, d\mu(t) \int_{-\infty}^{\infty} K_H^{(\mu)}(\rho, t, z) d\mu(z)\right)^{\frac{1}{p}} \times \\
&\quad \left(\int_{-\infty}^{\infty} |\, \phi(\sigma z)\, |^q\, d\mu(z) \int_{-\infty}^{\infty} K_H^{(\mu)}(\rho, t, z) d\mu(t)\right)^{\frac{1}{q}}
\end{aligned}$$

by using equation 2.3, it follows that

$$|\left(\tilde{H}_{\phi}^{(\mu)} f\right)(\rho, \sigma)| \leq \|f\|_{p,\mu}\|\phi(\sigma, \cdot)\|_{q,\mu};$$

so that

$$\| \left(\tilde{H}_{\phi}^{(\mu)} f\right)(\rho, \sigma)\|_{\infty,\mu} \leq \|f\|_{p,\mu}\|\phi(\sigma, \cdot)\|_{q,\mu}.$$

the inequality 1.1 follows from inequality 2.8. □

Theorem 3.2. *If ϕ is a basic Hermite wavelet and Ψ is any bounded function, then $(\phi * \Psi)$ is also a hermite wavelet.*

Proof.

$$\begin{aligned} C_{(\phi*\Psi)} &= \sum_{n=0}^{\infty} \frac{|(\phi*\Psi)^\wedge(n)|^2}{n} \\ &= \sum_{n=0}^{\infty} \frac{|(\phi)^\wedge(n)(\Psi)^\wedge(n)|^2}{n} \\ &\leq |(\Psi)^\wedge(n)| \sum_{n=0}^{\infty} \frac{|(\phi)^\wedge(n)|^2}{n} \\ &< \infty \end{aligned}$$

Hence $(\phi * \Psi)$ is a Hermite wavelet. □

Basic Properties of Continuous Hermite Wavelet Transform

Theorem 3.3. *Let ϕ and Ψ be two wavelets and f, g be two functions belong to $L_{p,\mu}(-\infty,\infty)$, then*

(i) ***Linearity property:***

$$H_\phi^{(\mu)}(\eta f + \zeta g)(\sigma,\rho) = \eta H_\phi^{(\mu)}(f)(\sigma,\rho) + \zeta H_\phi^{(\mu)}(g)(\sigma,\rho)$$

where η and ζ are any two scalars.

(ii) ***Shift property***

$$\left(H_\phi^{(\mu)} f\right)(x-\tau)(\sigma,\rho) = \left(H_\phi^{(\mu)} f\right)(\sigma,\rho-\tau)$$

where τ is any scalar.

(iii) ***Scaling property*** *If $c \neq 0$ is any scalar, then the Hermite wavelet transform of the scaled function $f_c(x) = \frac{1}{c} f\left(\frac{1}{2}\right)$ is*

$$\left(H_\phi^{(\mu)} f_c\right)(\rho,\sigma) = H_\phi^{(\mu)} f\left(\frac{\sigma}{c}, \frac{\rho}{c}\right)$$

(iv) ***Symmetry property:***

$$\left(H_\phi^{(\mu)} f\right)(\sigma,\rho) = \left(H_\phi^{(\mu)} f\right)(\phi)\left(\frac{1}{\sigma}, \frac{-1}{\rho}\right)$$

(v) ***Parity property***

$$\left(H_{p\phi}^{(\mu)} p f\right)(\sigma,\rho) = \left(H_\phi^{(\mu)} f\right)(\sigma,-\rho)$$

where p is the parity operator defined by $pf(x) = f(-x)$.

Proof. The proof is the straight forward application of Hermite transform. □

Plancharel and Persevals relation for Continuous Hermite wavelet Transform

Let $f, g \in L_{2,\mu}(-\infty,\infty)$ and $\phi_1, \phi_2 \in L_{2,\mu}(-\infty,\infty)$ are two Hermite wavelets. Then we have

$$\langle \left(H_{\phi_1}^{(\mu)} f\right)(\sigma,\rho), \left(H_{\phi_2}^{(\mu)} g\right)(\sigma,\rho) \rangle_{L_{2,\mu((-\infty,\infty)\times(-\infty,\infty))}} = c_{\phi_1,\phi_2} \langle f, g \rangle_{L_{2,\mu(-\infty,\infty)}}, \tag{3.6}$$

where

$$c_{\phi_1,\phi_2} = \int_0^\infty \phi_1(\sigma,n)\phi_2(\rho,n)d\mu(\sigma).$$

Proof. Let $f, g \in L_{2,\mu}(-\infty,\infty)$ then from 3.4, we have

$$\int_0^\infty \int_{-\infty}^\infty \left(H_\phi^{(\mu)} f\right)(\sigma,\rho)\overline{\left(H_\phi^{(\mu)} g\right)(\sigma,\rho)}d\mu(\sigma)d\mu(\rho) = \int_0^\infty \int_{-\infty}^\infty H_\phi^{(\mu)-1}\left[\hat{f}(n)\phi_1(\sigma,n)\right](\rho)$$
$$\overline{H_\phi^{(\mu)-1}\left[\hat{g}(n)\phi_2(\sigma,n)\right](\rho)}d\mu(\sigma)d\mu(\rho)$$

now by using 2.10 we get

$$\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty f(x)\phi_1(\sigma,x)(\rho)\overline{g(x)\phi_2(\sigma,x)(\rho)}d\mu(\sigma)d\mu(\rho)\\
&= \sum_n \left[\tilde{h}_n^{(\mu)}\right]^{-1} \hat{f}(n)\overline{\hat{g}(n)} \int_{-\infty}^\infty \phi_1(\sigma,n)\phi_2(\sigma,n)d\mu(\sigma) \qquad (3.7)\\
&= c_{\phi_1,\phi_2}\sum_n \left[\tilde{h}_n^{(\mu)}\right]^{-1} \hat{f}(n)\overline{\hat{g}(n)}.
\end{aligned}$$

Hence by using the Parseval formula for Hermite transform, we get

$$\begin{aligned}
\int_0^\infty \int_{-\infty}^\infty \left(H_\phi^{(\mu)} f\right)(\sigma,\rho)\overline{\left(H_\phi^{(\mu)} g\right)(\sigma,\rho)}d\mu(\sigma)d\mu(\rho) &= c_{\phi_1,\phi_2}\hat{f}(n)\overline{\hat{g}(n)}\\
&= c_{\phi_1,\phi_2}\langle f,g\rangle_{L_{2,\mu(-\infty,\infty)}}. \qquad (3.8)
\end{aligned}$$

□

Theorem 3.4. ***(Inversion formula)*** *Let $f \in L_{2,\mu}(-\infty,\infty)$ and ϕ is Hermite wavelet defines continuous Hermite wavelet transform. Then,*

$$f(x) = \frac{1}{c_\phi}\int_{-\infty}^\infty \int_{-\infty}^\infty \left(H_\phi^{(\mu)} f\right)(\sigma,\rho)\phi_{\rho,\sigma}(t)d\mu(\sigma)d\mu(\rho),$$

where c_ϕ is the Admissible Hermite wavelet.

Proof. Let $h(x) \in L_{2,\mu}(-\infty,\infty)$ be any function, then by applying previous theorem, we have

$$\begin{aligned}
c_\phi\langle f,h\rangle_{L_{2,\mu(-\infty,\infty)}} &= \int_{-\infty}^\infty \int_{-\infty}^\infty \left(H_\phi^{(\mu)} f\right)(\sigma,\rho)\overline{\left(H_\phi^{(\mu)} h\right)(\sigma,\rho)}d\mu(\sigma)d\mu(\rho)\\
&= \frac{1}{2}\int_{-\infty}^\infty \int_{-\infty}^\infty \left(H_\phi^{(\mu)} f\right)(\sigma,\rho)\int_{-\infty}^\infty \overline{h(t)\overline{\phi_{\rho,\sigma}(t)}}dtd\mu(\sigma)d\mu(\rho)\\
&= \frac{1}{2}\int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \left(H_\phi^{(\mu)} f\right)(\sigma,\rho)\phi_{\rho,\sigma}(t)\overline{h(t)}dtd\mu(\sigma)d\mu(\rho)\\
&= \int_{-\infty}^\infty g(t)\overline{h(t)}dt\\
&= \langle g,h\rangle, \qquad (3.9)
\end{aligned}$$

where,

$$g = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(H_{\phi}^{(\mu)}f\right)(\sigma,\rho)\phi_{\rho,\sigma}(t)d\mu(\sigma)d\mu(\rho).$$

Then,

$$c_{\phi}\langle f,h\rangle = \langle g,h\rangle$$

$$f = \frac{1}{c_{\phi}}g = \frac{1}{2c_{\phi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(H_{\phi}^{(\mu)}f\right)(\sigma,\rho)\phi_{\rho,\sigma}(t)d\mu(\sigma)d\mu(\rho).$$

If $f = h$,

$$\|f\|^2_{L_{2,\mu(-\infty,\infty)}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \left(H_{\phi}^{(\mu)}f\right)(\sigma,\rho) \right|^2 d\mu(\sigma)d\mu(\rho).$$

Moreover the Hermite wavelet transform is isometry from $L_{2,\mu}(-\infty,\infty)$ to $L_{2,\mu}(-\infty,\infty)\times L_{2,\mu}(-\infty,\infty)$.

□

A General Reconstruction Formula

In this section, we show that the function f can be recovered from its Hermite wavelet transform. In derived the reconstruction formula, we need the following lemma.

Lemma 3.1. *Let $f \in L_{2,\mu}$ and $\phi \in L_{2,\mu}$ be a basic wavelet, which defines Hermite wavelet transform 3.5. Then*

$$\left(\tilde{H}_{\phi}^{(\mu)}f\right)^{\wedge}(\rho,\sigma) = \hat{f}(n)\overline{\hat{\phi}(\sigma,n)}, \tag{3.10}$$

where

$$\hat{\phi}(\sigma,n) = \int_{-\infty}^{\infty}\phi(\sigma z)\tilde{H}_n^{(\mu)}(z)d\mu(z). \tag{3.11}$$

Proof. Using representation 3.5, we have

$$\begin{aligned}
\left(\tilde{H}_{\phi}^{(\mu)}f\right)(\rho,\sigma) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\overline{\phi(\sigma z)}K_H^{(\mu)}(\rho,t,z)d\mu(z)d\mu(t)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\overline{\phi(\sigma z)}d\mu(z)d\mu(t)\left(\sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1}\tilde{H}_n^{(\mu)}(\rho)\tilde{H}_n^{(\mu)}(t)\tilde{H}_n^{(\mu)}(z)\right)\\
&= \sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1}\tilde{H}_n^{(\mu)}(\rho)\left(\int_{-\infty}^{\infty}f(t)\tilde{H}_n^{(\mu)}(t)d\mu(t)\int_{-\infty}^{\infty}\overline{\phi(\sigma z)}\tilde{H}_n^{(\mu)}(z)d\mu(z)\right)\\
&= \sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1}\tilde{H}_n^{(\mu)}(\rho)\hat{f}(n)\overline{\hat{\phi}(\sigma,n)}\\
&= \left(\hat{f}(n)\overline{\hat{\phi}(\sigma,n)}\right)^{\vee}(\rho).\\
\therefore \left(\tilde{H}_{\phi}^{(\mu)}f\right)^{\wedge}(\rho,\sigma) &= \hat{f}(n)\overline{\hat{\phi}(\sigma,n)}.
\end{aligned}$$

This completes the proof. □

Theorem 3.5. *Let $f \in L_{2,\mu}$ and ϕ be a basic wavelet which defines Hermite wavelet transform by equation 3.5. Let $q(\sigma) > 0$ be a weight function such that*

$$Q(n) = \int_0^\infty q(\sigma) \mid \hat{\phi}(\sigma, n) \mid^2 d\mu(\sigma) > 0. \tag{3.12}$$

Set

$$\hat{\phi}^{\rho,\sigma}(n) = \frac{\hat{\phi}_{\rho,\sigma}(n)}{Q(n)}. \tag{3.13}$$

Then

$$f(t) = \int_0^\infty \int_{-\infty}^\infty q(\sigma) \left(\tilde{H}_\phi^{(\mu)} f\right)(\rho, \sigma)\phi^{b,a}(t) d\mu(\sigma) d\mu(\rho). \tag{3.14}$$

Proof. From equation 3.10, we have

$$\left(\tilde{H}_\phi^{(\mu)} f\right)^\wedge (\rho, \sigma) = \hat{f}(n)\hat{\phi}(\sigma, n)$$
$$\Rightarrow \int_{-\infty}^\infty \left(\tilde{H}_\phi^{(\mu)} f\right)(\rho, \sigma)\tilde{H}_\phi^{(\mu)}(b) d\mu(b) = \hat{f}(n)\overline{\hat{\phi}(\sigma, n)}.$$

multiplying both sides by $\hat{\phi}(\sigma, n)$ and weight function $q(\sigma)$ and integrating with respect to $d\mu(\sigma)$, we have

$$\int_0^\infty q(\sigma)\hat{\phi}(\sigma, n) \left(\int_{-\infty}^\infty \left(\tilde{H}_\phi^{(\mu)} f\right)(\rho, \sigma)\tilde{H}_n^{(\mu)}(\rho) d\mu(\rho)\right) d\mu(\sigma)$$
$$= \int_0^\infty q(\sigma)\hat{f}(n)\hat{\phi}(\sigma, n)\overline{\hat{\phi}(\sigma, n)} d\mu(\sigma)$$
$$\Rightarrow \int_0^\infty q(\sigma)\hat{\phi}(\sigma, n) \left(\int_{-\infty}^\infty \left(\tilde{H}_\phi^{(\mu)} f\right)(\rho, \sigma)\tilde{H}_n^{(\mu)}(\rho) d\mu(\rho)\right) d\mu(\sigma)$$
$$= \int_0^\infty q(\sigma)\hat{f}(n) \mid \phi(\sigma, n) \mid^2 d\mu(\sigma). \tag{3.15}$$

Equation 3.11 and 3.15 gives

$$\begin{aligned} \hat{f}(n)Q(n) &= \int_0^\infty q(\sigma)\hat{\phi}(\sigma, n) d\mu(\sigma) \int_{-\infty}^\infty \left(\tilde{H}_\phi^{(\mu)} f\right)(\rho, \sigma)\tilde{H}_n^{(\mu)}(\rho) d\mu(\rho) \\ \hat{f}(n) &= \frac{1}{Q(n)} \int_0^\infty q(\sigma) \int_{-\infty}^\infty \left(\tilde{H}_\phi^{(\mu)} f\right)(\rho, \sigma)\hat{\phi}(\sigma, n)\tilde{H}_n^{(\mu)}(\rho) d\mu(\sigma) d\mu(\rho) \end{aligned} \tag{3.16}$$

We also have from equation 3.3,

$$\begin{aligned} \phi_{\rho,\sigma}(t) &= \int_{-\infty}^\infty \phi(\sigma z) \sum_{n=0}^\infty \left[h_n^{(\mu)}\right]^{-1} \tilde{H}_n^{(\mu)}(\rho)\tilde{H}_n^{(\mu)}(t)\tilde{H}_n^{(\mu)}(z) d\mu(z) \\ &= \sum_{n=0}^\infty \left[h_n^{(\mu)}\right]^{-1} \tilde{H}_n^{(\mu)}(\rho)\tilde{H}_n^{(\mu)}(t) \int_{-\infty}^\infty \phi(\sigma z)\tilde{H}_n^{(\mu)}(z) d\mu(z) \\ &= \sum_{n=0}^\infty \left[h_n^{(\mu)}\right]^{-1} \tilde{H}_n^{(\mu)}(\rho)\tilde{H}_n^{(\mu)}(t)\hat{\phi}(\sigma, n) \\ &= \left(\hat{\phi}(\sigma, n)\tilde{H}_n^{(\mu)}(\rho)\right)^\vee (t). \end{aligned}$$

Therefore

$$\phi_{\rho,\sigma}(t) = \left(\hat{\phi}(\sigma,n)\tilde{H}_n^{(\mu)}(\rho)\right)^{\vee}(t).$$
$$\hat{\phi}_{\rho,\sigma}(t) = \hat{\phi}(\sigma,n)\tilde{H}_n^{(\mu)}(\rho). \tag{3.17}$$

Using equation 3.17 in 3.16, we have

$$\hat{f}(n) = \frac{1}{Q(n)}\int_0^{\infty} q(\sigma)\int_{-\infty}^{\infty}\hat{\phi}_{\rho,\sigma}(n)\left(\tilde{H}_{\phi}^{(\mu)}f\right)(\rho,\sigma)d\mu(\sigma)d\mu(\rho). \tag{3.18}$$

From equation 3.13 it follows that

$$\hat{f}(n) = \int_0^{\infty} q(\sigma)\int_{-\infty}^{\infty}\hat{\phi}^{\rho,\sigma}(n)\left(\tilde{H}_{\phi}^{(\mu)}f\right)(\rho,\sigma)d\mu(\sigma)d\mu(\rho). \tag{3.19}$$

From equation 1.7 and 3.19, we have

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty}\left[h_n^{(\mu)}\right]^{-1}\tilde{H}_n^{(\mu)}(t)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}q(\sigma)\left(\tilde{H}_{\phi}^{(\mu)}f\right)(\rho,\sigma)\hat{\phi}^{\rho,\sigma}(n)d\mu(\sigma)d\mu(\rho) \\ &= \sum_{n=0}^{\infty}q(\sigma)\left(\tilde{H}_{\phi}^{(\mu)}f\right)(\rho,\sigma)\sum_{n=0}^{\infty}\left[h_n^{(\mu)}\right]^{-1}\hat{\phi}^{\rho,\sigma}(n)\tilde{H}_n^{(\mu)}(t)d\mu(\sigma)d\mu(\rho) \\ &= \int_0^{\infty}\int_{-\infty}^{\infty}q(\sigma)\left(\tilde{H}_{\phi}^{(\mu)}f\right)(\rho,\sigma)\phi^{\rho,\sigma}(n)d\mu(\sigma)d\mu(\rho). \end{aligned}$$

This completes the proof of theorem 3.5.

A characterization of $\phi^{\rho,\sigma}$ is given below. □

Theorem 3.6. *Assume that there exist positive constant A and B such that,*

$$0 < A \leq Q(n) \leq B < \infty \tag{3.20}$$

Let

$$\phi^{\sigma}(t) = \sum_{n=0}^{\infty}\frac{\left[h_n^{(\mu)}\right]^{-1}}{Q(n)}\hat{\phi}(\sigma,n)\tilde{H}_n^{(\mu)}(t). \tag{3.21}$$

Then

$$(i)\phi^{\rho,\sigma}(t) = \tau_{\rho}\phi^{\sigma}(t); \tag{3.22}$$

$$(ii)\|\phi^{\rho,\sigma}\|_{2,\mu} \leq A^{-1}\|\phi_{\rho,\sigma}\|_{2,\mu}. \tag{3.23}$$

Proof. (i) Using equations 1.7, 2.2, 3.13 and 3.16, we have

$$\begin{aligned}
\phi^{\rho,\sigma}(t) &= \frac{\phi_{\rho,\sigma}(n)}{Q(n)} \\
&= \frac{\phi(\sigma,n)\tilde{H}_n^{(\mu)}(\rho)}{Q(n)} \\
&= \frac{\sum_{n=0}^{\infty}\hat{\phi}(\sigma,n)\tilde{H}_n^{(\mu)}(t)\left[\tilde{h}_n^{(\mu)}\right]^{-1}\tilde{H}_n^{(\mu)}(\rho)}{Q(n)} \\
&= \frac{\sum_{n=0}^{\infty}\frac{\hat{\phi}_{\rho,\sigma}(n)}{\tilde{H}_n^{(\mu)}(\rho)}\tilde{H}_n^{(\mu)}(t)\left[\tilde{h}_n^{(\mu)}\right]^{-1}\tilde{H}_n^{(\mu)}(\rho)}{Q(n)} \\
&= \frac{\sum_{n=0}^{\infty}\hat{\phi}^{\rho,\sigma}(n)Q(n)\tilde{H}_n^{(\mu)}(t)\left[\tilde{h}_n^{(\mu)}\right]^{-1}}{Q(n)} \\
&= \sum_{n=0}^{\infty}\hat{\phi}_{\rho,\sigma}(n)\tilde{H}_n^{(\mu)}(t)\left[\tilde{h}_n^{(\mu)}\right]^{-1} \\
&= \sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1}\frac{\hat{\phi}_{\rho,\sigma}(n)}{Q(n)}\tilde{H}_n^{(\mu)}(t) \\
&= \sum_{n=0}^{\infty}\frac{\left[\tilde{h}_n^{(\mu)}\right]^{-1}}{Q(n)}\hat{\phi}(\sigma,n)\tilde{H}_n^{(\mu)}(\rho)\tilde{H}_n^{(\mu)}(t) \\
&= \sum_{n=0}^{\infty}\frac{\left[\tilde{h}_n^{(\mu)}\right]^{-1}}{Q(n)}\hat{\phi}(\sigma,n)\left(\int_{-\infty}^{\infty}K_H^{(\mu)}(x,y,z)\tilde{H}_n^{(\mu)}(z)d\mu(z)\right) \\
&= \int_{-\infty}^{\infty}K_H^{(\mu)}(x,y,z)\left(\sum_{n=0}^{\infty}\frac{\left[\tilde{h}_n^{(\mu)}\right]^{-1}}{Q(n)}\hat{\phi}(\sigma,n)\tilde{H}_n^{(\mu)}(z)\right)d\mu(z) \\
&= \int_{-\infty}^{\infty}\phi^{\sigma}K_H^{(\mu)}(x,y,z)d\mu(z) \\
&= \tau_\rho\phi^{\sigma}(z),
\end{aligned}$$

where $\phi^{\sigma}(t)$ is given in equation 3.21.

(ii) From equation 3.13, we have

$$|\ \hat{\phi}^{\rho,\sigma}\ |\leq A^{-1}\ |\ \phi_{\rho,\sigma}(n)\ |; \tag{3.24}$$

so that

$$\sum_{n=0}^{\infty}\left[h_n^{(\mu)}\right]^{-1}|\ \hat{\phi}^{\rho,\sigma}(n)\ |^2\leq A^{-2}\sum_{n=0}^{\infty}\left[h_n^{(\mu)}\right]^{-1}|\ \phi_{\rho,\sigma}(n)\ |^2 .$$

Using equation 2.10, we get

$$\|\phi^{\rho,\sigma}\|_{2,\mu}\leq\|\phi_{\rho,\sigma}\|_{2,\mu}.$$

□

4. The Discrete Hermite Wavelet Transform

The continuous Hermite wavelet transform of the function f in terms of two continuous parameters σ and ρ can be converted into a semi-discrete Hermite wavelet transform by assuming that $\sigma = 2^{-j}, j \in \mathbf{Z}$ and $\rho \in \mathbf{R}_+$.

In what follows we assume that $\phi \in L_{1,\mu} \cap L_{2,\mu}$ satisfies the so called 'stability condition'.

$$A \leq \sum_{j=-\infty}^{\infty} \mid \hat{\phi}(2^{-j}n) \mid^2 \leq B \qquad a.e. \tag{4.1}$$

for certain positive constants A and B, $-\infty < A \leq B < \infty$.

The function $\phi \in L_{1,\mu} \cap L_{2,\mu}$ satisfying condition 4.1 is called dyadic wavelet. Using definition 3.4, we define the semi discrete Hermite wavelet transform of any $f \in L_{1,\mu} \cap L_{2,\mu}$ by

$$\begin{aligned} \left(H_j^{\phi} f\right)(\rho) &= \left(H_j^{\phi} f\right)(\rho, 2^{-j}) \\ &= \langle f(t), \phi_{\rho,2^{-j}}(t)\rangle \\ &= \int_{-\infty}^{\infty} f(t)\overline{\phi_{\rho,2^{-j}}(t)}d\mu(t) \\ &= \int_{-\infty}^{\infty} f(t)\overline{\tau_\rho \phi(2^{-j}t)}d\mu(t) \\ &= \left(f * \overline{\phi_j}\right)(\rho), \end{aligned} \tag{4.2}$$

where $\phi_j(z) = \phi(2^{-j}z), j \in \mathbf{Z}$.

Theorem 4.1. *Assume that the semi discrete Hermite wavelet transform of any $f \in L_{1,\mu} \cap L_{2,\mu}$ is defined by the equation 4.2.*

Let us consider another wavelet ϕ^* defined by means of its Hermite transform

$$\hat{\phi}^*(n) = \frac{\hat{\phi}(n)}{\sum_{l\to-\infty}^{\infty} \mid \hat{\phi}(2^{-l}n) \mid^2}. \tag{4.3}$$

Then

$$f(t) = \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(H_j^{\phi} f\right)(\rho) \left(\hat{\phi}^*(2^{-j}n)\tilde{H}_n^{(\mu)}(t)\right)^{\vee}(\rho)d\mu(\rho). \tag{4.4}$$

Proof. In view of relations 1.7, 3.23 and 2.9,

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} & \int_{-\infty}^{\infty}\left(H_j^{\phi} f\right)(\rho)\left(\hat{\phi}^*(2^{-j} n)\tilde{H}_n^{(\mu)}(t)\right)^{\vee}(\rho) d\mu(\rho) \\
&= \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty}\left(H_j^{\phi} f\right)(\rho)\left[\sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1} \hat{\phi}^*(2^{-j} n)\tilde{H}_n^{(\mu)}(t)\tilde{H}_n^{(\mu)}(\rho)\right] d\mu(\rho) \\
&= \sum_{j=-\infty}^{\infty}\sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1} \hat{\phi}^*(2^{-j} n)\tilde{H}_n^{(\mu)}(t)\int_{-\infty}^{\infty}\left(H_j^{\phi} f\right)(\rho)\tilde{H}_n^{(\mu)}(\rho) d\mu(\rho) \\
&= \sum_{j=-\infty}^{\infty}\sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1} \hat{\phi}^*(2^{-j} n)\tilde{H}_n^{(\mu)}(t)\left(H_j^{\phi} f\right)^{\wedge}(n) \\
&= \sum_{j=-\infty}^{\infty}\sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1} \hat{\phi}^*(2^{-j} n)\tilde{H}_n^{(\mu)}(t)\left(f * \overline{\phi_j}\right)^{\wedge}(n) \\
&= \sum_{j=-\infty}^{\infty}\sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1} \hat{\phi}^*(2^{-j} n)\tilde{H}_n^{(\mu)}(t)\hat{f}(n)\overline{\hat{\phi}(2^{-j} n)} \\
&= \sum_{j=-\infty}^{\infty}\sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1} \hat{f}(n)\tilde{H}_n^{(\mu)}(t)\hat{\phi}^*(2^{-j} n)\overline{\hat{\phi}(2^{-j} n)} \\
&= \sum_{j=-\infty}^{\infty}\sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1} \hat{f}(n)\tilde{H}_n^{(\mu)}(t)\frac{\hat{\phi}(2^{-j} n)}{\sum_l \mid \hat{\phi}(2^{-j}2^{-l} n)\mid^2}\overline{\hat{\phi}(2^{-j} n)} \\
&= \sum_{j=-\infty}^{\infty}\sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1} \hat{f}(n)\tilde{H}_n^{(\mu)}(t)\frac{\mid\hat{\phi}(2^{-j} n)\mid^2}{\sum_l \mid \hat{\phi}(2^{-j}2^{-l} n)\mid^2} \\
&= \sum_{j=-\infty}^{\infty}\sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1} \hat{f}(n)\tilde{H}_n^{(\mu)}(t) \\
&= f(t).
\end{aligned}$$

The above theorem leads to the following definition of dyadic dual. □

Definition 4.1. *A function $\tilde{\phi} \in L_{1,\mu} \cap L_{2,\mu}$, is called a dyadic dual of a dyadic wavelet ϕ, if every $f \in L_{1,\mu} \cap L_{2,\mu}$ can be expressed as*

$$f(t) = \sum_j \int_{-\infty}^{\infty}\left(H_j^{\phi} f\right)(\rho)\left(\tilde{\phi}(2^{-j} n)\tilde{H}_n^{(\mu)}(t)\right)^{\vee}(\rho) d\mu(\rho). \tag{4.5}$$

So far we have considered semi-discrete Hermite wavelet transform of any $f \in L_{1,\mu} \cap L_{2,\mu}$ discretizing only variable a. Now, we discretize the translation parameter b also by restricting it to the discrete set of points:

$$\rho_{j,k} = \frac{k}{2^j}\rho_0; j \in \mathbf{Z}, k \in \mathbf{N}, \tag{4.6}$$

where $\rho_0 > 0$ is a fixed constant. We write,

$$\phi_{\rho_0;j,k}(t) = \phi_{\rho,j,k;\sigma_j}(t) = \phi(2^{-j}t, 2^{-j}k\rho_0). \tag{4.7}$$

Then the discrete Hermite wavelet transform of any $f \in L_{2,\mu}$ can be expressed as

$$\left(H_\phi^{(\mu)} f\right)(\rho_{j,k}, \sigma_j) = \langle f, \phi_{\rho_0;j,k} \rangle, j \in \mathbf{Z}, k, n \in \mathbf{N}. \tag{4.8}$$

The 'stability condition' for this reconstruction takes the form

$$A\|f\|_{2,\mu}^2 \leq \sum_{j \in \mathbf{Z}} |\langle f, \phi_{\rho_0;j,k} \rangle|^2 \leq B\|f\|_{2,\mu}^2, k \in \mathbf{N}, \tag{4.9}$$

where A and B are positive constant such that $0 \leq A \leq B < \infty$.

Theorem 4.2. *Assume that the discrete Hermite wavelet transform of any $f \in L_{2,\mu}$ is defined by 4.8 and stability condition 4.9 holds. Let T be a linear operator on $L_{2,\mu}$ defined by*

$$Tf = \sum_{j \in \mathbf{Z}, k \in \mathbf{N}_0} \langle f, \phi_{\rho_0;j,k} \rangle_\mu \phi_{\rho_0;j,k}. \tag{4.10}$$

Then

$$f = \sum \langle f, \phi_{\rho_0;j,k} \rangle_\mu \phi_{\rho_0}^{j,k},$$

where,

$$\phi_{\rho_0}^{j,k} = T^{-1}\phi_{\rho_0;j,k}; j \in \mathbf{Z}, k \in \mathbf{N}_0.$$

Proof. From the stability condition 4.9, it follows that the operator defined by equation 4.10 is a one-one bounded linear operator.

Set

$$g = Tf, f \in L_{2,\mu}.$$

Then from equation 4.10, we have

$$\langle Tf, f \rangle = \sum_{j \in \mathbf{Z}, k \in \mathbf{N}_0} | \; f, \phi_{\rho_0;j,k} \; |^2 .$$

Therefore, from condition 4.9,

$$\begin{aligned} A\|T^{-1}g\|_{2,\mu}^2 &= A\|f\|t_2^2 \leq \sum | \; \langle f, \psi_{\rho_0;j,k} \rangle \; |^2 = \langle Tf, f \rangle_\mu \\ &= \langle gT^{-1}g \rangle_\mu \leq \|g\|_{2,\mu} \|T^{-1}g\|_{2,\mu} \end{aligned}$$

by Schwartz equality.

Therefore,

$$\|T^{-1}g\|_{2,\mu} \leq \frac{1}{A}\|g\|_{2,\mu}.$$

Hence, every $f \in L_{2,\mu}$ can be constructed from its discrete Hermite wavelet transform given by 4.8. Thus

$$f = T^{-1}Tf = \sum_{j \in \mathbf{Z}, k \in \mathbf{N_0}} \langle f, \phi_{\rho_0;j,k} \rangle T^{-1}\phi_{\rho_0;j,k}. \tag{4.11}$$

Finally, set

$$\phi_{\rho_0}^{j,k} = T^{-1}\phi_{\rho_0;j,k}; j \in \mathbf{Z}, n \in \mathbf{N}_0.$$

Then the construction 4.11 can be expressed as

$$f = \sum_{j\in\mathbf{Z},k\in\mathbf{N}_0} \langle f, \phi_{\rho_0;j,k}\rangle \phi_{\rho_0}^{j,k},$$

which completes the proof of theorem 4.2. □

Frames and Riesz basis in $L_{2,\mu}$

In this section, using $\phi_{\rho_0;j,k}$ a frame is defined and Riesz basis of $L_{2,\mu}$ is studied.

Definition 4.2. *A function $f \in L_{2,\mu}$ is said to generate a frame $\{\phi_{\rho_0;j,k}\}$ of $L_{2,\mu}$ with sampling rate ρ_0 if condition 4.8 holds for some positive constant A and B. If $A = B$, then the frame is called a tight frame.*

Definition 4.3. *A function $f \in L_{2,\mu}$ is said to generate a Riesz basis of $\{\phi_{\rho_0;j,k}\}$ with sampling rate ρ_0 if the following two properties are satisfied.*

(i) *The linear span $\langle \phi_{\rho_0;j,k}; j \in \mathbf{Z}\rangle$ is dense in $L_{2,\mu}$.*

(ii) *There exist positive constants A and B with $0 < A \le B < \infty$ such that*

$$A\|c_{j,k}\|_{l^2}^2 \le \| \sum_{j,k\in\mathbf{Z}} c_{j,k}\phi_{\rho_0;j,k}\|_{2,\mu}^2 \le B\|c_{j,k}\|_{l^2}^2 \tag{4.12}$$

for all $\{c_{j,k}\} \in l^2(N^2)$. Here A and B are called the Riesz bounds of $\{\phi_{\rho_0;j,k}\}$.

Theorem 4.3. *Let $\phi \in L_{2,\mu}$, then the following statements are equivalent.*

(i) *$\{\phi_{\rho_0;j,k}\}$ is a Riesz basis of $L_{2,\mu}$.*

(ii) *$\{\phi_{\rho_0;j,k}\}$ is a frame of $L_{2,\mu}$ and is also an l^2 linearly independent family in the sense that if $\sum_{j,k} c_{j,k}\phi_{\rho_0;j,k} = 0$ and $\{c_{j,k}\} \in l^2$, then $c_{j,k} = 0$. Furthermore, the Riesz bounds and frame bounds agree.*

Proof. It follows from property 4.12 that any Riesz basis is l^2-linearly independent. Let $\{\phi_{\rho_0;j,k}\}$ be a Riesz basis with Riesz bounds A and B, and consider the matrix operator:

$$M = [\gamma_{l,m;j,k}]_{(l,m)(j,k)\in N\times N},$$

where the entries are defined by

$$\gamma_{l,m;j,k} = \langle \phi_{\rho_0;l,m}, \phi_{\rho_0;j,k}\rangle_\mu. \tag{4.13}$$

Then from property 4.12, we have

$$A\|\{c_{j,k}\}\|_{l^2}^2 \le \sum_{l,m,j,k} c_{l,m}\gamma_{l,m;j,k}c_{j,k} \le B\|\{c_{j,k}\}\|_{l^2}^2;$$

so that M is positive definite. We denote the inverse of M by

$$M^{-1}=[\mu_{l,m;j,k}]_{(l,m)(j,k)\in N\times N}, \tag{4.14}$$

which means that both

$$\sum_{r,s}\mu_{l,m;r,s}\gamma_{r,s;j,k}=\delta_{l,j}\delta_{m,k}; l,m,j,k\in N, \tag{4.15}$$

and

$$B^{-1}\|\{c_{j,k}\}\|_{l^2}^2\leq\sum_{l,m,j,k}c_{l,m}\mu_{l,m;j,k}\bar{c}_{j,k}\leq A^{-1}\|\{c_{j,k}\}\|_{l^2}^2, \tag{4.16}$$

are satisfied. This allows us to introduce

$$\phi^{l,m}(x)=\sum_{j,k}\mu_{l,m;j,k}\phi_{\rho_0;j,k}(x). \tag{4.17}$$

Clearly, $\phi^{l,m}\in L_{2,\mu}$ and it follows from equation 4.15 and 4.13 that

$$\langle\phi^{l,m},\phi_{\rho_0;j,k}\rangle_\mu=\delta_{l,j}\delta_{m,k;l,m,j,k}\in N,$$

which means that $\{\phi^{l,m}\}$ is the basis of $L_{2,\mu}$, which is dual to $\{\phi_{\rho_0;j,k}\}$.

Furthermore from equation 4.15 and 4.17, we conclude that

$$\langle\phi^{l,m},\phi^{j,k}\rangle_\mu=\mu_{l,m;j,k}$$

and the Reisz bounds of $\{\phi^{l,m}\}$ are B^{-1} and A^{-1}.

In particular, for any $f\in L_{2,\mu}$ we may write

$$f(x)=\sum_{j,k}\langle f,\phi_{\rho_0;j,k}\rangle_\mu\phi^{j,k}(x)$$

and

$$B^{-1}\sum_{j,k}\mid\langle f,\phi_{\rho_0;j,k}\rangle_\mu\mid^2\leq\|f\|_{2,\mu}^2\leq A^{-1}\sum_{j,k}\mid\langle f,\phi_{\rho_0;j,k}\rangle_\mu\mid^2. \tag{4.18}$$

Since condition 4.18 is equivalent to condition 4.8, therefore statement (i) implies statement (ii).

To prove the converse part, we recall Theorem 3.5 and we have any $g\in L_{2,\mu}$ and $f=T^{-1}g$,

$$g(x)=\sum_{m\in\mathbf{Z},n\in N}\langle f,\phi_{\rho_0;j,k}\rangle_\mu\phi_{\rho_0;j,k}.$$

Also by the l^2-linear independence of $\{\phi_{\rho_0;j,k}\}$, this representation is unique.

From the Banach-Steinhaus and open mapping theorem it follows that $\{\phi_{\rho_0;j,k}\}$, is a Riesz basis of $L_{2,\mu}$. □

Example 4.1. *Let the mother wavelet be*

$$\phi(t)=\begin{cases}1, & 0\le t\le \frac{1}{2}\\ -1, & -\frac{1}{2}\le t<1\\ 0, & otherwise\end{cases}. \tag{4.19}$$

This mother wavelet is called Haar wavelet. This is piecewise continuous. Using this wavelet we have following expression for $\phi(\sigma t)$.

$$\phi(\sigma t)=\begin{cases}1, & 0\le t\le \frac{1}{2\sigma}\\ -1, & \frac{1}{2\sigma}\le t<\frac{1}{\sigma}\\ 0, & otherwise\end{cases} \tag{4.20}$$

Let $f(t)=t^{-2\mu}e^{-2t}$. Then Hermite transform of $f(t)$ is given by

$$\begin{aligned}H\left[t^{-2\mu}e^{-2t}\right] &= \int_{-\infty}^{\infty} f(t)\tilde{H}_n^{(\mu)}(t)d\mu(t)\\ &= \int_{-\infty}^{\infty} f(t)e^{-t^2}\mid t\mid^{2\mu}\tilde{H}_n^{(\mu)}(t)dt\\ &= \int_{-\infty}^{\infty} t^{-2\mu}e^{-2t}e^{\frac{t^2}{2}}H_n^{(\mu)}(t)\mid t\mid^{2\mu}e^{-t^2}dt\\ &= e^{-\left(t^2+2t-\frac{t^2}{2}\right)}H_n^{(\mu)}(t)dt\\ &= \sqrt{\pi}(2\sigma)^n e^{\alpha^2}\end{aligned} \tag{4.21}$$

Now,

$$\begin{aligned}\int_{-\infty}^{\infty}\overline{\phi(\sigma z)}\tilde{H}_n^{(\mu)}(z)d\mu(z) &= \int_0^{\frac{1}{2\sigma}}\tilde{H}_n^{(\mu)}(z)d\mu(z)-\int_{\frac{1}{2\sigma}}^{\frac{1}{\sigma}}\tilde{H}_n^{(\mu)}(z)d\mu(z)\\ &= 2\int_0^{\frac{1}{\sigma}}\tilde{H}_n^{(\mu)}(z)d\mu(z)-\int_0^{\frac{1}{2\sigma}}\tilde{H}_n^{(\mu)}(z)d\mu(z)\\ &= 2\phi_1(n,\mu)-\phi_2(n,\mu),\end{aligned} \tag{4.22}$$

where $\phi_1(n,\mu)=\int_0^{\frac{1}{\sigma}}\tilde{H}_n^{(\mu)}(z)d\mu(z)$ and $\phi_2(n,\mu)=\int_0^{\frac{1}{2\sigma}}\tilde{H}_n^{(\mu)}(z)d\mu(z)$.

Using representation 3.5 and 2.1, we have

$$\begin{aligned}\left(\tilde{H}_\phi^{(\mu)}f\right)(\rho,\sigma) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\overline{\phi(\sigma z)}K_H^{(\mu)}(\rho,t,z)d\mu(z)d\mu(t)\\ &= \sum_{n=0}^{\infty}\left[\tilde{h}_n^{(\mu)}\right]^{-1}\tilde{H}_n^{(\mu)}(\rho)\left(\int_{-\infty}^{\infty} f(t)\tilde{H}_n^{(\mu)}(z)d\mu(z)\right)\left(\int_{-\infty}^{\infty}\overline{\phi(\sigma z)}\tilde{H}_n^{(\mu)}(z)d\mu(z)\right).\end{aligned}$$

From equations 4.21 and 4.22, it follows that

$$\left(\tilde{H}_\phi^{(\mu)}f\right)(\rho,\sigma)=\sum_{n=0}^{\infty}2^{2n}\Gamma\left(\left[\frac{n}{2}\right]+1\right)\Gamma\left(\left[\frac{n+1}{2}\right]+\mu+1\right)\tilde{H}_n^{(\mu)}(\rho)\sqrt{\pi}(2\sigma)^n e^{\alpha^2}(2\phi_1(n,\mu)-\phi_2(n,\mu)).$$

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