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INVARIANT SUMMABILITY AND UNCONDITIONALLY CAUCHY SERIES

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Abstract. In this study, we will give new characterizations of weakly unconditionally Cauchy series and unconditionally convergent series through summability obtained by the invariant convergence.

1. Introduction

Let σ be a mapping of the positive integers into itself. A continuous linear functional φ on m, the space of real bounded sequences, is said to be an invariant mean or a σ mean, if and only if,

(1) $\phi(x) \geq 0$, when the sequence $x = (x_j)$ is such that $x_j \geq 0$ for all j,
(2) $\phi(e) = 1$,where $e = (1, 1, 1....)$,
(3) $\phi(x_{\sigma(j)}) = \phi(x)$ for all $x \in m$.

The mappings ϕ are assumed to be one-to-one and such that $\sigma^i(j) \neq j$ for all positive integers j and i, where $\sigma^i(j)$ denotes the ith iterate of the mapping σ at j. Thus ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In case σ is translation mappings $\sigma(j) = j + 1$, the σ mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

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It can be shown that

$$V_\sigma = \{x = (x_j) : \lim_i t_{ij}(x) = \ell \; uniformly \; in \; j, \ell = \sigma - \lim x\}$$

where,

$$t_{ij}(x) = \frac{x_{\sigma(j)} + x_{\sigma^2(j)} + \cdots + x_{\sigma^i(j)}}{i+1}$$

Several authors including Raimi [19], Schaefer [20], Mursaleen and Edely [10], Mursaleen [12], Savaş [22,23], Nuray and Savaş [14], Pancaroğlu and Nuray [16,17] and some authors have studied invariant convergent sequences. The concept of strongly σ-convergence was defined by Mursaleen [11]. Savaş and Nuray [24] introduced the concepts of σ-statistical convergence and lacunary σ-statistical convergence and gave some inclusion relations.

Now, we recall the basic concepts and some definitions and notations (See [1,3–5,7–9,13,15,21]).

Let X be a normed space. For any given series $\sum_i x_i$ in X, let us consider the sets

$$S(\sum_i x_i) = \{(a_i) \in \ell_\infty : \sum_i a_i x_i \quad \text{convergent}\}$$

$$S_w(\sum_i x_i) = \{(a_i) \in \ell_\infty : \sum_i a_i x_i \quad \text{convergent for the weak topology}\}.$$

The above sets endowed with the sup norm and they will be called the space of convergence and the space of weak convergence associated to the series $\sum_i x_i$.

Definition 1.1. *A series $\sum_i x_i$ in a normed space X is said to be a weakly unconditionally Cauchy(wuc) if for each $\varepsilon > 0$ and $f \in X^*$, an $n_0 \in \mathbb{N}$ can be found such that for each finite subset $F \subset \mathbb{N}$ with $F \cap \{1, \ldots, n_0\} \neq \emptyset$ is $\sum_{i \in F} |f(x_i)| < \varepsilon$.*

As a consequence, $\sum_i x_i$ is a *wuc* series in X if and only if each functional $f \in X^*$ satisfies that $\sum_{i=1}^\infty |f(x_i)| < \infty$.

In [18] it is proved that a normed space X is complete if and only if for every weakly unconditionally Cauchy (wuc) series $\sum_i x_i$, the space $S(\sum_i x_i)$ is also complete.

Diestel [6] proved the following characterization that will be used throughout the paper.

Theorem 1.1. *Let $\sum_i x_i$ be a series in a normed space X. Then, the series $\sum_i x_i$ is wuc if and only if there exists $H > 0$ such that*

$$\begin{aligned} H &= \sup\{\|\sum_{i=1}^{n} a_i x_i\| : n \in \mathbb{N}, |a_i| \le 1, i \in \{1,\dots,n\}\} \\ &= \sup\{\|\sum_{i=1}^{n} \varepsilon_i x_i\| : n \in \mathbb{N}, \varepsilon_i \in \{-1,1\}, i \in \{1,\dots,n\}\} \\ &= \sup\{\sum_{i=1}^{n} |f(x_i)| : f \in B_{X^*}\} \end{aligned}$$

where B_{X^*} is denotes the closed unit ball in X^*

2. Main Results

Proposition 2.1. *Let X be a normed space and (x_n) an invariant convergent sequence in X. Then $(x_n) \in \ell_\infty(X)$.*

Proof. Let (x_n) be a sequence in X such that $\sigma - \lim_n x_n = x_0$ for some $x_0 \in X$. We can fix $\varepsilon > 0$ and $i_0 \in \mathbb{N}$ satisfying that

$$\left\| \frac{1}{i+1} \sum_{k=0}^{i} x_{\sigma^k(j)} \right\| \le \|x_0\| + \varepsilon$$

for every $i \ge i_0$ and $j \in \mathbb{N}$. Then we have that for every $j \in \mathbb{N}$ is

$$\|x_j\| = \|x_{\sigma^0(j)}\| = \left\| \frac{i_0+2}{i_0+1} \sum_{k=0}^{i_0+1} \frac{x_{\sigma^k(j)}}{i_0+2} - \sum_{k=1}^{i_0+1} \frac{x_{\sigma^k(j)}}{i_0+1} \right\| \le \left(\frac{i_0+2}{i_0+1} + 1 \right) (\|x_0\| + \varepsilon)$$

where the last term is a fixed constant, what concludes the proof. □

Definition 2.1. *A series $\sum_i x_i$ in X is said to be invariant convergent to $x_0 \in X$ if $\sigma - \lim_n s_n = x_0$, where $s_n = \sum_{i=1}^n x_i$ is sequence of partial sums, and we will denote it by $V_\sigma - \sum_i x_i = x_0$. Therefore, $V_\sigma - \sum_i x_i = x_0$ if and only if*

$$\lim_{i\to\infty} \left(\sum_{k=1}^{j} x_k + \frac{1}{i+1} \sum_{k=1}^{i} \left[(i-k+1) x_{\sigma^k(j)} \right] \right) = x_0$$

uniformly in $j \in \mathbb{N}$.

Definition 2.2. *x_0 is said to be weak invariant limit of a sequence (x_n) if each function $f \in X^*$ verifies that $\sigma - \lim f(x_n) = f(x_0)$ and we will write $w\sigma - \lim x_n = x_0$.*

Let X be a normed space and $\sum_i x_i$ a series in X. We define following sets:

$$S_\sigma(\sum_i x_i) = \{(a_i) \in \ell_\infty : V_\sigma - \sum_i a_i x_i \quad \textit{exists}\}$$

$$S_{w\sigma}(\sum_i x_i) = \{(a_i) \in \ell_\infty : wV_\sigma - \sum_i a_i x_i \quad \textit{exists}\}.$$

These spaces are the vector subspaces of ℓ_∞ and we consider them endowed with the sup norm.

Theorem 2.1. *Let X be a Banach space and $\sum_i x_i$ a series in X. Then $\sum_i x_i$ is wuc(weakly unconditionally Cauchy) if and only if $S_\sigma(\sum_i x_i)$ is complete.*

Proof. Let $\sum_i x_i$ be a wuc series. We will prove that $S_\sigma(\sum_i x_i)$ is closed in ℓ_∞. Let (a^n) be a sequence in $S_\sigma(\sum_i x_i)$, $a^n = (a_i^n)$ for each $n \in \mathbb{N}$ and let also be $a_0 \in \ell_\infty$ such that $\lim_n \|a^n - a^0\| = 0$. We will show that $a^0 \in S_\sigma(\sum_i x_i)$. Let $H > 0$ be such that

$$H \geq \sup\{\|\sum_{i=1}^{n} a_i x_i\| : n \in \mathbb{N}, |a_i| \leq 1, i \in \{1, \dots, n\}\}.$$

For each natural n there exists $y_n \in X$ such that $y_n = V_\sigma - \sum_i a_i^n x_i$. We will see that (y_n) is a Cauchy sequence.

If $\varepsilon > 0$ is given, there exists an n_0 such that if $p, q \geq n_0$,then $\|a^p - a^q\| < \frac{\varepsilon}{3H}$. If $p, q \geq n_0$ are fixed, there exists $i \in \mathbb{N}$ verifying

$$\left\| y_p - \left(\sum_{k=1}^{j} a_k^p x_k + \frac{1}{i+1} \sum_{k=1}^{i} \left[(i-k+1) a^p_{\sigma^k(j)} x_{\sigma^k(j)} \right] \right) \right\| < \frac{\varepsilon}{3} \tag{2.1}$$

$$\left\| y_q - \left(\sum_{k=1}^{j} a_k^q x_k + \frac{1}{i+1} \sum_{k=1}^{i} \left[(i-k+1) a^q_{\sigma^k(j)} x_{\sigma^k(j)} \right] \right) \right\| < \frac{\varepsilon}{3} \tag{2.2}$$

for each $j \in \mathbb{N}$. Then, if $p, q \geq n_0$ we have that

$$\|y_p - y_q\| \leq (2.1) + (2.2) + \left\| \sum_{k=1}^{j} (a_k^p - a_k^q) x_k + \sum_{k=1}^{i} \left[\frac{i-k+1}{i+1} (a^p_{\sigma^k(j)} - a^q_{\sigma^k(j)}) x_{\sigma^k(j)} \right] \right\|, \tag{2.3}$$

where $(2.3) \leq \frac{\varepsilon}{3}$. Therefore, since X is Banach space, there exists $y_0 \in X$ such that $\lim_n \|y_n - y_0\| = 0$. We will check that $\sigma \sum_i a_i^0 x_i = y_0$, that is,

$$\lim_{i\to\infty} \left(\sum_{k=1}^{j} a_k^0 x_k + \frac{1}{i+1} \sum_{k=1}^{i} \left[(i-k+1) a^0_{\sigma^k(j)} x_{\sigma^k(j)} \right] \right) = y_0,$$

uniformly in $j \in \mathbb{N}$.

If $\varepsilon > 0$ is given, we can fix a natural n such that $\|a^n - a^0\| < \frac{\varepsilon}{3H}$ and $\|y_n - y_0\| < \frac{\varepsilon}{3}$. Now, we can also fix i_0 such that for every $i \geq i_0$ is

$$\left\| y_n - \left(\sum_{k=1}^{j} a_k^n x_k + \frac{1}{i+1} \sum_{k=1}^{i} \left[(i-k+1) a^n_{\sigma^k(j)} x_{\sigma^k(j)} \right] \right) \right\| < \frac{\varepsilon}{3}$$

for every $j \in \mathbb{N}$. Then, if $i \geq i_0$ it is satisfied that

$$\begin{aligned}
&\left\| y_0 - \left(\sum_{k=1}^{j} a_k^0 x_k + \frac{1}{i+1} \sum_{k=1}^{i} \left[(i-k+1) a^0_{\sigma^k(j)} x_{\sigma^k(j)} \right] \right) \right\| \leq \|y_0 - y_n\| \\
&+ \left\| y_n - \left(\sum_{k=1}^{j} a_k^n x_k + \frac{1}{i+1} \sum_{k=1}^{i} \left[(i-k+1) a^n_{\sigma^k(j)} x_{\sigma^k(j)} \right] \right) \right\| \\
&+ \left\| \sum_{k=1}^{j} (a^n - a^0) x_k + \frac{1}{i+1} \sum_{k=1}^{i} \left[(i-k+1)(a^n_{\sigma^k(j)} - a^0_{\sigma^k(j)}) x_{\sigma^k(j)} \right] \right\| \leq \frac{2\varepsilon}{3} \\
&+ \|a^n - a^0\| \left(\sum_{k=1}^{\sigma(j)} \frac{(a_k^n - a_k^0)}{\|a^n - a^0\|} x_k + \sum_{k=1}^{i} \left[\frac{(i-k+1)}{i+1} \frac{(a^n_{\sigma^k(j)} - a^0_{\sigma^k(j)})}{\|a^n - a^0\|} x_{\sigma^k(j)} \right] \right) \\
&\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3H} H \leq \varepsilon
\end{aligned}$$

for every $j \in \mathbb{N}$. Thus $(a_n^0) \in S_\sigma(\sum_i x_i)$.

Conversely, if $S_\sigma(\sum_i x_i)$ is closed, since $c_{00} \subset S_\sigma(\sum_i x_i)$, we deduce that $c_0 \subset S_\sigma(\sum_i x_i)$. Suppose that $\sum_i x_i$ is not *wuc* series. Then there exists $f \in X^*$ verifying $\sum_{i=1}^{\infty} |f(x_i)| = +\infty$.

We can choose a natural n_1 such that $\sum_{i=1}^{n_1} |f(x_i)| > 2.2$ and for $i \in \{1, \ldots, n_1\}$ we define $a_i = \frac{1}{2}$ if $f(x_i) \geq 0$ or $a_i = \frac{-1}{2}$ if $f(x_i) < 0$.

There exists $n_2 > n_1$ such that $\sum_{i=n_1+1}^{n_2} |f(x_i)| > 3.3$ and for $i \in \{n_1+1, \ldots, n_2\}$ we define $a_i = \frac{1}{3}$ if $f(x_i) \geq 0$ or $a_i = \frac{-1}{3}$ if $f(x_i) < 0$.

In this manner we obtain an increasing sequence $(n_k)_k$ in $\mathbb{N}$ and a sequence $a = (a_i)_i$ in c_0 such that $\sum_{i=1}^{\infty} a_i f(x_i) = +\infty$. Since $(a_i)_i \in S_\sigma(\sum_i x_i)$, it follows that $\sigma \sum_i a_i x_i$ exists and therefore $\left(\sum_{i=1}^{n} a_i f(x_i) \right)_n$ is bounded sequence, which is a contradiction. □

Then we have the following result.

Corollary 2.1. *Let X be a Banach space and $\sum_i x_i$ a series in X. Then $\sum_i x_i$ is a wuc(weakly unconditionally Cauchy) series if and only if for each sequence $(a_i)_i \in c_0$ it is satisfied that $V_\sigma - \sum_i a_i x_i$ exists.*

Proof. Let $\sum_i x_i$ be a wuc series in X. Then, we have that $S_\sigma(\sum_i x_i)$ is complete. Since $c_{00} \subset S_\sigma(\sum_i x_i)$, we deduce that $c_0 \subset S_\sigma(\sum_i x_i)$, that is, $V_\sigma - \sum_i a_i x_i$ exists for every sequence $(a_i) \in c_0$. The converse is proved similar to the end of the previous demonstration. □

Remark 2.1. *Let X be a normed space and $\sum_i x_i$ a series in X. We consider the linear map $T : S_\sigma(\sum_i x_i) \to X$ defined by $T(a) = V_\sigma - \sum_i a_i x_i$.*

Suppose that $\sum_i x_i$ *is a wuc series and consider* $H = sup\{\|\sum_{i=1}^n a_i x_i\| : n \in \mathbb{N}, |a_i| \leq 1, i \in \{1, \dots, n\}\}$. *Then, it is easy to check that if* $a \in S_\sigma(\sum_i x_i)$ *then* $\|T(a)\| = \|V_\sigma - \sum_i a_i x_i\| \leq H\|a\|$ *and therefore* T *is continuous.*

Conversely if T *is continuous and* $\{a_1, \dots, a_j\} \subset [-1, 1]$, *it is satisfied that* $\|\sum_{i=1}^j a_i x_i\| = \|V_\sigma - \sum_{i=1}^\infty a_i x_i\| \leq \|T\|$ *(considering* $a_i = 0$ *if* $i > j$*), which implies that* $\sum_i x_i$ *is a wuc series.*

In the next theorem we study the completeness of space $S_{w\sigma}(\sum_i x_i)$.

Theorem 2.2. *Let* X *be a Banach space and* $\sum_i x_i$ *a series in* X. *Then* $\sum_i x_i$ *is a wuc series if and only if* $S_{w\sigma}(\sum_i x_i)$ *is complete.*

Proof. Consider $\sum_i x_i$ to be a wuc series. It will be enough to prove that $S_{w\sigma}(\sum_i x_i)$ is closed in ℓ_∞. Let (a^n) be sequence in $S_{w\sigma}(\sum_i x_i)$, $a^n = (a_i^n)_i$ for each $n \in \mathbb{N}$ and let also be $a^0 \in \ell_\infty$ such that $\lim_n \|a^n - a^0\| = 0$. We will show that $a^0 \in S_{w\sigma}(\sum_i x_i)$. Let $H > 0$ be such that

$$H \geq \sup\{\|\sum_{i=1}^n a_i x_i\| : n \in \mathbb{N}, |a_i| \leq 1, i \in \{1, \dots, n\}\}$$

For each natural n there exists $y_n \in X$ such that $y_n = wV_\sigma - \sum_i a_i^n x_i$. We will check that $(y_n)_n$ is Cauchy sequence.

If $\varepsilon > 0$ is given, there exists an n_0 such that if $p, q \geq n_0$,then $\|a^p - a^q\| < \frac{\varepsilon}{3H}$. We fix $p, q \geq n_0$ and we have that there exists $f \in S_{X^*}$ (unit sphere in X^*)verifying $\|y_p - y_q\| = |f(y_p - y_q)|$. Since $V_\sigma - \sum_i a_i^p f(x_i) = f(y_p)$ and $V_\sigma - \sum_i a_i^q f(x_i) = f(y_q)$, there exists $i \in \mathbb{N}$ such that

$$\left| f(y_p) - \left(\sum_{k=1}^j a_k^p f(x_k) + \frac{1}{i+1} \sum_{k=1}^i \left[(i-k+1) a_{\sigma^k(j)}^p f(x_{\sigma^k(j)}) \right] \right) \right| < \frac{\varepsilon}{3} \tag{2.4}$$

$$\left| f(y_q) - \left(\sum_{k=1}^j a_k^q f(x_k) + \frac{1}{i+1} \sum_{k=1}^n \left[(i-k+1) a_{\sigma^k(j)}^q f(x_{\sigma^k(j)}) \right] \right) \right| < \frac{\varepsilon}{3} \tag{2.5}$$

for each $j \in \mathbb{N}$. Then, if $p, q \geq n_0$ we have that

$$\|y_p - y_q\| = |f(y_p) - f(y_q)| \leq (2.4) + (2.5) \tag{2.6}$$

$$+ \left| \sum_{k=1}^j (a_k^p - a_k^q) f(x_k) + \sum_{k=1}^i \left[\frac{i-k+1}{i+1} (a_{\sigma^k(j)}^p - a_{\sigma^k(j)}^q) f(x_{\sigma^k(j)}) \right] \right|, \tag{2.7}$$

where (2.6) $\leq \frac{\varepsilon}{3}$. Therefore, since X is Banach space, there exists $y_0 \in X$ such that $\lim_n \|y_n - y_0\| = 0$. We will check that $wV_\sigma - \sum_i a_i^0 x_i = y_0$.

If $\varepsilon > 0$ is given, we can fix a natural n such that $\|a^n - a^0\| < \frac{\varepsilon}{3H}$ and $\|y_n - y_0\| < \frac{\varepsilon}{3}$. Consider a functional $f \in B_{X^*}$. We have that there exists $i_0 \in \mathbb{N}$ such that if $i \geq i_0$ is

$$\left|f(y_n) - \left(\sum_{k=1}^{j} a_k^n f(x_k) + \frac{1}{i+1}\sum_{k=1}^{i}\left[(i-k+1)a_{\sigma^k(j)}^n f(x_{\sigma^k(j)})\right]\right)\right| < \frac{\varepsilon}{3}$$

for every $j \in \mathbb{N}$. Then, if $i \geq i_0$ and $j \in \mathbb{N}$, we have that

$$\begin{aligned}\left|f(y_0) - \left(\sum_{k=1}^{j} a_k^0 f(x_k) + \frac{1}{i+1}\sum_{k=1}^{i}\left[(i-k+1)a_{\sigma^k(j)}^0 f(x_{\sigma^k(j)})\right]\right)\right| &\leq |f(y_0 - y_n)| \\ + \left|f(y_n) - \left(\sum_{k=1}^{j} a_k^n f(x_k) + \frac{1}{i+1}\sum_{k=1}^{i}\left[(i-k+1)a_{\sigma^k(j)}^n f(x_{\sigma^k(j)})\right]\right)\right| \\ + \left|\sum_{k=1}^{j}(a^n - a^0)f(x_k) + \frac{1}{i+1}\sum_{k=1}^{i}\left[(i-k+1)(a_{\sigma^k(j)}^n - a_{\sigma^k(j)}^0)f(x_{\sigma^k(j)})\right]\right| &\leq \varepsilon\end{aligned}$$

that is, $wV_\sigma - \sum_i a_i^0 x_i = y_0$ and $a^0 \in S_{w\sigma}(\sum_i x_i)$.

Conversely, if $S_{w\sigma}(\sum_i x_i)$ is complete, which implies that $c_0 \subset S_{w\sigma}(\sum_i x_i)$. Suppose that there exists $f \in X^*$ verifying $\sum_{i=1}^{\infty} |f(x_i)| = +\infty$.

Then, as we did in Theorem 2.1, a sequence $a = (a_i)$ in c_0 can be obtained such that $\sum_i a_i f(x_i) = +\infty$ since $a \in S_{w\sigma}(\sum_i x_i)$, there will exists $x_0 \in X$ such that $wV_\sigma - \sum_i a_i x_i = x_0$ and it will be $V_\sigma - \sum_i a_i f(x_i) = x_0$. But this implies that the sequence $\left(\sum_{i=1}^{n} a_i f(x_i)\right)_n$ is bounded which is a contradiction. □

Remark 2.2. *Let X be a Banach space $\sum_i x_i$ a series in X. We consider the linear map $T{:}S_{w\sigma}(\sum_i x_i) \to X$ defined by $T(a) = wV_\sigma - \sum_i a_i x_i$. We will show that $\sum_i x_i$ is wuc series if and only if T is continuous.*

We define $H = \sup\{\|\sum_{i=1}^{n} a_i x_i\| : n \in \mathbb{N}, |a_i| \leq 1, i \in \{1, \dots, n\}\}$ and take $a \in S_{w\sigma}(\sum_i x_i)$. Then $wV_\sigma - \sum_i a_i x_i = x_0$ exists and we can take $f \in S_{X^*}$ such that $|T(a)| = |f(T(a))| = |V_\sigma - \sum_i a_i f(x_i)| \leq H\|a\|$.

Conversely if T is continuous.Then if $\{a_1, \dots, a_n\} \subset [-1, 1]$, we have that $\|\sum_{i=1}^{n} a_i x_i\| = \|wV_\sigma - \sum_{i=1}^{\infty} a_i x_i\| \leq \|T\|$ (considering $a_i = 0$ if $i > n$), which implies that $\sum_i x_i$ is a wuc series.

From the previous theorem and its proof the following corallary can be easily proved.

Corollary 2.2. *Let X be a Banach space $\sum_i x_i$ a series in X. Then the following are equivalent:*

(1) *$\sum_i x_i$ is a wuc series.*
(2) *$S_{w\sigma}(\sum_i x_i)$ is complete.*
(3) *$c_0 \subset S_{w\sigma}(\sum_i x_i)$ ($wV_\sigma - \sum_i a_i x_i$ exists for each $a = (a_i) \in c_0$).*

Let us see that the hypothesis of completeness in the two previous theorems is completely necessary.

Let X be a non-complete normed space. Then it is easy to prove that there exists a sequence $\sum_i x_i$ in X such that $\|x_i\| < \dfrac{1}{i2^i}$ and $\sum_i x_i = x^{**} \in X^{**} \backslash X$. Then we have that $V_\sigma - \sum_i x_i = x^{**}$. If we consider the series $\sum_i z_i$ defined by $z_i = ix_i$ for each $n \in \mathbb{N}$, we have that $\sum_i z_i$ is wuc series. Consider the sequence $a = (a_i) \in c_0$ given by $a_i = \dfrac{1}{i}$. It is satisfied that $V_\sigma - \sum_i a_i z_i \in X^{**} \backslash X$ and therefore $a \notin S_\sigma(\sum_i z_i)$ and $a \notin S_{w\sigma}(\sum_i z_i)$.

Let X be a normed space and X^* its dual space. Let also $\sum_i f_i$ be a series in X^*. It is known that [6], $\sum_i f_i$ is wuc if and only if $\sum_i |f_i(x)| < \infty$ for each $x \in X$.

Now we consider the vector space

$$S_{*w\sigma}(\sum_k f_i) = \{a = (a_i) \in \ell_\infty : *wV_\sigma - \sum_i a_i f_i \ exists\}$$

, where $*wV_\sigma - \sum_i a_i f_i = f_0$ if and only if $V_\sigma - \sum_i a_i f_i(x) = f_0(x)$ for each $x \in X$.

Theorem 2.3. *Let X be a normed space. It is satisfied that $1 \Rightarrow 2 \Rightarrow 3$, where:*

(1) *$\sum_i f(i)$ is a wuc series.*

(2) *$S_{*w\sigma}(\sum_i f_i) = \ell_\infty$.*

(3) *If $x \in X$ and $M \subset \mathbb{N}$, then $V_\sigma - \sum_{i \in M} f_i(x)$ exists.*

Besides, if X is a barelled normed space, the three items are equivalent.

Proof. From the $*$ weak compacity of B_{X^*} we deduce that $1 \Rightarrow 2$. It is clear that $2 \Rightarrow 3$.

We suppose now that X is barelled and we will prove that $3 \Rightarrow 1$. Effectively, our goal is to prove that $E = \{\sum_{i=1}^n a_i f_i : n \in \mathbb{N}, |a_i| \leq 1, i \in \{1, \ldots, n\}\}$ is pointwise bounded for each $x \in X$ and therefore E is bounded, which implies that $\sum_i f_i$ is wuc series. Suppose that E is not pointwise bounded, that is, there exists $x_0 \in X$ such that $\sum_i |f_i(x_0)| = +\infty$. Then, we can choose a subset $M \subset \mathbb{N}$ such that $\sum_{i \in M} f_i(x_0) = + - \infty$. But, by hypothesis, $V_\sigma - \sum_{i \in M} f_i(x_0)$ exists, which is a contradiction. □

When $\sigma(j) = j + 1$, we have the almost all definitions and theorems in [2] concerning almost summability.

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