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A COMMUTATIVE AND COMPACT DERIVATIONS FOR W* ALGEBRAS

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ABSTRACT. In this paper, we study the compact derivations on W* algebras. Let M be W*-algebra, let $\mathrm{LS}(\mathrm{M})$ be algebra of all measurable operators with M, it is show that the results in the maximum set of orthogonal predictions. We have found that W* algebra A contains the Center of a W* algebra ß and is either a commutative operation or properly infinite. We have considered derivations from W* algebra two-sided ideals.

1. INTRODUCTION

Let M be a W*-algebra and let $Z(M)$ be the center of M. Fix $a \in M$ and consider the inner derivation δ_a on M generated by the component a, which is $\delta_a(\cdot) := [a, \cdot]$.

The norm closing two sided ideal $f(B)$ generated by the finite projections of a W* algebra B behaves somewhat similar to the idealized compact operators of $B(H)$ (see [11],[8],[9]). Therefore, it is natural to ask about any sub-algebras d of B that is any derivation from A into $f(B)$ implemented from an element of $y(B)$.

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We perform two main difficulties: the presence of the center of B and the fact that the main characteristic in [8] proof (that is, if Q_n, is a sequence of mutually orthogonal projections and $T \in B(H)$ hence $\|Q_n T Q_n\| > \alpha > 0$ for all n implies that T is not compact) failure to generalize to the case in which g is of Type II_∞.

Finally, we have considered derivations from d at the two-sided $C_{1+\varepsilon}(B,\tau) = B \cap L^{1+\varepsilon}(B, \tau)(1 \le 1+\varepsilon < \infty)$ to obtain faithful finite normal trace τ on B.

2. Notations Preliminary

Lemma (1). Let B be a semi-finite algebra, let $Q_0 \in p(B)$ and $x_0 \in Q_0$ be such that ω_{x_0}, is a faithful trace on B_{Q_0}. Assume there are $Q_n \in p(B)$, $F_n \in p(\ell)$ and $U_n \in B$ for $n = n_i, n_{i+1}, \ldots$, such that the projections Q_n are mutually orthogonal and $Q_n = U_n U_n^*$, $Q_o F_n = U_n^* U_n$ for all n (i.e., $Q_n \sim Q_0 F_n$). Let $x_n = U_n F_n x_0$. Then $x_n \to_{JRW} O$.

Proof. Assume that $\sum_{n=n_i}^{\infty} Q_n = n_i$. Let τ be a faithful semi-finite normal (fsn) trace on B^+ to be agreed on B_{Q_0} with ω_{x_0}. Then for all $B \in B_{Q_n}^+$ we have

$$\begin{aligned}\tau(B) &= \tau\left(U_n U_n^* B U_n U_n^*\right)\\ &= \tau\left(U_n^* U_n U_n^* B U_n\right)\\ &= \tau\left(Q_o F_n U_n^* B U_n F_n Q_o\right)\\ &= \omega_{x_0}\left(F_n U_n^* B U_n F_n\right)\\ &= \omega_{x_n}(B).\end{aligned}$$

Let $P \in p(B)$ be any semi-finite projection. Then by [11] there is a central decomposition of the identity $\sum_{\gamma\in\Gamma} E_\gamma = 1, E_\gamma \in p(\ell), E_\gamma E_{\gamma'} = 0$ for $\gamma \neq \gamma'$ such that $\tau(PE_\gamma) < \infty$ for all $\gamma \in \Gamma$. Then

$$\begin{aligned}\tau(PE_\gamma) &= \sum_{n=1}^{\infty} \tau\left(Q_n P E_\gamma Q_n\right)\\ &= \sum_{n=1}^{\infty} \omega_{x_n}\left(Q_n P E_\gamma Q_n\right)\end{aligned}$$

$$\sum_{n=1}^{\infty} \left\|P E_\gamma x_n\right\|^2 < \infty$$

whence $\|PE_\gamma x_n\| < 0$ for all $\gamma \in \Gamma$. Let $\varepsilon > 0$ and let $\Lambda \subset \Gamma$ be a finite index set such that $\sum_{\gamma \notin \Lambda} \|E_\gamma x_0\|^2 < \varepsilon$. Then for all n,

$$\begin{aligned} \sum_{\gamma \notin \Lambda} \|PE_\gamma x_n\|^2 &= \sum_{\gamma \notin \Lambda} \|PE_\gamma U_n F_n x_0\|^2 \\ &= \sum_{\gamma \notin \Lambda} \|PU_n F_n E_\gamma x_0\|^2 \\ &\leq \sum_{\gamma \notin \Lambda} \|E_\gamma x_0\|^2 < \varepsilon \end{aligned}$$

Hence from $\|Px_n\|^2 \leq \sum_{\gamma \in \Lambda} \|PE_\gamma x_n\|^2 + \varepsilon$ where $\|Px_n\| \to 0$, to completes the proof.

Lemma (2). Let $T \notin f\ (P)$, then there is an $\alpha > 0$ and $0 \neq E \in p(\ell)$ such that for every $0 \neq F \in p(\ell)$ with $F \leq E$ we have $\|\pi(TF)\| > \alpha$.

Proof. Let $\alpha = \frac{1}{2}\|\pi(T)\| \neq 0$ and let G be the sum of a maximal family of mutually orthogonal central projections G_γ such that $\|\pi(TG_\gamma)\| \leq \alpha$. Then

$\|\pi(TG)\| = \sup_\gamma \|\pi(TG_\gamma)\| \leq \alpha$, hence $G \neq 1$. Let $E = Z - G$ and let $0 \neq F \in P(\ell)$ with $F \leq E$. Since $FG = 0$, by the maximally of the family we have $\|\pi(TF)\| > \alpha$.

3. Relatively Compact Derivation

Let M be a W^*-algebra and let $Z(M)$ be the center of M. Fix $a \in M$ and consider the inner derivation δ_a on M generated by the element a, that is $\delta_a(\cdot) := [a, \cdot]$. Obviously, δ_a there is a linear bounded operator on $(M, \|\cdot\|_M)$, where $\|\cdot\|_M$ is a C^* -norm on M. It is known that there exists $c \in Z(M)$ such that the following estimate holds: $\|\delta_a\| \geq \|a - c\|_M$. In view of this result, it is natural to ask whether there exists is an element $y \in M$ with $\|y\| \leq 1$ and $c \in Z(M)$ such that $\|[a, y]\| \geq |a - c|$.

Definition (3). A linear subspace I in the W* algebra M equipped with a norm $\|\cdot\|_I$ is said to be a symmetric operator ideal if

(i) $\| S \|_I \geq \| S \| \, for \ all \ S \in I$,

(ii) $\| S^* \|_I = \|S\|_I \ for \ all \ S \in I$,

(iii) $\|ASB\|_I \leq \|A\| \, \|S\|_I \, \|B\| \ for \ all \ S \in I, \ A, B \in M$.

Observe, that every symmetric operator ideal I is a two-sided ideal in M, and therefore by [13], it follows from $0 \leq S \leq T$ and $T \in I$ that $S \in I$ and $\|S\|_I \leq \|T\|_I$.

Corollary (4). Let M be a W^*-algebra and let I be an ideal in M. Let $\delta : M \to I$ be a derivation. Then there exists an element $a \in I$, such that $\delta = \delta_a = [a, \cdot]$.

Proof. Since δ is a derivation on a W^*-algebra, it is necessarily inner [8]. Thus, there exists an element $d \in M$, such that $\delta(\cdot) = \delta d(\cdot) = [d, \cdot]$. It follows from the hypothesis that $[d, M] \subseteq I$.

Using [22] (or [20]), we obtain $[d^*, M] = -[d, M]^* \subseteq I^* = I$ and $[d_k, M] \subseteq I, k = 1, 2$, where $d = d_1 + id_2, \ dk = d_k^* \in M$, for $k = 1, 2$. It follows now, that there exist $c_1, \ c_2 \in Z(M)$ and $u_1, u_2 \in U(M)$, such that $\|[d_k, u_k]\| \geq 1/2 |d_k - c_k|$ for $k = 1, 2$. Again applying [20], we obtain $d_k - c_k \in I$, for $k = 1, 2$. Setting $a := (d_1 - c_1) + i(d_2 - c_2)$, we deduce that $a \in I$ and $\delta = [a, \cdot]$.

Corollary (5). Let M be a semi-finite W* -algebra and let E be a symmetric operator space. Fix $a = a^* \in S(M)$ and consider inner derivation $\delta = \delta_a$ on the algebra $LS(M)$ given by $\delta(x) = [a, x], \ x \in LS(M)$. If $\delta(M) \subseteq E$, then there exists $d \in E$ satisfying the inequality $\|d\|_E \leq \|\delta\|_{M \to E}$ and such that $\delta(x) = [d, x]$.

Proof. The existence of $d \in E$ such that $\delta(x) = [d, x]$. Now, if $u \in U(M)$, then $\|\delta(u)\|_E = \|du - ud\|_E \leq \|du\|_E + \|ud\|_E = 2\|d\|_E$. Hence, if $x \in M_1 = \{x \in M : \|x\| \leq 1\}$, then $x = \sum_{i=1}^4 \alpha_i u_i$, where $u_i \in U(M)$ and $|\alpha_i| \leq 1$ for $i = 1, 2, 3, 4$, and so $\|\delta(x)\|_E \leq \sum_{i=1}^4 \|\delta(\alpha_i u_i)\|_E \leq 8\|d\|_E$, that is $\|\delta\|_{M \to E} \leq 8\|d\|_E < \infty$.

4. A COMMUTATIVE OPERATION ON W* SUB-ALGEBRAS

When A a commutative operation is is crucial because it provides the following explicit way to find an operator $T \in B$ implementing the derivation.

For the rest of this section let A be any a commutative operation sub-algebras of B and $\delta : \mathrm{A} \to B$ be any derivation. Let u be the unitary group of A and M be a given invariant mean on u, i.e., a linear functional on the algebra of bounded complex-valued functions on u such that

(i) For all real f, $inf\left\{ f\left(U\right) \middle| U \in u \right\} \le Mf \le sup\left\{ f\left(U\right) \middle| U \in u \right\}$

(ii) For all $U \in u,\ Mf_U = MS,\ where\ f_U\left(V\right) = f\left(UV\right)\ for\ V \in u$.

Thus M is bounded and $\left|Mf\right| \le sup\left\{ \left|\mathrm{f}\left(U\right)\right| \ \middle| \ \mathrm{U} \in \mathrm{u} \right\}$ for all f (see [8] for the existence and properties of M).

For each $\phi \in B_*$ the map

$$\phi \to M\phi\left(U^*\delta\left(U\right)\right)$$

is linear and bounded and hence defines an element $T \in \left(B_*\right)^*$. Explicitly,

$$\phi\left(T\right) \to M\phi\left(U^*\delta\left(U\right)\right) \quad for\ all\ \phi \in B_*$$

The same easy computation as in [8] shows that $\delta = a\mathrm{A}T$. Notice that for all $A \in B$ the map

$$\phi \to M\phi\left(U^*BU\right) = \phi\left(E\left(B\right)\right)$$

defines an element $E\left(B\right)$ which clearly belongs to $\mathrm{A} \cap B$. Moreover it is easy to see that E is a conditional expectation (i.e., a projection of norm one) from B onto $\mathrm{A} \cap B$ (see [6]).

Theorem (6). Let A be a commutative operation W* sub-algebras of B containing the center ℓ of B. For every derivation $\delta : \mathrm{A} \to f\ \left(B\right)$ there is a $T\ \in f\ \left(B\right)$ such that $\delta = a\mathrm{A}\ T$.

We have seen that given an invariant mean M on u there is a unique $T \in B$ such that $\delta = a\mathrm{A}\ T$ and $E\left(T\right) = 0$. We are going to show that $T\ \in \mathrm{A}\left(B\right)$. Reasoning by contradiction assume that $T\ \notin \mathrm{A}\left(B\right)$. We proof requires several reductions to the restricted derivation

$\delta_E : A_E \to f\ (B)$ for some $0 \neq E \in p(\ell)$. To simplify notations we shall assume each time that $E = 1$.

Let us start by noticing that if $Q_i \in p(A)$ for $i = n, n+1,\ Q_n, Q_{n+1} = 0$ and $P = Q_n + Q_{n+1}$, then

$$PTP = \sum_{i=n}^{n+1} Q_i T Q_i + \delta(Q_{n+1}) Q_n + \delta(Q_n) Q_{n+1}$$

hence

$$\|\pi(PTP)\| = \left\| \sum_{i=n}^{n+1} \pi(Q_i T Q_i) \right\| + \max_i\ \pi(Q_i T Q_i)$$

Definition (7). For every $Q \in p(A)$ define $[Q] = [Q,\ \varepsilon]$ to be the central projection. Set

$$P = \{P \in p(A) \,|\, [P] = 1\}.$$

Thus $P \in p$ iff $\|\pi(PTPG)\| = \|\pi(TG)\|$ for all $G \in p(\ell)$. We collect several properties of $[Q]$.

Corollary (8). Let B be a semi-finite W* algebra with a trace τ, let A be a properly infinite W* sub-algebras of B and let $1 \le 1+\varepsilon < \infty$. Then for every derivation $\delta : A \to C_{1+\varepsilon}(B, \tau)$ there is $a\,T \in C_{1+\varepsilon}(B, \tau)$ such that $\delta = a A T$.

In the notations introduced there, it is easy to see that $\phi\left(C_{1+\varepsilon}(B,\tau)\right) = C_{1+\varepsilon}\left(\tilde{B}, \tilde{\tau}\right)$), where $\tau = \tau \oplus \tau_0$ and τ_0 is the usual trace on $B(H_0)$. We can actually simplify the proof by choosing $\tilde{A}_n = I \otimes \ell$ since the condition $\ell \subset A$ is no longer required.

Corollary (9). Let $P = Q_n + Q_{n+1}$. Then there is a largest central projection $[Q_n, Q_{n+1}]$ such that for every $G \in p(\ell)$ with $G \le [Q_n, Q_{n+1}]$, we have $\|\pi(Q_1 T Q_1 G)\| = \|\pi(PTPG)\|$.

Proof. Let $G_i = \{G \in p(\ell) \| \|\pi(Q_i T Q_i G)\| = \|\pi(PTPG)\|\}$ and $\Xi = \{G + \varepsilon \in p(\ell)|$ if $G \in p(\ell)$ and $\varepsilon \ge 0$ then $G \in G_n\}$. Since $\|\pi(PTPG)\| = \max_i \|\pi(Q_i T Q_i G)\|$ for all $G \in p(\ell)$, we see that $G_n \cup G_{n+1} = p(\ell)$. Notice that Ξ is hereditary (i.e., $G - \varepsilon \in \Xi$ and $F \in p(\ell)$, $F \le G + \varepsilon$ imply $F \in \Xi$).

Let $[Q_n, Q_{n+1}] = \sup \Xi$. We have only to show that $[Q_n, Q_{n+1}] \in \Xi$. Let $G + \varepsilon = \sum_\gamma (G+\varepsilon)_\gamma$ be the sum of a maximal collection of mutually orthogonal projections $(G+\varepsilon)_\gamma \in \Xi$. Then for every $F \in \Xi$ we have $([Q_n, Q_{n+1}] - (G+\varepsilon))F = 0$ because of the maximal of the collection of Ξ. Then $[Q_n, Q_{n+1}] = G + \varepsilon$. Consider now any $G \in p(\ell)$, $\varepsilon \geq 0$, then $G = \sum_\gamma G\,(G+\varepsilon)_\gamma$ and since $G\,(G+\varepsilon)_\gamma \leq (G+\varepsilon)_\gamma \in \Xi$, we have $\left\| \pi\left(Q_n T Q_n G\,(G+\varepsilon)_\gamma\right)\right\| = \left\| \pi\left(PTPG\,(G+\varepsilon)_\gamma\right)\right\|$ for all γ. Since $\pi(Q_n T Q_n G)(resp.\ \pi(\ PTPG\,))$ is the direct sum of then

$\pi\left(Q_n T Q_n G\,(G+\varepsilon)_\gamma\right)\left(resp.\ \pi\left(\ PTPG\,(G+\varepsilon)_\gamma\right)\right)$, then we have

$$\begin{aligned} \left\|\pi(Q_n T Q_n G\,)\right\| &= \sup_\gamma \left\| \pi\left(Q_n T Q_n G\,(G+\varepsilon)_\gamma\right)\right\| \\ &= \sup_\gamma \left\| \pi\left(PTPG\,(G+\varepsilon)_\gamma\right)\right\| \\ &= \left\|\pi(PTPG\,)\right\| \end{aligned}$$

whence $G \in G_n$. Since $\varepsilon \geq 0$ is arbitrary, we have $G + \varepsilon = [Q_n, Q_{n+1}] \in \Xi$ which completes the proof.

Corollary (10). (i) If $Q_n Q_{n+1} = 0$ with $Q_i \in p(\mathrm{A})$ then $1 - [Q_n, Q_{n+1}] \leq [Q_n, Q_{n+1}]$.

(ii) If $Q_n \leq Q_{n+1}$ with $Q_i \in p(\mathrm{A})$ then $[Q_n] \leq [Q_{n+1}]$.

(iii) If $\varepsilon \geq 0$ with $Q \in p(\mathrm{A}), Q + \varepsilon \in p$ then $[Q] = [Q,\ \varepsilon]$ and $1 - [Q] \leq [\varepsilon]$

If $\pi(TG) \neq 0$ for all $0 \neq E \in p(\ell)$ then the following hold:

(iv) If $E \in p(\ell)$ then $E = [E]$.

(v) If $Q \in p(\mathrm{A})$ then $[Q] \leq c(Q)$, where $c(Q)$ is the central support of Q.

Proof. We have to show that for every $G \in p(\ell)$, $G \leq 1 - [Q_n,\ Q_{n+1}]$ we have $G \in G_{n+1}$. Let $E + \varepsilon$ be the sum $\sum_\gamma E_\gamma$ of a maximal collection of mutually orthogonal projections of G_{n+1} that are majored by G. Then

$$\begin{aligned}\left\|\pi\left(Q_nTQ_nF\right)\right\| &= \sup_\gamma\left\|\pi\left(Q_nTQ_nF_\gamma\right)\right\| \\ &= \sup_\gamma\left\|\pi\left(Q_{n+1}+Q_{n+1}\right)T\left(Q_{n+1}+Q_{n+1}\right)F_\gamma\right\| \\ &= \left\|\pi\left(Q_{n+1}+Q_{n+1}\right)T\left(Q_{n+1}+Q_{n+1}\right)F\right\|\end{aligned}$$

whence $E+\varepsilon\in G_{n+1}$. By the maximalist of the collection, $0\le G-\left(E+\varepsilon\right)$ does not majority any nonzero projection of G_{n+1} and since $p\left(\ell\right)=G_n\cup G_{n+1}$, any central projection $G'\le G-\left(E+\varepsilon\right)$ must be in G_n. By definition of Ξ, this implies that $G-\left(E+\varepsilon\right)\in\Xi$ whence $G-\left(E+\varepsilon\right)\le\left[Q_n,Q_{n+1}\right]$. So,

$G-\left(E+\varepsilon\right)\le G\le 1-\left[Q_n,Q_{n+1}\right]$ and hence $G=E+\varepsilon\in G_{n+1}$ which completes the proof.

(ii) Let $G\in p\left(\ell\right)$ and $G\le\left[Q_n\right]$. Then $\left\|\pi\left(TG\right)\right\|=\left\|\pi\left(Q_nTQ_nG\right)\right\|\le\left\|\pi\left(Q_{n+1}TQ_{n+1}G\right)\right\|$ $\le\left\|\pi\left(TG\right)\right\|$ whence equality holds and $\left[Q_n\right]\le\left[Q_{n+1}\right]$ by the maximalist of $\left[Q_{n+1}\right]$.

(iii) $\left[Q,\ \varepsilon\right]$ is maximal under the condition: if $G\in p\left(\ell\right)$ and $G\le\left[Q,\ \varepsilon\right]$ then

$$\left\|\pi\left(QTQG\right)\right\|\le\left\|\pi\left(\left(Q+\varepsilon\right)T\left(Q+\varepsilon\right)G\right)\right\|=\left\|\pi\left(TG\right)\right\|$$

which is the same condition defining $\left[Q,I-Q\right]=\left[Q\right]$. Thus $\left[Q\right]=\left[Q,\ \varepsilon\right]$. Applying this to ε we have $\left[\varepsilon\right]=\left[\varepsilon,Q\right]$ and thus by (i) we have $\left[\varepsilon\right]\ge 1-\left[Q,\varepsilon\right]=1-\left[Q\right]$.

(ii) Let $E+\varepsilon,E\in p\left(\ell\right)$ then $\left\|\pi\left(ETE\left(E+\varepsilon\right)\right)\right\|=\left\|\pi\left(TE\left(E+\varepsilon\right)\right)\right\|$. This implies that if $\varepsilon\ge 0$, then $E+\varepsilon\le\left[E\right]$ so $E\le\left[E\right]$ and if $E+\varepsilon=\left[E\right]-E\le\left[E\right]$ then

$0=\left\|\pi\left(ETE\left(E+\varepsilon\right)\right)\right\|=\left\|\pi\left(T\left(E+\varepsilon\right)\right)\right\|$ whence $E=\left[E\right]$.

(v) Follows at once from (ii) and (iv).

The condition that $\left\|\pi\left(TE\right)\right\|\ne 0$ for all $0\ne E\in p\left(\ell\right)$ is of course meaningless unless B is properly infinite. Hence, we may assume without loss of generality that:

B is properly infinite and semi-finite.

There is an $\alpha>0$ such that $\left\|\pi\left(TE\right)\right\|>\alpha$ for all $0\ne E\in p\left(\ell\right)$.

Lemma (11). Let $P \in p$ and $R_n = X_{PTP}[\alpha, \infty)$, $R_{n+1} = X_{PTP}(-\infty, -\alpha]$, where $X_{PTP}(\ )$ denotes the spectral measure of the self-adjoint operator PTP. Then there is an $E_n \in p(\ell)$, with $E_n = I - E$ such that $R_i E_i$ are properly infinite and $c(R_i E_i) = E_j$ for $i = n, \text{n+1}$.

Proof. Let $R = R_n + R_{n+1} = X_{|PTP|}[\infty, \alpha)$ and let $F \neq 0$ be any central projection. If RF were finite, we would have

$$\begin{aligned} \|\pi(TF)\| &= \|\pi(PTPF)\| \\ &= \|\pi(PTP(1-R)F)\| \\ &= \|\pi(|PTP|(1-R)F)\| \\ &\leq \alpha \end{aligned}$$

Thus RF is infinite and nonzero. Hence R is properly infinite and $c(R) = n$. Now let E_1 be the maximal central projection majored by $c(R_n)$, such that $R_n F_n$ is properly infinite. Then $c(R_n,\ E_n) = E_n$ and $R_n(n - E_n)$ is finite, hence $R_{n+}(n - E_n) = R_{n+1}E_{n+1}$ is properly infinite and $c(R_{n+1},\ E_{n+1}) = E_{n+1}$.

End of the Proof of Theorem (6). Take any $0 \neq Q_0 \in p(B)$ such that B_{Q_0} has a faithful trace ω_{x_0} with $x_0 \in Q_0 H$ and assume $\|x_0\| = 1$. Let $P_\gamma \in p, \gamma \in \Gamma$ be the not decreasing to zero. We are going to construct inductively a sequence $\gamma_n \in \Gamma, F_n \in p(\ell), Q_n \in p(B), U_n$ partial

isometrics in B, $x_n \in H$ such that

(a) $U_n U_n^* = Q_n,\ U_n^* U_n = Q_0 F_n,\ i.e., Q_n \sim Q_0 F_n$

(b) $x_n = U_n F_n x_0 \in Q_n H$

(c) $Q_n Q_m = 0 \quad for\ n \neq m$

(d) $\gamma_n > \gamma_m\ \left(hence\ P_{\gamma_n} < P_{\gamma_m}\right)\ for\ n > m$

(e) $Q_n \leq p_{\gamma_n}$

(f) $\left\| p_{\gamma_{n+1}} x_n \right\| < 1/n$

(g) $|Tx_n, x_n| \geq \alpha/2$.

The induction can be started with an arbitrary P_γ; assume we have the construction for $n-1$. Let us apply Lemma(11) to $P=P_{\gamma_n}$ and obtain $E_i \in p(\ell)$, $R_i \in p(B)$ *for* $i = n, n+1$ as defined there. Then

$$1=\|x_o\|^2=\|E_n x_0\|^2+\|E_{n+1}x_0\|^2$$

Let F_n be (any of) the projection E_n or E_{n+1} for which $\|E_i x_0\|^2 \geq \frac{1}{2}$ and let i be the corresponding index. Then $R_i F_n$ is properly infinite and has central support F_n. Now Q_0 is finite having a finite faithful trace ω_{x_0}, hence so is $Q_j \sim F_j Q_0 \leq Q_0$ *for* $1 \leq j \leq n-1$ and $\left(\sum_{j=1}^{n-1} Q_j\right) F_n$. Let $S_n = inf\left\{R_i F_n,\ \left(1-\sum_{j=1}^{n-1} Q_j\right) F_n\right\}$. By the parallelogram law (see [2]) applied to F_n we have that

$$R_i F_n - S_n \sim \left(\sum_{j=1}^{n-1} Q_j\right) F_n - \inf\left\{\left(\sum_{j=1}^{n-1} Q_j\right) F_n, (1-R_i) F_n\right\}$$

whence $R_i F_n - S_n$ is finite and hence S_n is properly infinite and $c(S_n)=F_n$. Since $Q_0 F_n$ is finite and $c(Q_0 F_n) \leq F_n$ we have $Q_0 F_n \prec S_n$, i.e., there is a partial isometry $U_n \in B$ and a $Q_n \in p(B), Q_n \leq S_n$ such that (a) holds. Let x_n be defined by (b) and choose $\gamma_{n+1} \succ \gamma_n$ so that (d) and (f) hold. -Since $Q_n \leq R_i \leq P_{\gamma_n}$ we have (e), since $Q_n \leq \left(1-\sum_{j=1}^{n-1} Q_j\right) F_n$ we have (c). Finally $x_n = R_i x_n = P_{\gamma_n} x_n$ hence (g) follows from

$$\begin{aligned}
|(Tx_n, x_n)| &= \left|\left(P_{\gamma_n} T P_{\gamma_n} x_n, x_n\right)\right| \\
&= \left|\left(P_{\gamma_n} T P_{\gamma_n} R_i x_n, R_i x_n\right)\right| \\
&\geq \alpha \left|(R_i x_n, R_i x_n)\right| \\
&= \alpha \|x_n\|^2 \\
&= \alpha \|F_n x_0\|^2 \\
&\geq \tfrac{1}{2}\alpha.
\end{aligned}$$

Let now $y_n = x_n - P_{\gamma_{n+1}} x_n$. B is semi-finite, hence we can apply Lemma (1) to obtain that $x_n \to_{BEW} 0$. Since $\left\|P_{\gamma_{n+1}} x_n\right\| \to 0$ we thus have $y_n \to_{BEW} 0$ and $y_n \in P_n H$, where

$P_n = P_{\gamma_n} - P_{\gamma_{n+1}} \in p(d)$ and are mutually orthogonal by (d). Clearly for n large enough, $|(Ty_n, y_n)| = |\omega_{y_n}(T)| > \frac{1}{4}\alpha$. Since $\omega_{y_n}(T) = M\omega_{y_n}(U^*\delta(U))$, by the properties of the invariant mean mentioned, we have that $\sup\{|\omega_{y_n}(U^*\delta(U))| \, |U \in u\} > \frac{1}{4}\alpha$. Thus we can find for every n, a unitary

$V_n \in u$ such that $|(V_n^*\delta(V_n)y_n, y_n)| > \frac{1}{4}\alpha$. Let $A = \sum_{n=1}^{\infty} V_n P_n$, then $A \in d$ and

$$\begin{aligned} A^*\delta(A)y_n, y_n &= |(P_n A^*\delta(A)P_n y_n, y_n)| \\ &= |(P_n(A^*AT - A^*TA)P_n y_n, y_n)| \\ &= |(P_n V_n^*\delta(V_n)P_n y_n, y_n)| \\ &= |(V_n^*\delta(V_n)y_n, y_n)| \\ &= \tfrac{1}{4}\alpha \end{aligned}$$

for all n. Therefore $\|\delta(A)y_n\| \not\to 0$. But because of (Π), we have $\delta(A) \notin f(B)$, which completes the proof.

5. THE PROPERTY OF INFINITE W* SUB-ALGEBRA

Lemma (12). Let $0 < b \in Z(M)$, $s(b) = 1$; $e_z^a(0,\infty)$ be a properly infinite projection and $c(e_z^a(0,\infty)) = 1$. Let projection $q \in P(M)$ be finite or properly infinite, $c(q) = 1$ and $q \prec\prec e_z^a(0,\infty)$. Let $\mathbb{R} \ni \mu_n \downarrow 0$. For every $n \in \mathbb{N}$ we denote by z_n such a projection that $1 - z_n$ is the largest central projection, for which $(1 - z_n)q \geq (1 - z_n)e_z^a(\mu_n b, +\infty)$ holds. We have $z_n \uparrow_n 1$ and for

$$d := \left[\mu_1 z_1 + \sum_{n=1}^{\infty} \mu_{n+1}(z_{n+1} - z_n)\right] b$$

the following relations $hold: q \prec\prec e_z^a(d, +\infty)$, $0 < d \leq \mu_1 b$ and $s(d) = 1$. Moreover, if all projections $e_z^a(\mu_n b, +\infty)$, $n \geq 1$ are finite then $e_z^a(d, +\infty)$ is a finite projection as well.

Proof. Since, $e_z^a(\mu_{n+1} b, +\infty) \geq e_z^a(\mu_n b, +\infty)$ we have

$e_z^a(1-z_{n+1})q \geq (1-z_{n+1})e_z^a(\mu_{n+1}b,+\infty) \geq (1-z_{n+1})e_z^a(\mu_n b,+\infty)$. Hence, $z_{n+1} \geq z_n$ for every $n \in \mathbb{N}$. In addition, $e_z^a(\mu_n b,+\infty) \uparrow_n e_z^a(0,+\infty)$ and $e_z^a(0,+\infty)$ is properly infinite projection. Hence, in the case when q is finite projection, it follows that $z_n \uparrow_n 1$. Let us consider the case when q is a properly infinite projection with $c(q)=1$ and such that $q \prec\prec e_z^a(0,\infty)$. In this case, with $p=q,\ q=e_z^a(0,+\infty),\ q_n=e_z^a(\mu_n b,+\infty)$ and deduce $\vee_{n=1}^{\infty} z_n \geq c(q)=1$.

All other statements follow from the form of element d. Since, $z_1 d=\mu_1 z_1 b$, $(z_{n+1}-z_n)=\mu_{n+1}(z_{n+1}-z_n)b$ and $z_n q \prec\prec z_n e_z^a(\mu_n b,+\infty)$ for every $n \in \mathbb{N}$. Observe also that $s(d)=s(b)\left(z_1+\sum_{n=1}^{\infty}(z_{n+1}-z_n)\right)=1$.

Finally, let all projections $e_z^a(\mu_n b,+\infty),\ n \geq 1$ be finite. Since

$dz_1=\mu_1 b, d(z_{n+1}-z_n)=\ \mu_{n+1}b(z_{n+1}-z_n)$, we have

$$e_z^a(d,+\infty)z_1=e_z^a(\mu_1 b,+\infty)z_1,$$

$$e_z^a(d,+\infty)(z_{n+1}-z_n)=e_z^a(\mu_{n+1}b,+\infty)(z_{n+1}-z_n)$$

for every $n \in \mathbb{N}$. There projections standing on the right-hand sides are finite. Hence, $e_z^a(d,+\infty)$ is finite projection as a sum of the left-hand sides [22].

We shall use a following well-known implication

$$p \prec\prec q \quad \Rightarrow \quad zp \prec\prec zq, \quad \forall z \in P(Z(M)),\ 0<z \leq c(p) \vee c(q).$$

We supply here a straightforward argument. Let $z' \in z \in Z(M)$ be such that $0<z' \leq c(pz) \vee c(qz) z(c(p) \vee c(q))$. Then $z' \leq c(p) \vee c(q)$ and therefore $z'(zp)=z'p \prec z'q=z'(zq)$. This means $zp \prec\prec zq$.

As in [6] we can use Theorem (6) to extend the result to the properly infinite case.

Theorem (13). Let A be a properly infinite W* sub-algebra of B containing the center ℓ of B. For every derivation $\delta : \mathrm{A} \to f\ (B)$ there is a$T\ \in f\ (B)$ such that $\delta = a\mathrm{A}\ T$.

Before we start the proof let us recall that if A is properly infinite there is an infinite countable decomposition of the identity into mutually orthogonal projections of A, all equivalent in A to I, and thus a fortify equivalent in B to 1 [8]. Therefore there is a spatial isomorphism

$$\phi : B \to \tilde{B} = B \otimes B(H_0)$$

with $H_0 = l^{n+1}(\mathbb{Z})$ and

$$\phi(\mathrm{A}) = \tilde{\mathrm{A}} = \mathrm{A} \otimes B\,(H_0)$$

[5]. Recall also that the elements B of $\tilde{B}$ (or $\tilde{\mathrm{A}}$) are represented by bounded matrices $\left[B_{ij}\right], i, j \in \mathbb{Z}$ with entries in B (or A) by the formula

$$\left(I \otimes E_{ij}\right) T\left(I \otimes E_{kl}\right) = T_{jk} \otimes E_{il}$$

where E_{ij} is the canonical matrix unit of $B(H_0)$. In particular if ℓ, $\wp$ are the maximal a commutative operation subalgebras of $B(H_0)$of Laurent (resp. diagonal) matrices, then $B \in B \otimes \ell\,(resp.\, B \in B \otimes \wp)$ iff $\left[B_{ij}\right]$ is a Laurent matrix with entries in B, i.e., $B_{ij} = B_{i-j}$,where B_k, denotes the entry along the kth diagonal$\left(resp.\, B_{ij} = \delta_{ij} B_{ii}\right)$ for all $i, j \in \mathbb{Z}$.

Proof. Let $\tilde{\delta} = \phi \circ \delta \circ \phi^{-1}$ then

$$\tilde{\delta} : \tilde{d} \to \phi(f\ (B)) = f\ (\tilde{B})$$

is a relative compact derivation. Let us define the following W* algebras: $\tilde{\ell} = \tilde{B} \cap \tilde{B},\ \tilde{\mathrm{A}}_n = \ell \otimes \ell,\ \mathrm{A}_n = \phi^{-1}\left(\tilde{\mathrm{A}}_n\right),\ \tilde{\mathrm{A}}_{n+1} = \mathrm{A} \otimes \ell$, and $\tilde{\mathrm{A}}_{n+2} = \mathrm{A}_n \otimes \wp$. First, let us notice that

$$\begin{aligned}\tilde{\mathrm{A}}_n \cap f\left(\tilde{B}\right) &= \left(\ell' \otimes \ell\right) \cap \left(B \otimes B(H_0)\right) \cap f\left(\tilde{B}\right)\\ &= \left(B \otimes \ell\right) \cap f\left(\tilde{B}\right)\\ &= \{0\}\end{aligned}$$

by [22]. Therefore

$$\mathrm{A}'_n \cap f\,(B) = \phi^{-1}\left(\tilde{A}'_n\right) \cap f\,(B) = \phi^{-1}\left(\tilde{\mathrm{A}}'_n \cap f\left(\tilde{B}\right)\right) = \{0\}$$

because ϕ is spatial Now

$$\begin{aligned}\tilde{\ell} &= \left(B \otimes B(H_0)\right) \cap \left(B' \otimes I\right)\\ &= \ell \otimes I \subset \tilde{\mathrm{A}}_n \subset \tilde{\mathrm{A}}.\end{aligned}$$

Thus we can apply Theorem(6) to the derivation $\tilde{\delta}$ restricted to the a commutative operation sub-algebra $\tilde{\mathrm{A}}_n$ of $\tilde{B}$ and we obtain a $T_n \in f\left(\tilde{B}\right)$ such that $\tilde{\delta}_n = \tilde{\delta} - a\mathrm{A}T_n$ vanishes on $\tilde{\mathrm{A}}_n$. Now

$$\tilde{\mathrm{A}}_{n+1} \subset B \otimes \ell \subset \ell' \otimes \ell = \tilde{\mathrm{A}}'_n .$$

Therefore, for all $A_n \in \tilde{\mathrm{A}}_n$ and $A_{n+1} \in \tilde{\mathrm{A}}_{n+1}$ we have

$$\tilde{\delta}_n\left(A_n A_{n+1}\right) = A_n\, \tilde{\delta}_n\left(A_{n+1}\right) = \tilde{\delta}_n\left(A_{n+1} A_n\right) = \tilde{\delta}_n\left(A_{n+1}\right) A_n$$

i.e., $\tilde{\delta}_n\left(A_{n+1}\right)$ and A_n commute and hence

$$\tilde{\delta}_n\left(\tilde{\mathrm{A}}_{n+1}\right) \subset \tilde{\mathrm{A}}'_n \cap f\left(\tilde{B}\right) = \{0\}$$

Thus $\tilde{\delta}_n$ also vanishes on $\tilde{\mathrm{A}}_{n+1}$. Now $\tilde{\mathrm{A}}_n$ is a commutative operation and hence so are A_n and $\tilde{\mathrm{A}}_{n+2}$. Moreover,

$$\tilde{\ell} \subset \tilde{\mathrm{A}}_n \subset \tilde{\mathrm{A}} \subset \tilde{B}$$

Implies

$$\ell = \phi^{-1}\left(\tilde{\ell}\right) \subset \mathrm{A}_n \subset \mathrm{A} \subset B$$

and hence

$$\tilde{\ell} = \ell \otimes I \subset \mathrm{A}_n \otimes I \subset \tilde{\mathrm{A}}_{n+2} \subset \tilde{\mathrm{A}} \subset \tilde{B}$$

Thus we can apply again Theorem(6) to the relative compact derivation $\tilde{\delta}_n$ restricted to $\tilde{\mathrm{A}}_{n+2}$. Let $T_{n+1} \in f\left(\tilde{B}\right)$ be such that $\tilde{\delta}_n$ agrees with ad T_{n+1} on $\tilde{\mathrm{A}}_{n+2}$. Since

$$\mathrm{A}_n \otimes I \subset \mathrm{A} \otimes I \subset \mathrm{A} \otimes \ell = \tilde{\mathrm{A}}_{n+1}$$

and $\tilde{\delta}_n$ vanishes on $\tilde{\mathrm{A}}_{n+1}$, we see that ad T_{n+1} vanishes on $\mathrm{A}_n \otimes I$, i.e.,

$$T_{n+1} \in \left(\mathrm{A}_n \otimes I\right)' \cap f\left(\tilde{B}\right) = \left(\mathrm{A}'_n \otimes B\left(H_0\right)\right) \cap fg\left(\tilde{B}\right)$$

Then for all $i, j \in \mathbb{Z}, \left(T_{n+1}\right)_{ij} \in \mathrm{A}'_n$ and

$$\left(T_2\right)_{ij} \otimes E_{nn} = \left(I \otimes E_{ni}\right) T_{n+1} \left(I \otimes E_{jn}\right) \in f\left(\tilde{B}\right)$$

whence by Lemma(12)(a) $(T_{n+1})_{ij} \in f(B)$. But we saw that $d'_n \cap f(B) = \{0\}$, hence $(T_{n+1})_{ij} = 0$ for all $i, j \in \mathbb{Z}$, so $T_{n+1} = 0$. Therefore $\tilde{\delta}_n$ vanishes also on $\tilde{\mathrm{A}}_{n+2}$ and hence on $I \otimes \wp$. Now ℓ and $\wp$ generate $B(H_0)$, whence $\tilde{\mathrm{A}}_{n+1} = \mathrm{A} \otimes \ell$ and $I \otimes \wp$ generate $\tilde{\mathrm{A}}$. Thus by the σ-weak continuity of $\tilde{\delta}_n$ (see [6]) we see that

$\tilde{\delta}_n = \tilde{\delta} - a\mathrm{A}T_n = 0, i.e., \tilde{\delta} = a\mathrm{A}T_n$. Clearly $\delta = ad\phi^{-1}(T_n)$ and $\phi^{-1}(T_n) \in \mathrm{A}(B)$.

Let us assume in this part that B is semi-finite and let τ be a fsn trace on it. Beside the closed ideal $f(B)$ we can also consider the (non closed) two-sided norm-ideals $C_{1+\varepsilon}(B,\tau)$ *for* $1 \le 1+\varepsilon < \infty$ defined by

$$C_{1+\varepsilon}(B,\tau) = \left\{ B \in B \middle| \tau\left(|B|^{1+\varepsilon}\right) < \infty \right\}$$

$$\|B\|_{1+\varepsilon} = \tau\left(|B|^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \quad \textit{for } B \in C_{1+\varepsilon}(B,\tau).$$

Obviously,

$$C_{1+\varepsilon}(B,\tau) = B \cap L^{1+\varepsilon}(B,\tau),$$

where the latter is the non commutative $L^{1+\varepsilon}$-space of B relative to τ (see [14]).

Recall the following facts about $L^{1+\varepsilon}(M)$ spaces in the case of a general W* algebra M and $1 \le 1+\varepsilon < \infty$ ($L^{\infty}(M)$ is identified with M): $L^{1+\varepsilon}(M)$ is a Banach space, its dual is isomorphic to $L^{\frac{\varepsilon}{1+\varepsilon}}(M)$ (with $1/_{1+\varepsilon} + {}^{1+\varepsilon}/_{\varepsilon} = 1$), and the duality is established by the functional tr on $L^1(M)$, where if $A \in L^{1+\varepsilon}(M)$, $B \in L^{\frac{\varepsilon}{1+\varepsilon}}(M)$ we have $AB,\ BA \in L^n(M)$ and $tr(AB) = tr(BA)$, $|tr(AB)| \le \|A\|_{1+\varepsilon} \|B\|_{\frac{\varepsilon}{1+\varepsilon}}$,

$$\|A\|_{1+\varepsilon} = \left(tr|A|^{1+\varepsilon}\right)^{1/_{1+\varepsilon}} = \max\left\{ |trAB| \middle| B \in L^{\frac{\varepsilon}{1+\varepsilon}}(M), \|B\|_{\frac{\varepsilon}{1+\varepsilon}} \le 1 \right\}$$

(see [14]). Of course, if $M = B$ we can identify $L^{1+\varepsilon}(M)$ with $L^{1+\varepsilon}(B,\tau)$ and tr with τ. The following inequality will be used here only in the semi-finite case and in the context of $C_{1+\varepsilon}$-ideals, but since the same proof holds for $L^{1+\varepsilon}$-spaces, we shall consider the general case.

Corollary (14). Let M be a W* algebra, $\varepsilon \geq 0, A \in L^{1+\varepsilon}(M)$ and

$Q_n, Q_{n+1} \in p(M)$, $Q_n Q_{n+1} = 0$, $Q_n + Q_{n+1} = 1$. Then

$$\|A\|_{1+\varepsilon}^{1+\varepsilon} \geq \|Q_n A Q_n\|_{1+\varepsilon}^{1+\varepsilon} + \|Q_{n+1} A Q_{n+1}\|_{1+\varepsilon}^{1+\varepsilon}$$

Proof. Let us first note that

$$\left|\sum_{i=n}^{n+1} Q_i A Q_i\right|^{1+\varepsilon} = \sum_{i=n}^{n+1} |Q_i A Q_i|^{1+\varepsilon}$$

And

$$\left\|\sum_{i=n}^{n+1} Q_i A Q_i\right\|_{1+\varepsilon}^{1+\varepsilon} = \sum_{i=n}^{n+1} \|Q_i A Q_i\|_{1+\varepsilon}^{1+\varepsilon}$$

Consider first $1+\varepsilon = n$ and take the polar decomposition's

$$Q_i A Q_i = U_i |Q_i A Q_i|, \ i = n, n+1.$$

Then $U_i U_i^*$ and $U_i^* U_i$ are majored by Q_i and hence U_i commutes with Q_j. Therefore $B = (U_n + U_{n+1})^*$ commutes with Q_i and $\|B\| = 1$. Then

$$\begin{aligned} \|A\|_1 &\geq |trAB| \\ &= \left| tr\left(\sum_{i=n}^{n+1} Q_i B A Q_i \right) \right| \\ &= tr\left(\sum_{i=n}^{n+1} Q_i A Q_i \right) \\ &= \sum_{i=n}^{n+1} \|Q_i A Q_i\|_n . \end{aligned}$$

Consider now $\varepsilon > 0$. Let $B \in L^{\frac{\varepsilon}{1+\varepsilon}}(M)$ be such that $\|B\|_{\frac{\varepsilon}{1+\varepsilon}} \leq 1$ and

$$\left\|\sum_{i=n}^{n+1} Q_i A Q_i\right\|_{1+\varepsilon} = tr\left(\left(\sum_{i=n}^{n+1} Q_i A Q_i\right) B\right).$$

Take the polar decomposition's $A = U|A|$ and $B = V|B|$, then VU are in M and $|A|, |B|$ are in $L^{1+\varepsilon}(M), L^{\frac{\varepsilon}{1+\varepsilon}}(M)$, respectively. Let

$$f\ (z\ )=tr\left(\sum_{i=n}^{n+1} Q_i U\ |A|^{(1+\varepsilon)z}\ Q_i V\ |B|^{\left(\frac{\varepsilon}{1+\varepsilon}\right)(1-z\ )}\right).$$

Then by standard arguments, it is easy to see that f is analytic on $0 < Re\ z < n$ and continuous and bounded on $0 \le Re\ z \le n$. Then by the three-line theorem (see [4]) we have

$$f\left(\tfrac{1}{1+\varepsilon}\right)\ \le \underset{t\in\mathbb{R}}{Max}\ f\left(it\right)^{\varepsilon/_{1+\varepsilon}}\ \underset{t\in\mathbb{R}}{Max}\ f\left(1+it\right)^{1/_{1+\varepsilon}}$$

Now $f\ \left(\dfrac{1}{1+\varepsilon}\right)=\left\|\sum_{i=n}^{n+1} Q_i A Q_i\right\|_{1+\varepsilon}$ and by Holder's inequality

$$\begin{aligned}
\left|f\ (it\ )\right| &= tr\left|\left(\sum_{j=n}^{n+1} Q_j U\ |A|^{i\,(1+\varepsilon)t}\ Q_j V\ |B|^{-i\left(\frac{\varepsilon}{1+\varepsilon}\right)t}\ |B|^{\frac{\varepsilon}{1+\varepsilon}}\right)\right| \\
&\le \left\|\sum_{j=n}^{n+1} Q_j U\ |A|^{i\,(1+\varepsilon)t}\ Q_j V\ |B|^{-i\left(\frac{\varepsilon}{1+\varepsilon}\right)t}\right\| \left\||B|^{\frac{\varepsilon}{1+\varepsilon}}\right\|_1 \\
&\le \left(\max_j \left\|Q_j U\,|A|^{i(1+\varepsilon)t}\ Q_j\right\|\right)\left\|V\,|B|^{-i\left(\frac{\varepsilon}{1+\varepsilon}\right)t}\right\|\left\|B\right\|_{\frac{\varepsilon}{1+\varepsilon}}^{\frac{\varepsilon}{1+\varepsilon}} \\
&\le n.
\end{aligned}$$

Again by Holder's inequality applied twice and by the result already obtained in the $\varepsilon = 0$ case,

$$\begin{aligned}
\left|\,f\left(1+it\right)\right| &= tr\left|\left(\sum_{j=n}^{n+1} Q_j U\,|A|^{i(1+\varepsilon)t}\ |A|^{1+\varepsilon}\ Q_j V\,|B|^{-i\left(\frac{\varepsilon}{1+\varepsilon}\right)t}\right)\right| \\
&\le \left\|\sum_{j=n}^{n+1} Q_j U\,|A|^{i(1+\varepsilon)t}\ |A|^{1+\varepsilon}\ Q_j\right\|_1 \left\|V\,|B|^{-i\left(\frac{\varepsilon}{1+\varepsilon}\right)t}\right\| \\
&\le \left\|U\,|A|^{i(1+\varepsilon)t}\ |A|^{1+\varepsilon}\right\|_1 \\
&\le \left\|U\,|A|^{i(1+\varepsilon)t}\right\|\left\||A|^{1+\varepsilon}\right\|_1 \\
&\le \left\|A\right\|_{1+\varepsilon}^{1+\varepsilon}
\end{aligned}$$

Thus $f\ \left(\frac{1}{1+\varepsilon}\right) \le \|A\|_{1+\varepsilon}$ whence by the second equality in this proof,

$$\|A\|_{1+\varepsilon}^{1+\varepsilon} \ge \left\|\sum_{i=n}^{n+1} Q_i A Q_i\right\|_{1+\varepsilon}^{1+\varepsilon} = \sum_{i=n}^{n+1}\|Q_i A Q_i\|_{1+\varepsilon}^{1+\varepsilon}$$

Data Availability

No data were used to support this study.

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