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GEOMETRIC SINGULARITIES OF THE POISSON'S EQUATION IN A NON-SMOOTH DOMAIN WITH APPLICATIONS OF WEIGHTED SOBOLEV SPACES

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Abstract. The solution fields of the elliptic boundary value problems may exhibit singularities near the corners, edges, crack tips, and so forth of the physical domain. This paper deals with the boundary singularities of weak solutions of boundary value problems governed by the Poisson equation in a two-dimensional non-smooth domain with singular points on the boundary. The presence of these points on the boundary, generally, generates local singularities in the solution. The applications of Fourier transform and weighted Sobolev spaces make it possible to describe the qualitative properties of the solution including its regularity. The general theory of V. A. Kondratiev is followed to obtain these results.

1. Introduction

Let $\mathcal{D}$ be a 2-dimensional bounded plane polygonal domain $\mathcal{D} \subset \mathbb{R}^2$, (see Figure-1) whose boundary $\partial\mathcal{D}$ comprises the corner points ($\omega \neq \pi$) and the points where the type of boundary conditions changes ($\omega = \pi$). Let $\mathcal{N}$ denote the set of these boundary points which consists of $\{P_1, ..., P_N\} \subset \partial\mathcal{D}$. Note that a point $P \in \partial\mathcal{D}$ is said to be a corner point if there exists a neighborhood $\eta(P)$ of the point P such that $\mathcal{D} \cap \eta(P)$ is diffeomorphic to an angle κ intersected with the unit circle.

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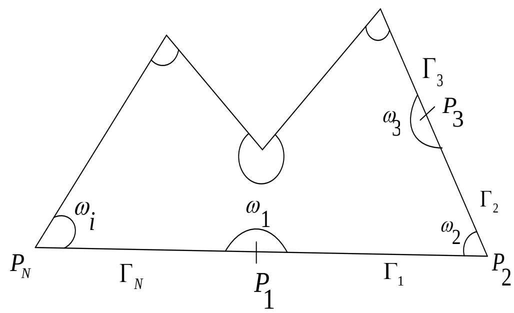

FIGURE 1. A Polygonal domain.

Let P_i denote the vertices of the polygon and the open edges Γ_i connecting the vertices P_{i+1} and P_i, $1 \le i \le N$. Let $P_{N+1} = P_1$, $\Gamma_{N+1} = \Gamma_1$, $\Gamma_0 = \Gamma_N$, and the interior angles are $\omega_i = \Gamma_i - \Gamma_{i-1}$. Suppose that the boundary $\partial\mathcal{D} = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$ with meas$(\Gamma_0) > 0$ (Lebesgue measure). Further, we assume that $\mathcal{J}_1$ and $\mathcal{J}_2$ be the disjoint subsets of $\{1, 2, ..., N\}$, where we can set $\Gamma_0 = \bigcup_{i\in\mathcal{J}_1} \bar{\Gamma}_i$ and $\Gamma_1 = \bigcup_{i\in\mathcal{J}_2} \bar{\Gamma}_i$, respectively, denote the union of the boundary parts, where the Dirichlet boundary conditions and the Neumann boundary conditions are given. As well, the combinations of the boundary points with different boundary conditions are considered. To characterize them, we denote the Dirichlet-Dirichlet boundary conditions by (DD), i.e., $P_i \in \mathcal{J}_1$, the Neumann-Neumann boundary conditions by (NN), $P_i \in \mathcal{J}_2$ and the Dirichlet-Neumann (mixed) boundary conditions by (DN), $P_i \in \mathcal{J}_1\mathcal{J}_2$. Note that $\mathcal{N} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_1\,\mathcal{J}_2$.

To describe the problem mathematically, let us consider the mixed boundary value problem for the Poisson equation

$$\begin{cases} -\Delta v = f \quad \text{in} \quad \mathcal{D}, \\ v = 0 \quad \text{on} \quad \Gamma_0, \\ \frac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \Gamma_1, \end{cases} \tag{1.1}$$

where $\mathbf{n} = (n_1, n_2)$ is known as the unit outward normal vector to the boundary, $f \in L^2(\mathcal{D})$ and $\Delta = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ is the Laplacian operator.

It is known from the theory of elliptic boundary value problems in domains with boundary irregularities, like corners, conic vertices, edges, and cracks, etc., the solution may exhibit singularities. Numerous interesting results about the regularity of the solution cannot be extended if one of the following situations appears: the domain has corners, edges or angular points on the boundary, the change of the boundary conditions at some points, the discontinuities of the solution and the singularities of the coefficients.

Principally, the theory for smooth domains cannot be applied directly to non-smooth domains having corners or edges on the boundary, and the points where the type of boundary conditions changes. The asymptotic expansion of the solution near the conical or angular points plays an important role to describe the regularity behavior of the solution accurately. Moreover, the information of the singularity functions in non-smooth domains can help to improve the rate of convergence of the numerical methods for approximations, for instance, the finite element approximation, singular function method or the dual singular function method, and the graded mesh refinement [13, 19, 20]. Presently, there exists a wide-ranging theory for parabolic, hyperbolic, and elliptic boundary value problems having a smooth boundary. Generally, the results of this theory conclude that if the boundary of the domain, the boundary operators, the coefficients of the equations, and the right-hand sides are sufficiently smooth, then the solution of the considered problem is itself sufficiently smooth [12, 13, 18, 29].

Generally, three types of singularities arise in elliptic type problems: the angular type singularities, the interface, and the infinity type singularities in unbounded solution domains. This paper deals with the angular type singularities and several approaches to find these singularities are discussed in [5, 10, 27, 30]. In recent, [4] has comprehensively discussed the methods to find the singular behavior of the solution structure of the elliptic boundary value problems in a polygonal domain with convex and non-convex vertices. Further, it is noted from the general theory on H^2-regularity for linear elliptic boundary value problems [14, 15, 25], the general solution $\mathbf{u}$ for two or three-dimensional domain $\mathcal{D}$ with corner or edge singularities and any right-hand side function $\mathbf{f} \in L^2(\mathcal{D})$ can be broken down as a sum of a singular and a regular part

$$\mathbf{u} = \sum_{m=1}^{N} c_m s_m + \mathbf{u}_R, \tag{1.2}$$

where $\mathbf{u}_R \in H^2(\mathcal{D})$. The second part is the locally acting singular part that is a combination of explicit model singular solutions s_m and the unknown coefficients c_m. The special singular functions s_m rely on the geometry of the model problem, the differential operator, and the characteristic boundary conditions. The unknown coefficients c_m relating to singularity functions are some real numbers or unique scalar constants which are stated as the stress intensity factors. The rigorous formulas for their derivations are of constant interest and a challenging task [7, 13, 14, 25]. The mathematical analysis like well-posedness and regularity results of such type of elliptic boundary value problems in non-smooth domains have attracted many mathematicians and scientists to examine the singular behavior of the solution structure near the singular points [9, 15, 16, 21, 23].

The main purpose of this paper is the derivation and the computation of the singular terms of the solutions of the generalized boundary eigenvalue problem for the Poisson equation in a bounded plane polygonal domain with singular points on its boundary. The theory developed by Kondratiev [22, 23] and further extended by [28] for scalar problems is used in the context of weighted Sobolev spaces. Generally, the Sobolev spaces are not suitable to define the regularity results of the boundary value problems in non-smooth domains. So, [22, 28] have introduced weighted Sobolev spaces with Kondratiev type weights for parabolic and elliptic problems in polygonal domains. In [25], where the method of special ansatzes and spherical coordinates are used to calculate the singular terms for the Dirichlet problem of the Poisson equation.

Analogous to [6], where the Mellin transform and the method of the special ansatzes is used to obtain the asymptotic singular representations of the solution of the biharmonic operator on a bounded domain with angular corners. The technique of Fourier transform is used here to obtain the generalized form of the boundary eigenvalue problem for the Poisson equation with the mixed boundary conditions. The achieved eigenvalues and eigensolutions generate singular terms. The information about the singular terms allows us to evaluate the optimal regularity of the corresponding weak solution of the considered boundary value problem.

The rest of this paper is organized as follows: Section 2 is dedicated to present the weak formulation of the problem and introduce some function spaces. In Section 3, determine a parametric boundary eigenvalue problem with a complex parameter ξ, the Poisson equation is considered in an infinite cone with various combinations of the boundary conditions. Furthermore, the distribution of the eigenvalues and the eigenfunctions are discussed. In Section 4, the regularity and expansion results for the corresponding problem with various conditions are investigated. Some concluding remarks are given in the last Section 5.

2. Analytical Preliminaries

Besides the strong formulation, let us consider the weak formulation of the mixed boundary value problem (1.1) which reads: Find $v \in U(\mathcal{D}) = \{v \in H^1(\mathcal{D}) : v = 0 \quad \text{on} \quad \Gamma_0\}$ such that

$$a(v,u) = f(v) \qquad \forall u \in U(\mathcal{D}), \tag{2.1}$$

where

$$a(v,u) = \int_{\mathcal{D}} \nabla v \cdot \nabla u \, d\mathrm{x} \quad \text{and} \quad f(v) = \int_{\mathcal{D}} f \cdot u \, d\mathrm{x}.$$

The Lax-Milgram theorem [12–14] deduces that the variational problem (2.1) has a unique solution. Hence, we have to analyze the smoothness of the weak solution v and see how it depends on the size of the angle ω_i, $i = 1, ..., N$.

2.1. **Weighted Sobolev spaces.**

To analyze the regularity results of the weak solution of the corresponding boundary value problem in a non-smooth domain with singular points, firstly, we introduce some function spaces in line with [1, 11, 22, 28].

Let $\mathcal{N}$ be the set of singular points on the boundary, i.e., $\mathcal{N} \subset \partial\mathcal{D}$. Denote

$$C^\infty_{\mathcal{N}} = \left\{ v \in C^\infty(\overline{\mathcal{D}}),\, \operatorname{supp} v \cap \overline{\mathcal{N}} = \emptyset \right\},$$

where the $\operatorname{supp} v$ is bounded. We assume that $D^\beta v$ be the multi-index notation for higher-order derivatives and in cartesian coordinates is defined by

$$D^\beta v = \frac{\partial^{|\beta|} v}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}, \qquad \beta = (\beta_1,\, \beta_2), \qquad |\beta| = \beta_1 + \beta_2.$$

Let $\alpha = \big(\alpha_1, ..., \alpha_N\big)$ be an $N-$tuple of real numbers which satisfying $0 < \alpha_i < 1$ for $1 \le i \le N$. Therefore, the weight function is characterized by

$$\Phi_{\alpha+m}(x) = \prod_{i=1}^{N} \big(r_i(x)\big)^{\alpha_i+m},$$

where m is an any integer and $\big(r_i(x)\big) = \text{dist}\,(x, P_i)$. Let $\mathcal{W}_{\alpha}^{m,\,p}(\mathcal{D})$ be the weighted Sobolev spaces and is the closure of $C_N^{\infty}(\mathcal{D})$ equipped with the norm

$$\|v\|_{\mathcal{W}_{\alpha}^{m,\,p}(\mathcal{D})} = \Big(\sum_{|\beta|\le m} \int_{\mathcal{D}} |x|^{p(\alpha-m+|\beta|)} \big|D^{\beta} v\big|^{p} dx\Big)^{\frac{1}{p}}. \tag{2.2}$$

Let $Q = \big\{(\tau, \theta) : -\infty < \tau < \infty,\, 0 < \theta < \omega_0\big\}$ denote the infinite strip with positive width ω_0. For any real $h > 0$, and for an integer $m \ge 0$, the spaces are defined as

$$\mathcal{W}_{h}^{m}(Q) = \Big\{u \in L^2(Q) : \sum_{|\beta|\le m} \int_{Q} e^{2h\tau} \big|D^{\beta} u\big|^{2} d\tau d\,\theta < \infty\Big\},$$

where

$$\|u\|_{\mathcal{W}_{h}^{m}(Q)} = \Big(\sum_{|\beta|\le m} \int_{Q} e^{2h\tau} \big|D^{\beta} u\big|^{2} d\tau\, d\theta\Big)^{\frac{1}{2}}.$$

3. The boundary value problem in an infinite cone

In this section, we will see the occurrence of the singular terms near the singular points and the structure which they have. So, to analyze these results, the following steps are followed.

(1) We localize the model problem in the neighborhood of the corner point or a point where the boundary conditions changes (known as a singular point), and then the model problem is considered in an infinite cone.

(2) The model problem is transformed in the form of local polar coordinates $(r,\, \theta)$ and then the variable transformation $r = e^{\tau}$ is used. Afterward, the complex Fourier transform respecting the variable τ is applied to attain a boundary value problem which depends on the complex parameter ξ. Moreover, the operator $\hat{\mathcal{V}}(\xi)$ is used to represent the generalized form of this parametric boundary eigenvalue problem.

(3) The eigenvalues and the generalized eigensolutions of this parametric boundary eigenvalue problem with various kinds of boundary conditions are obtained. They exhibit the asymptotic development of the solution of the model problem near the singular points. Finally, the regularity results can be followed by the general theory of ellipticity.

3.1. **Localization of the model problem.** The regularity analysis of the solutions of mixed boundary value problems in a bounded plane polygonal domain is a local problem. If we presume that $\mathcal{D}$ is a polygonal domain, then the regularity principles work well in the interior of the domain and also on $\partial\mathcal{D} \setminus \bigcup_{i=1}^{N} \eta(P_i)$, where $\eta(P_i)$ is the neighborhood of the corner points or the points where the type of boundary conditions changes. Usually, these points are called singular points. To show that the weak solution v is regular, we have to investigate its behavior near the corner points P_i, $i = 1, 2, ..., N$. Let us consider one corner point P_N as an origin with an angle ω_0, and an appropriate infinite differentiable cut-off function $\chi(|\mathrm{x}|) = \chi(r)$ is defined as

$$\chi(r) = \begin{cases} 1 & \text{for} \quad 0 \le r \le \epsilon, \\ 0 & \text{for} \quad r \ge 2\epsilon, \end{cases} \tag{3.1}$$

and it depends on the distance from the point P_N. The number ϵ is so small that P_N is the only corner point of the domain $\mathcal{D}$ that lies inside the circle $\{\mathrm{x} : |\mathrm{x}| \le 2\epsilon\}$. Afterward, multiplying the smooth cut-off function χ on both sides of (1.1), then substituting $u = \chi v$ in (1.1). The derivatives are considered in the distribution sense. Thus, the boundary value problem is transformed into an infinite cone

$$S = \big\{(r,\, \theta) : 0 < r < \infty,\, 0 < \theta < \omega_0\big\},$$

which coincides with the original problem near the corner point P_N. Then the system (1.1) become

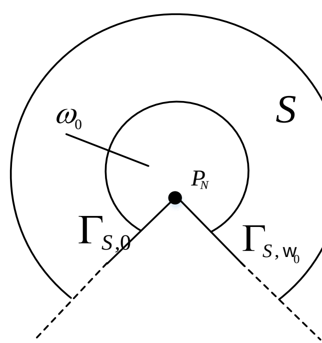

FIGURE 2. Infinite cone S with opening angle ω_0.

$$\begin{cases} -\Delta u &= F \quad \text{in} \quad S, \\ u &= 0 \quad \text{on} \quad \Gamma_{S,\,0},\, \Gamma_{S,\,\omega_0} \quad \text{if} \quad \Gamma_{S,\,0},\, \Gamma_{S,\,\omega_0} \subset \Gamma_0, \\ \frac{\partial u}{\partial \mathbf{n}} &= G \quad \text{on} \quad \Gamma_{S,\,0},\, \Gamma_{S,\,\omega_0} \quad \text{if} \quad \Gamma_{S,\,0},\, \Gamma_{S,\,\omega_0} \subset \Gamma_1, \end{cases} \tag{3.2}$$

where $F = \chi\, f - 2\nabla\chi \cdot \nabla v - v\, \Delta\chi$ and $G(x) = 0$ for $r < \epsilon$ and $r > 2\epsilon$. The behavior of u near the point P_N illustrate the regularity of the solution v in the neighborhood of P_N. If we suppose that the right-hand side in (1.1) is $f \in L^2(\mathcal{D})$, then $F \in L^2(S)$. The following boundary conditions are prescribed on the subsequent edges $\Gamma_{S,0}\,(\theta=0)$ and $\Gamma_{S,\omega_0}\,(\theta=\omega_0)$ of the cone (see Figure-2). Just one condition is considered per edge to differentiate between the mixed boundary conditions.

To analyze the regularity results of the boundary value problem (3.2), we rewrite the operators in the structure of polar coordinates (r,θ). Hence, the transformed form is

$$\begin{aligned} -\Big(\frac{\partial^2 \breve{u}}{\partial r^2}+\frac{1}{r}\,\frac{\partial \breve{u}}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2 \breve{u}}{\partial \theta^2}\Big) =& \breve{F}(r,\theta) \quad \text{in} \quad \hat{S},\\ \breve{u}(r,\theta)\big|_{\theta=0,\,\omega_0} =& 0,\\ \frac{\partial \breve{u}}{\partial \theta}(r,\theta)\big|_{\theta=0,\,\omega_0} =& \breve{G}(r,\theta)\big|_{\theta=0,\,\omega_0}, \end{aligned} \tag{3.3}$$

where $\hat{S}$ is the infinite half-strip in the (r,θ)-plane and $\breve{u}(r,\theta)=u(x,y)$, $\breve{F}(r,\theta)=F(x,y)$ and $\breve{G}(r,\theta)=G(x,y)$. Now, a variable τ with the relation $r=e^{\tau}$ is introduced, then (3.3) is transformed to the infinite strip with the width ω_0 as

$$\begin{aligned} -\Big(\frac{\partial^2 \tilde{u}}{\partial \tau^2}+\frac{\partial^2 \tilde{u}}{\partial \theta^2}\Big) =& \tilde{F}(\tau,\theta) \quad \text{in} \quad \bar{S},\\ \tilde{u}(\tau,\theta)\big|_{\theta=0,\,\omega_0} =& 0,\\ \frac{\partial \tilde{u}}{\partial \theta}(\tau,\theta)\big|_{\theta=0,\,\omega_0} =& \tilde{G}(\tau,\theta)\big|_{\theta=0,\,\omega_0}. \end{aligned} \tag{3.4}$$

Here, $\bar{S}=\big\{(\tau,\,\theta):-\infty<\tau<\infty,\,0<\theta<\omega_0\big\}$ and $\tilde{u}=\breve{u}(e^{\tau},\theta)$, $\tilde{F}=e^{2\tau}\,\breve{F}(e^{\tau},\theta)$ and $\tilde{G}=e^{\tau}\,\breve{G}(e^{\tau},\theta)$.

To obtain the boundary eigenvalue value problem, some basic properties of the complex Fourier transform respecting variable τ in line with [16, 23, 28] are described as

$$\mathcal{F}[u](\xi)=\hat{u}(\xi)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{-i\xi\tau}\,u(\tau)d\tau, \qquad \xi\in\mathbb{C}, \tag{3.5}$$

and the inverse Fourier transform is

$$\mathcal{F}^{-1}[u](\xi)=u(\tau)=(2\pi)^{-\frac{1}{2}}\int_{-\infty+ih}^{\infty+ih}e^{i\xi\tau}\hat{u}(\xi)\,d\xi. \tag{3.6}$$

It defines an isomorphic mapping, i.e.,

$$\mathcal{F}[u](\xi)=\Big\{u(\tau):\int_{-\infty}^{\infty}e^{2h\tau}|u(\tau)|^2d\tau<\infty\Big\}\to L^2(\mathbb{R}+ih), \tag{3.7}$$

for $\xi=s+ih$, where $h=$ constant, $\mathbb{R}+ih=\Big\{\xi\in\mathbb{C}:\operatorname{Im}\xi=h\Big\}$. Therefore, the subsequent Parseval identity holds

$$\int_{-\infty}^{\infty}e^{2h\tau}\,|u(\tau)|^2\,d\tau=\int_{-\infty+ih}^{\infty+ih}|\hat{u}(\xi)|^2\,d\xi. \tag{3.8}$$

We have

$$\mathcal{F}\big(\frac{d^m}{d\tau^m}u(\tau)\big)(\xi)=(i\xi)^m\mathcal{F}\big(u(\tau)\big)(\xi). \tag{3.9}$$

Moreover, it is noted that if $h_1 < h_2$ and the following properties are satisfied

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{2h_1\tau}\left|u(\tau)\right|^2 d\tau < \infty,\\ \int_{-\infty}^{+\infty} e^{2h_2\tau}\left|u(\tau)\right|^2 d\tau < \infty,\end{aligned} \tag{3.10}$$

then $\hat{u}(\xi)$ is holomorphic in the strip $h_1 < \operatorname{Im}\xi < h_2$.

Now, by applying (3.5) to (3.4) with respect to τ, the parametric boundary value problem for the unknown function $\hat{u}$ is obtained that depend on the complex parameter ξ and holds in the interval $I = (0, \omega_0)$. Consequently, the transformed form of (3.4) is

$$\begin{aligned}\xi^2\hat{u} - \frac{\partial^2\hat{u}}{\partial\theta^2} =& \hat{F}(\xi,\theta),\\ \hat{u}(\xi,\theta)\big|_{\theta=0,\,\omega_0} =& 0,\\ \frac{\partial\hat{u}}{\partial\theta}(\xi,\theta)\big|_{\theta=0,\,\omega_0} =& \hat{G}(\xi,\theta)\big|_{\theta=0,\,\omega_0}.\end{aligned} \tag{3.11}$$

Let $\hat{\mathcal{V}}(\xi)$ represent the operator of (3.11) and it maps from $\hat{\mathcal{V}}(\xi) : W^{2,\,2}(0,\omega_0)$ into $L^2(0,\omega_0)\times\mathbb{C}\times\mathbb{C}$. Note that the operator $\hat{\mathcal{V}}(\xi)$ can be defined for every boundary point in the sense of [2, 3]. So, the operator $\hat{\mathcal{V}}(\xi)(\xi,\theta) = 0$ is used to describe a generalized eigenvalue problem and the solvability of these type of problems is discussed in [24]. The operator $\hat{\mathcal{V}}(\xi)$ realizes an isomorphism for all $\xi \in \mathbb{C}$ apart from some isolated points (known as the eigenvalues of $\hat{\mathcal{V}}(\xi)$). So, the resolvent $\mathcal{R}(\xi) = \left[\hat{\mathcal{V}}(\xi)\right]^{-1}$ is an operator-valued, meromorphic function of ξ has poles of finite multiplicity.

To compute the eigenvalues ξ_μ (generally referred for multiple eigenvalues) and the corresponding eigenfunctions, we proceed as.

Definition 3.1. *A complex number $\xi = \xi_0$ is known as the eigenvalue of $\hat{\mathcal{V}}(\xi)$ if there exists a nontrivial solution which is holomorphic at ξ_0, i.e., $\hat{u}(.,\xi_0) \neq 0$, and $\hat{\mathcal{V}}(\xi_0)\,\hat{u}(\theta,\xi_0) = 0$, where $\hat{u}(\theta,\xi_0)$ is an eigenfunction of $\hat{\mathcal{V}}(\xi_0)$ corresponding to the eigenvalue ξ_0. The set of fields $\left\{\hat{u}_0(\theta,\xi_0),\, \hat{u}_{0,1}(\theta,\xi_0), ...,\, \hat{u}_{0,s}(\theta,\xi_0)\right\}$ with $\hat{u}_{0,0} = \hat{u}_0$ is said to be a Jordan chain corresponding to the eigenvalue ξ_0, if the equation*

$$\sum_{q=0}^{\sigma}\frac{1}{q!}\Big(\frac{\partial}{\partial\xi}\Big)^q\,\hat{\mathcal{V}}(\xi)\,\hat{u}_{0,m-q}(\theta,\xi)\Big|_{\xi=\xi_0} = 0 \quad for \quad m = 1, 2, ..., s,$$

is satisfied. The number $s+1$ is called the length of the Jordan chain.

Remark 3.1. *It is noted from [22–24] that if the complex number ξ is not an eigenvalue of the operator $\hat{\mathcal{V}}(\xi)$, then $\hat{\mathcal{V}}(\xi)$ is an isomorphism among the spaces $\hat{\mathcal{V}}(\xi) : W^{2,\,2}(0,\omega_0)$ and $L^2(0,\omega_0)\times\mathbb{C}\times\mathbb{C}$.*

Theorem 3.1. *Let $l_h = \{\xi \in \mathbb{C} : \operatorname{Im}\xi = h\}$. If no eigenvalues of $\hat{\mathcal{V}}(\xi)$ lies on the line l_h, then the system (3.11) admits a unique solution $\hat{u} \in W^{2,2}(0,\omega_0)$ provided $(\hat{F}, 0, \hat{G}) \in L^2(0,\omega_0)\times\mathbb{C}\times\mathbb{C}$, and it holds for all*

$\xi \in l_h$:

$$\|\hat{u}\|^2_{W^{2,2}(0,\omega_0)} \leq c\Big\{\|\hat{F}\|^2_{L^2(0,\omega_0)} + |\xi||\hat{G}|^2\Big\}, \tag{3.12}$$

with the constant c is independent of ξ.

Proof. A similar theorem is proved in ([17], Theorem 4.9). So, we omit its proof. □

Taking note from the above-mentioned results, and [22,23,28], we can derive a fundamental regularity and expansion theorem, based on the Fourier transform, for the mixed boundary value problem for the Poisson equation in a two-dimensional bounded domain with singular points on the boundary. By considering the substitution $\operatorname{Re}\alpha = -\operatorname{Im}\xi - 2$ for $\alpha \in \mathbb{C}$, it improves theorems ([24], Theorem 8.2.1 and Theorem 8.2.2.) which are based on the Mellin transform technique and used for the solvability of the elliptic systems.

Theorem 3.2. ***(Regularity and expansion theorem)****. Let α_1 and α_2 be real numbers and satisfying $\alpha_1 - 1 < \alpha_2 < \alpha_1$. Let $v \in \mathcal{W}^{m,2}_{\alpha_1}(\mathcal{D})$ be a solution of the mixed boundary value problem (1.1) and $f \in \mathcal{W}^{m_1-2,p}_{\alpha_2}(\mathcal{D}) \cap \mathcal{W}^{m-2,2}_{\alpha_1}(\mathcal{D})$, where $1 \leq p < \infty$, $m_1 \geq m \geq 2$ and $\alpha_1 \geq \alpha_2 \geq 0$. Subsequently, the following implications holds:*

(1) *If the strip $\alpha_2 + \frac{2}{p} - m_1 \leq \operatorname{Im}\xi \leq \alpha_1 + 1 - m$, is free of eigenvalues of the operator $\hat{\mathcal{V}}(\xi)$, then the solution $v \in \mathcal{W}^{m_1,p}_{\alpha_2}(\mathcal{D})$ and holds the following estimate*

$$\|u\|_{\mathcal{W}^{m_1,p}_{\alpha_2}(\mathcal{D})} \leq c(\mathcal{D})\, \|f\|_{\mathcal{W}^{m_1-2,p}_{\alpha_2}(\mathcal{D})}.$$

(2) *Let $\xi_1, \xi_2, ..., \xi_M$ be the eigenvalues of the operator $\hat{\mathcal{V}}(\xi)$ and suppose that no eigenvalue lie on the lines $\operatorname{Im}\xi = \alpha_2 + \frac{2}{p} - m_1$ and $\operatorname{Im}\xi = \alpha_1 + 1 - m$. If the eigenvalues $\xi_1, \xi_2, ..., \xi_M$ are situated in the strip $\alpha_2 + \frac{2}{p} - m_1 < \operatorname{Im}\xi < \alpha_1 + 1 - m$, then the solution v admits the following expansion in the neighborhood P_δ of the corner point P, i.e.,*

$$v = \chi(r)\Big[\sum_{\mu=1}^{M}\sum_{\sigma=1}^{I_\mu}\sum_{\kappa=0}^{\kappa_{\mu\sigma}-1} c_{\mu,\sigma,\kappa}\,\Phi_{\mu,\sigma,\kappa}(r,\theta)\Big] + v_r(r,\theta), \tag{3.13}$$

where $v_r(r,\theta) \in \mathcal{W}^{m_1,p}_{\alpha_2}(P_\delta)$. Here, we set M be the number of eigenvalues of the operator $\hat{\mathcal{V}}(\xi)$ in the strip, the constants $c_{\mu,\sigma,\kappa}$ depends on the data and the singular functions, $I_\mu = \dim\operatorname{Ker}\hat{\mathcal{V}}(\xi_\mu)$ represents the geometrical multiplicity of ξ_μ, $\kappa_{\mu\sigma}$ is the length of the Jordan chains of $\hat{\mathcal{V}}(\xi_\mu)$, and the corresponding singular function is described as

$$\Phi_{\mu,\sigma,\kappa}(r,\theta) = r^{i\xi_\mu}\sum_{j=0}^{\kappa}\frac{(i\,\log r)^j}{j!}\,\psi^{\sigma,\kappa-j}_{\mu}(\theta), \tag{3.14}$$

where $\psi^{\sigma,\kappa-j}_{\mu}(\theta)$ is a canonical system of Jordan chains of $\hat{\mathcal{V}}(\xi)$ respecting ξ_μ. It is noted from (3.13) and (3.14) that the eigenvalues $\xi_\mu = 0$ does not yield singularities in the development of the solution in the neighborhood P_δ.

It is recognized for elliptic boundary value problems that the eigenvalues of the operator $\hat{\mathcal{V}}(\xi)$ which lies in the strip have a significant role in the regularity results. The assertions 1 and 2 of Theorem 3.2 represent the regularity and the expansion of the solution of the system (1.1) near the singular points.

3.2. **The calculation of the eigenvalues.** In this section, the eigenvalues and eigenfunctions of the boundary value problem (3.11) with relationships of various boundary conditions are evaluated. Generally, no unique solution exists for different boundary conditions, and the multiple solutions are marked with an index μ or l. The computed eigenvalues ξ_l and corresponding eigenfunctions $\Phi_l(\theta)$ of the considered problem with various conditions are defined as follows.

Dirichlet boundary conditions (DD)

For Dirichlet boundary conditions, the eigenvalues of the operator $\hat{\mathcal{V}}(\xi)$ are $\xi_l = i\,\frac{l\pi}{\omega_0}$, $l = \pm1, \pm2, ...$, and the corresponding eigenfunctions are $\Phi_l(\theta) = \sin\frac{l\pi}{\omega_0}\,\theta$.

Neumann boundary conditions (NN)

For Neumann boundary conditions, the eigenvalues of the operator $\hat{\mathcal{V}}(\xi)$ are $\xi_l = i\,\frac{l\pi}{\omega_0}$, $l = 0, \pm1, \pm2, ...$, and the corresponding eigenfunctions are $\Phi_l(\theta) = \cos\frac{l\pi}{\omega_0}\,\theta$.

Mixed boundary conditions (ND)

Similar to the latter cases of the boundary conditions, the eigenvalues of the corresponding operator $\hat{\mathcal{V}}(\xi)$ are $\xi_l = i(l + \frac{1}{2})\,\frac{\pi}{\omega_0}$, $l = 0, \pm1, \pm2, ...$, and the eigenfunctions are $\Phi_l(\theta) = \cos(l + \frac{1}{2})\,\frac{\pi}{\omega_0}\,\theta$.

Remark 3.2. *It is noted that if the versed boundary conditions are used which means that the Dirichlet condition is at $\theta = 0$ and the Neumann condition is at $\theta = \omega_0$, then the same eigenvalues of $\hat{\mathcal{V}}(\xi)$ like the mixed boundary conditions (ND) are obtained but the corresponding eigenfunctions are $\Phi_l(\theta) = \sin(l+\frac{1}{2})\,\frac{\pi}{\omega_0}\,\theta$.*

4. The Regularity results

In this section, the regularity results and the expansion of the solution u or v of the boundary value problem (3.2) or (1.1) are defined.

To analyze the regularity results of the boundary value problem (3.11), the combinations of the boundary points with different boundary conditions are considered. First of all, it is to be determined that the right-hand sides functions in (3.4) are Fourier transform in the sense of (3.7). We know from (3.2) that $F \in L^2(S)$, and further note that for all $\alpha \geq 0$, $F \in \mathcal{W}^{0,\,2}_{\alpha}(S)$. Since, $F \in \mathcal{W}^{0,\,2}_{\alpha}(S)$, we have

$$\int_S |F(\mathrm{x})|^2\,|\mathrm{x}|^{2\,\alpha}\,d\mathrm{x} = \int_{\bar{S}} e^{2(\tau\alpha+\tau)}|\tilde{F}(\tau,\,\theta)|^2 d\tau\,d\theta < \infty, \tag{4.1}$$

where $h = \alpha - 1$ for all $\alpha \geq 0$ and it is meaningful according to (3.7). Consequently, the Fourier transform of $\tilde{F}(\tau, \theta)$ is meaningful in the half plane $h = \operatorname{Im}\xi \geq -1$ for almost all $\theta \in (0,\,\omega_0)$.

The following regularity results of the boundary value problem (3.2) for various combinations of the boundary conditions are achieved as a direct consequence of Theorem 3.2 and the contemplations in Section 3.

Dirichlet boundary conditions (DD)

Let $\hat{\mathcal{V}}(\xi)$ denote the operator of the problem (3.11) for the Dirichlet-Dirichlet boundary conditions (DD) and $\hat{\mathcal{V}}(\xi) : W^{2,2}(0,\omega_0) \to L^2(0,\omega_0) \times \mathbb{C} \times \mathbb{C}$. If ξ is no eigenvalue of $\hat{\mathcal{V}}(\xi)$, then for any $\hat{F} \in L^2(0,\omega_0)$ a unique weak solution $\hat{u}$ of (3.11) exists. We write

$$\hat{u} = \hat{\mathcal{V}}^{-1}(\xi)[\hat{F}, 0, 0], \tag{4.2}$$

where $\hat{\mathcal{V}}^{-1}(\xi)$ represent the inverse (or resolvent) operator and $\hat{\mathcal{V}}^{-1}(\xi) : L^2(0,\omega_0) \times \mathbb{C} \times \mathbb{C} \to W^{2,2}(0,\omega_0)$. Moreover, the inverse Fourier transform of $\hat{u}$ produces the solution $\tilde{u}(\tau,\theta) = u(x,y)$ of (3.2) and the subsequent regularity result holds.

If no eigenvalues of $\hat{\mathcal{V}}(\xi)$ are lie on the line $h = \operatorname{Im}\xi = \alpha - 1$, $\alpha \geq 0$, then the inverse Fourier transform which can be read as follows in formula (3.6) exists and $\tilde{u}_h(\tau,\theta) = u_h(\mathrm{x}) \in \mathcal{W}^{2,2}_{\alpha}(S)$. Further, $u_h(\mathrm{x})$ be the unique solution of (3.2) from $\mathcal{W}^{2,2}_{\alpha}(S)$. It follows from the theory of Kondratiev in [22, 23], a regularity result yields that $u \in \mathcal{W}^{2,2}_{1}(S)$. Therefore, we have $u(\mathrm{x}) = u_0(\mathrm{x})$.

Let us derive an expansion of the solution $u(\mathrm{x})$ in S, the main question is the inverse Fourier transformation of the right-hand sides of (3.11) which can be read as follows

$$\tilde{u}_h(\tau,\theta) = (2\pi)^{-\frac{1}{2}} \int_{-\infty+ih}^{\infty+ih} e^{i\xi\tau}\, \hat{\mathcal{V}}^{-1}(\xi)[\hat{F},0,0]\, d\xi. \tag{4.3}$$

Using the Cauchy theorem, yields

$$\begin{aligned}\tilde{u}_h(\tau,\theta) = (2\pi)^{-\frac{1}{2}} \lim_{n\to\infty} \bigg\{ &\int_{-n+ih}^{-n+i\delta} e^{i\xi\tau}\, \hat{\mathcal{V}}^{-1}(\xi)[\hat{F},0,0]\, d\xi + \int_{-n+i\delta}^{n+i\delta} e^{i\xi\tau}\, \hat{\mathcal{V}}^{-1}(\xi)[\hat{F},0,0]\, d\xi \\ &+ \int_{n+i\delta}^{n+ih} e^{i\xi\tau}\, \hat{\mathcal{V}}^{-1}(\xi)[\hat{F},0,0]\, d\xi \bigg\} + \frac{1}{\sqrt{2\pi}} 2\pi i \sum_{\substack{-1<\operatorname{Im}\xi<0 \\ -n<\operatorname{Re}<n}} Res\left(e^{i\xi\tau}\, \hat{\mathcal{V}}^{-1}(\xi)[\hat{F},0,0]\right).\end{aligned} \tag{4.4}$$

As $n \to \infty$, the first integral and the third integral tend to zero by [22] and the second integral yields $w_h(\mathrm{x}) \in \mathcal{W}^{2,2}_{\alpha}(S)$, the calculation of the residue gives the singular terms. If the eigenvalues of the operator $\hat{\mathcal{V}}(\xi)$ does not lie on the line $\operatorname{Im}\xi = \alpha - 1$, $\forall\, \alpha \geq 0$, then the residue vanishes and the inverse Fourier transform exist.

Now, we ought to calculate the residue of $e^{i\xi\tau}\, \hat{\mathcal{V}}^{-1}(\xi)[\hat{F},0,0]$ and therefore to look at the singularities of (4.2). We know that $\hat{\mathcal{V}}^{-1}(\xi)$ is an operator-valued, meromorphic function of ξ with poles at $\xi_l = i\,\frac{l\pi}{\omega_0}$, $l = \pm 1, \pm 2, \ldots$. So as to calculate the residuum, the only poles ξ_l that lies in $-1 < \operatorname{Im}\xi_l < 0$ plays a significant role in the regularity results. Hence, if $\omega_0 > \pi$, then $\xi_{-1} = -i\,\frac{\pi}{\omega_0}$ lies in this strip.

Further from [26], the operator $\hat{\mathcal{V}}^{-1}(\xi)$ has the following expansion in the neighborhood of ξ_{-1}, i.e.,

$$\hat{\mathcal{V}}^{-1}(\xi) = \frac{q_1}{(\xi - \xi_{-1})} + \Gamma(\xi), \tag{4.5}$$

where q_1 and $\Gamma(\xi)$ map $L^2(0,\omega_0)\times\mathbb{C}\times\mathbb{C}$ into $W^{2,2}(0,\omega_0)$. The operator q_1 behaves as the space of eigenfunctions of $\hat{\mathcal{V}}(\xi)$ corresponding to ξ_{-1} and $\Gamma(\xi)$ is holomorphic. Moreover, (3.10) and (4.1) imply that $\hat{F}(\xi,\theta)$ is holomorphic respecting ξ in the strip $-1<\operatorname{Im}\xi<0$. Hence, we can write

$$[\hat{F},0,0]=\sum_{m=0}^{\infty} b_m(\theta)\,(\xi-\xi_{-1})^m, \tag{4.6}$$

in a neighborhood of ξ_{-1}, where the coefficients $b_m(\theta)$ are elements of $L^2(0,\omega_0)\times\mathbb{C}\times\mathbb{C}$. Finally, we have

$$e^{i\xi\tau}=e^{i\xi_{-1}}\big[1+i(\xi-\xi_{-1})\tau+...+\frac{[i(\xi-\xi_{-1})\tau]^m}{m!}+...\big]. \tag{4.7}$$

From (4.5)-(4.7), it follows that

$$e^{i\xi\tau}\,\hat{\mathcal{V}}^{-1}(\xi)\big[\hat{F},0,0\big]=e^{i\xi_{-1}\tau}\big[1+...\big]\sum_{m=0}^{\infty}\big[q_1 b_m(\theta)\,\frac{(\xi-\xi_{-1})^m}{(\xi-\xi_{-1})}+\Gamma(\xi)b_m(\theta)\,(\xi-\xi_{-1})^m\big]. \tag{4.8}$$

We conclude that

$$\begin{aligned} Res\left[e^{i\xi\tau}\,\hat{\mathcal{V}}^{-1}(\xi)\big[\hat{F},0,0\big]\right]\Big|_{\xi=\xi_{-1}} &= e^{i\xi_{-1}\tau}q_1 b_0(\theta),\\ &= e^{\frac{\pi\tau}{\omega_0}}\,\mathcal{C}_1\sin\frac{\pi}{\omega_0}\,\theta, \end{aligned} \tag{4.9}$$

where $\mathcal{C}_1$ is a complex constant and the formulas for its precise computation can be found in [7, 8, 20]. For $\omega_0>\pi$, (4.4) yields

$$u(\mathrm{x})=\tilde{u}_0(\tau,\,\theta)=e^{\frac{\pi\tau}{\omega_0}}\,\mathcal{C}_1\sin\frac{\pi}{\omega_0}\,\theta+w(\mathrm{x}), \tag{4.10}$$

where $w(\mathrm{x})\in\mathcal{W}_0^{2,\,2}(S)$ and $u(\mathrm{x})\in\mathcal{W}_0^{1,\,2}(S)$ is the solution of the boundary value (3.2). Now, substituting $r=e^{\tau}$, we get

$$u(\mathrm{x})=\tilde{u}_0(\tau,\,\theta)=\mathcal{C}_1\,r^{\frac{\pi}{\omega_0}}\,\sin\frac{\pi}{\omega_0}\,\theta+w(\mathrm{x}). \tag{4.11}$$

If $\omega_0\le\pi$, then $u(\mathrm{x})=w(\mathrm{x})\in\mathcal{W}_0^{2,\,2}(S)$.

Neumann boundary conditions (NN)

Let P_i be a boundary point at which the Neumann-Neumann (NN) conditions appear. Using the same approach which is used for the Dirichlet-Dirichlet conditions, the following Fourier transformed form of (3.2) is obtained as

$$\begin{aligned} \xi^2\hat{u}-\frac{\partial^2\hat{u}}{\partial\theta^2} &=\hat{F}(\xi,\theta),\\ \frac{\partial\hat{u}}{\partial\theta}(\xi,0) &=\hat{G}(\xi,0),\\ \frac{\partial\hat{u}}{\partial\theta}(\xi,\omega_0) &=\hat{G}(\xi,\omega_0). \end{aligned} \tag{4.12}$$

Let $\hat{\mathcal{V}}(\xi)$ denote the operator of the problem (4.12) for the Neumann-Neumann conditions (NN) and $\hat{\mathcal{V}}(\xi) : W^{2,2}(0,\omega_0) \to L^2(0,\omega_0) \times \mathbb{C} \times \mathbb{C}$. If ξ is no eigenvalue of $\hat{\mathcal{V}}(\xi)$, then for any $\hat{F} \in L^2(0,\omega_0)$ a unique weak solution $\hat{u}$ of (4.12) exists. We write

$$\hat{u} = \hat{\mathcal{V}}^{-1}(\xi)[\hat{F}, \hat{G}(0), \hat{G}(\omega_0)], \tag{4.13}$$

where $\hat{\mathcal{V}}^{-1}(\xi)$ represent the inverse (or resolvent) operator and $\hat{\mathcal{V}}^{-1}(\xi) : L^2(0,\omega_0) \times \mathbb{C} \times \mathbb{C} \to W^{2,2}(0,\omega_0)$. Besides, the inverse Fourier transform of $\hat{u}$ yields the solution $\tilde{u}(\tau,\theta) = u(x,y)$ of (3.2) and the subsequent regularity result holds.

If no eigenvalues of $\hat{\mathcal{V}}(\xi)$ are lie on the line $h = \operatorname{Im}\xi = \alpha - 1$, $\alpha \geq 0$, then the inverse Fourier transform which can be read as follows in formula (3.6) exists and $\tilde{u}_h(\tau,\theta) = u_h(\mathrm{x})$ is the unique solution of (4.12) from $\mathcal{W}^{2,2}_{\alpha}(S)$. It follows from the theory of Kondratiev in [22, 23], a regularity result yields that $u \in \mathcal{W}^{2,2}_{\gamma+1}(S)$ where γ is a small positive real number.

To derive an expansion of the solution $u(\mathrm{x})$ in S, where $v \in W^{1,2}(\mathcal{D})$ is the unique weak solution of the boundary value problem (1.1). The main question is the inverse Fourier transformation of the right-hand sides of (4.12) which can be read as follows

$$u(\mathrm{x}) = u_\gamma(\mathrm{x}) = (2\pi)^{-\frac{1}{2}} \int_{-\infty+i\gamma}^{\infty+i\gamma} e^{i\xi\tau}\, \hat{\mathcal{V}}^{-1}(\xi)[\hat{F}, \hat{G}(0), \hat{G}(\omega_0)]\, d\xi. \tag{4.14}$$

The integral (4.14) can be calculated in the same way by considering the Cauchy theorem and the approach used for calculating the regularity results of Dirichlet boundary conditions.

Hence, we conclude that for $\omega_0 > \pi$, the following expansion of the solution of the boundary value (3.2) is obtained

$$u(\mathrm{x}) = \mathcal{C}_1 + \mathcal{C}_2\, e^{\frac{\pi\tau}{\omega_0}}\, \cos\frac{\pi}{\omega_0}\theta + w(\mathrm{x}), \tag{4.15}$$

where $w(\mathrm{x}) \in \mathcal{W}^{2,2}_0(S)$ and $u(\mathrm{x}) \in \mathcal{W}^{1,2}_0(S)$ is the solution of the boundary value (3.2). Now, substituting $r = e^{\tau}$, we get

$$u(\mathrm{x}) = \mathcal{C}_1 + \mathcal{C}_2\, r^{\frac{\pi}{\omega_0}}\, \cos\frac{\pi}{\omega_0}\theta + w(\mathrm{x}). \tag{4.16}$$

Mixed boundary conditions (ND)

Let P_i be a boundary point at which the Neumann-Dirichlet conditions (ND) appear. Using the same approach which is used for the latter cases, i.e., (Dirichlet and Neumann) conditions, the following Fourier transformed form of (3.2) is obtained

$$\begin{aligned} \xi^2\hat{u} - \frac{\partial^2\hat{u}}{\partial\theta^2} &= \hat{F}(\xi,\theta), \\ \frac{\partial\hat{u}}{\partial\theta}(\xi,0) &= \hat{G}(\xi,0), \\ \hat{u}(\xi,\omega_0) &= 0. \end{aligned} \tag{4.17}$$

Let $\hat{\mathcal{V}}(\xi)$ denote the operator of the problem (4.17) and $\hat{\mathcal{V}}(\xi) : W^{2,2}(0,\omega_0) \to L^2(0,\omega_0) \times \mathbb{C} \times \mathbb{C}$. If ξ is no eigenvalue of the operator $\hat{\mathcal{V}}(\xi)$, then for any $\hat{F} \in L^2(0,\omega_0)$ a unique weak solution $\hat{u}$ of (4.17) exists. We can write

$$\hat{u} = \hat{\mathcal{V}}^{-1}(\xi)[\hat{F}, \hat{G}(0), 0)], \tag{4.18}$$

where $\hat{\mathcal{V}}^{-1}(\xi)$ represent the resolvent operator and $\hat{\mathcal{V}}^{-1}(\xi) : L^2(0,\omega_0) \times \mathbb{C} \times \mathbb{C} \to W^{2,2}(0,\omega_0)$. Further, the inverse Fourier transform of $\hat{u}$ yields the solution $\tilde{u}(\tau,\theta) = u(x,y)$ of (3.2) and the subsequent regularity result holds.

Again we have, if no eigenvalues of $\hat{\mathcal{V}}(\xi)$ lie on the line $h = \operatorname{Im}\xi = \alpha - 1$, $\alpha \geq 0$, then the inverse Fourier transform which can be read as follows

$$\tilde{u}_h(\tau,\,\theta) = (2\pi)^{-\frac{1}{2}} \int_{-\infty+ih}^{\infty+ih} e^{i\xi\tau}\, \hat{\mathcal{V}}^{-1}(\xi)[\hat{F}, \hat{G}(0), 0]\, d\xi = u_h(\mathrm{x}) \in \mathcal{W}_\alpha^{2,\,2}(S), \tag{4.19}$$

exists and $u_h(\mathrm{x})$ is the uniquely determined solution from $\mathcal{W}_\alpha^{2,\,2}(S)$ of (3.2) for mixed conditions (ND). From [22, 23], a regularity result yields that $u \in \mathcal{W}_{\gamma+1}^{2,\,2}(S)$ where γ is a sufficiently small positive real number.

To derive an expansion of the solution $u(\mathrm{x})$ in S, the main question is the inverse Fourier transformation of the right-hand sides of (4.17) which can be read as

$$u(\mathrm{x}) = u_\gamma(\mathrm{x}) = (2\pi)^{-\frac{1}{2}} \int_{-\infty+i\gamma}^{\infty+i\gamma} e^{i\xi\tau}\, \hat{\mathcal{V}}^{-1}(\xi)[\hat{F}, \hat{G}(0), 0]\, d\xi. \tag{4.20}$$

The integral (4.20) can be calculated using the Cauchy theorem as same in (4.4) and the approach used for calculating the regularity results of the latter conditions. Moreover, the rectangle choosing here have the corner points $-n+i\gamma$, $-n-i$, $n-i$, $n+i\gamma$. Since, we have the eigenvalues $\xi_l = i(l+\frac{1}{2})\,\frac{\pi}{\omega_0}$, $l = 0, \pm 1, \pm 2, ...,$. For $\omega_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$, we have $\xi_{-1} = (\frac{-i}{2})\,\frac{\pi}{\omega_0}$ and $\omega_0 > \frac{3\pi}{2}$ yield $\xi_{-2} = (\frac{-3i}{2})\,\frac{\pi}{\omega_0}$. Let these eigenvalues lie in the rectangle and the following expansion of the solution of the boundary value (3.2) is obtained

$$\begin{aligned} u(\mathrm{x}) &= \mathcal{C}_1\, e^{\frac{\pi\tau}{2\omega_0}}\, \cos\frac{\pi}{2\omega_0}\,\theta + w(\mathrm{x}), \quad \text{for} \quad \omega_0 \in (\frac{\pi}{2}, \frac{3\pi}{2}), \\ u(\mathrm{x}) &= \mathcal{C}_1\, e^{\frac{\pi\tau}{2\omega_0}}\, \cos\frac{\pi}{2\omega_0}\,\theta + \mathcal{C}_2\, e^{\frac{3\pi\tau}{2\omega_0}}\, \cos\frac{3\pi}{2\omega_0}\,\theta + w(\mathrm{x}), \quad \text{for} \quad \omega_0 > \frac{3\pi}{2}, \end{aligned} \tag{4.21}$$

where $w(\mathrm{x}) \in \mathcal{W}_0^{2,\,2}(S)$ and $u(\mathrm{x}) \in \mathcal{W}_0^{1,\,2}(S)$ is the solution of the boundary value (3.2). Now, substituting $r = e^\tau$, we get

$$\begin{aligned} u(\mathrm{x}) &= \mathcal{C}_1\, r^{\frac{\pi}{2\omega_0}}\, \cos\frac{\pi}{2\omega_0}\,\theta + w(\mathrm{x}), \quad \text{for} \quad \omega_0 \in (\frac{\pi}{2}, \frac{3\pi}{2}), \\ u(\mathrm{x}) &= \mathcal{C}_1\, r^{\frac{\pi}{2\omega_0}}\, \cos\frac{\pi}{2\omega_0}\,\theta + \mathcal{C}_2\, r^{\frac{3\pi}{2\omega_0}}\, \cos\frac{3\pi}{2\omega_0}\,\theta + w(\mathrm{x}), \quad \text{for} \quad \omega_0 > \frac{3\pi}{2}. \end{aligned} \tag{4.22}$$

Remark 4.1. *It is observed that if the versed boundary conditions are used which means that the Dirichlet condition is at $\theta = 0$ and the Neumann condition is at $\theta = \omega_0$, then the similar regularity results can be* *obtained like the mixed conditions (ND) but the eigenvalues and the corresponding eigenfunctions discussed in Remark 3.2 are used.*

4.1. **The regularity of the boundary value problem in a polygonal domain.** In Section 1, we have described that $\mathcal{D}$ is a polygonal domain and $\mathcal{N}$ denote the set of the boundary points which consists of $\{P_1, ..., P_N\} \subset \partial\mathcal{D}$. To investigate the regularity of the solution v of the boundary value problem (1.1) in $\mathcal{D}$, the following set of boundary points of $\mathcal{N}$ are considered. Let we denote

(1) $\mathcal{J}_1$ be the corresponding index set for the boundary points with the Dirichlet-Dirichlet boundary conditions (DD), where $\omega_i > \pi$,
(2) $\mathcal{J}_2$ be the corresponding index set for the boundary points with the Neumann-Neumann boundary conditions (NN), where $\omega_i > \pi$,
(3) $\mathcal{J}_1\mathcal{J}_2$ be the corresponding index set for the boundary points with the Neumann-Dirichlet boundary conditions (ND), where $\omega_i \in (\frac{\pi}{2}, \frac{3\pi}{2})$,
(4) $\mathcal{J}_2\mathcal{J}_1$ be the corresponding index set for the boundary points with the Neumann-Dirichlet boundary conditions (ND), where $\omega_i > \frac{3\pi}{2}$.

Let $v \in W^{1,2}(\mathcal{D})$ be the uniquely determined weak solution of (1.1). We investigate its regularity which can be written in the form

$$\begin{aligned} v = &\sum_{i\in\mathcal{J}_1} \chi_i^2 v + \sum_{i\in\mathcal{J}_2} \chi_i^2 v + \sum_{i\in\mathcal{J}_1\mathcal{J}_2} \chi_i^2 v + \sum_{i\in\mathcal{J}_2\mathcal{J}_1} \chi_i^2 v \\ &+ \big(1 - \sum_{i\in\mathcal{J}_1\cup\mathcal{J}_2\cup\mathcal{J}_1\mathcal{J}_2\cup\mathcal{J}_2\mathcal{J}_1} \chi_i^2 v\big), \end{aligned} \tag{4.23}$$

where the function χ is defined in (3.1) and $u = \chi v$. Using the expansions (4.11), (4.16) and (4.22) into (4.23), we get

$$\begin{aligned} v = &\sum_{i\in\mathcal{J}_1} \chi_i \mathcal{C}_i\, r_i^{\frac{\pi}{\omega_i}} \sin\frac{\pi}{\omega_i}\theta_i + \sum_{i\in\mathcal{J}_2} \chi_i \mathcal{C}_i\, r_i^{\frac{\pi}{\omega_i}} \cos\frac{\pi}{\omega_i}\theta_i + \sum_{i\in\mathcal{J}_1\mathcal{J}_2} \chi_i \mathcal{C}_i\, r_i^{\frac{\pi}{2\omega_i}} \cos\frac{\pi}{2\omega_i}\theta_i \\ &+ \sum_{i\in\mathcal{J}_2\mathcal{J}_1} \Big(\chi_i \mathcal{C}_i\, r_i^{\frac{\pi}{2\omega_i}} \cos\frac{\pi}{2\omega_i}\theta_i + \acute{\mathcal{C}}_i \chi_i\, r_i^{\frac{3\pi}{2\omega_i}} \cos\frac{3\pi}{2\omega_i}\theta_i\Big) + w(\mathrm{x}), \end{aligned} \tag{4.24}$$

where $w(\mathrm{x}) \in W^{2,2}(\mathcal{D})$. Let θ_i represents the locally variable angle, i.e., $0 < \theta_i < \omega_i$, whereas $\mathcal{C}_i$, $\acute{\mathcal{C}}_i$ are the singularity coefficients and their computations can be found in [7, 8, 20].

Finally, (4.24) completely describe the regularity of the solution v of the boundary value problem (1.1).

5. Conclusion

It is well-known from the theory of elliptic boundary value problems in domains with boundary irregularities, like corners, conic vertices, edges, and cracks, etc., the solution may exhibit singularities. Generally, the flows over corners usually change their behaviors and properties as a result of a rapid geometrical change in the shape. In this article, we have studied the boundary singularities and regularity of the weak solution of the mixed boundary value problem for the Poisson equation in a non-smooth domain with singular points on the boundary. The singular structure of the solution of the considered problem near the corner points is investigated through the Fourier transform and the suitable weighted Sobolev spaces that best characterize the singular behavior of the solution are presented. It is observed for Dirichlet and Neumann boundary conditions that if $\mathcal{D}$ has reentrant corners ($\omega_i > \pi : i = 1, 2, ...N$), then the weak solution $v \in W_0^{1,2}(\mathcal{D})$ of the considered problem has the form (4.11) and (4.16). If the domain $\mathcal{D}$ is a convex polygonal domain, then the solution $v \in W^{2,2}(\mathcal{D})$. For the mixed boundary conditions, the general solution is presented in the form of (4.22). Moreover, it is shown that the solution of the given problem can be decomposed into the singular and regular parts near the corner points for the values of $\omega_i \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $\omega_i > \frac{3\pi}{2}$ and does not belong locally to space H^2. Finally, (4.24) completely describe the regularity result of the original boundary value problem in a domain $\mathcal{D}$ with singular points on the boundary.

The results to be achieved here can be further extended to three-dimensional domains, for instance, polyhedral domain, etc. with straight edges to analyze the edge singularities and the regularity expansion of the solutions. Additionally, the technique to be presented here can be modified for investigating and treating numerous linear boundary value problems in two-dimensional domains with corners, such as Lame's equations, Stokes equations and so forth.

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