



RULED SURFACES WITH CONSTANT SLOPE RULING ACCORDING TO DARBOUX FRAME IN MINKOWSKI SPACE

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ABSTRACT. In this study, three different types of ruled surfaces are defined. The generating lines of these ruled surfaces are given by points on a curve X in Minkowski Space, while the position vector of X have constant slope with respect to the planes $(t, y), (t, n), (n, y)$. It is observed that the Lorentzian casual characters of the ruled surfaces with constant slope can be timelike or spacelike. Furthermore, striction lines of these surfaces are obtained and investigated under various special cases. Finally, new investigations are obtained on the base curve of these types of ruled surfaces.

1. Introduction

A ruled surface is a special surface which is formed by moving a line along a given curve in 3-dimensional Minkowski space. The line is called the generating line and the curve is called the direction curve of the surface. Thus, a ruled surface has a parametrization

$$M(u, v) = \alpha(u) + vX(u)$$

where α and X are curves. The curve α is called the directrix or base curve and X is called the director curve of the ruled surface. Thus, the ruled surfaces in Minkowski space can be classified according to the Lorentzian character of their ruling and surface normal. Developable ruled surfaces are surfaces which can

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be made isometric to part of the plane. The necessary and sufficient conditions for these surfaces to become developable are characterized by vanishing Gaussian curvature. In this study “developable” and “torsal” are used as synonyms, since a surface is developable if and only if it is a torsal ruled surface. Cylindrical, conical, torse surfaces, a plane and surfaces of polyhedrons are examples of torsal surfaces. These surfaces can be developed on a plane without any lap break. On the ground that their isometrics with planes are becoming interesting to discover more ways to use these surfaces in different applications in [5].

A regular curve in E_1^3 , whose position vector is obtained by Frenet frame vectors on another regular curve, is called Smarandache curve [1]. In this study, it is shown that if developable surface's generating line is a Smarandache curve and asymptotic or geodesic curve, then the basic curve is a general helix.

K. Malecek and others defined the surfaces with a constant slope with respect to the given surface in Euclidean Space in [7]. At the same time in [10], Yavuz, Ateş and Yaylı investigated surface with a constant slope ruling with respect to osculating plane by using Frenet Frame according to casual characters in Minkowski space. By the definition of surfaces with a constant slope ruling with respect to the given surface in study [7], in this study new surface definitions are obtained. These surfaces in three types were studied according to the Darboux frame in Minkowski Space. Furthermore, necessary and sufficient conditions are given for these surfaces to become developable in Minkowski 3-space. Striction lines of the surfaces are obtained and investigated under various special cases. Finally, the ruled surfaces with constant slope ruling visualized of given curves as examples, separately.

2. Preliminaries

A tangent vector v on a semi-Riemannian manifold M is spacelike if $g(v, v) > 0$ or $v = 0$, is null if $g(v, v) = 0$ and $v \neq 0$, timelike if $g(v, v) < 0$. The norm of a tangent vector v is given by $|v| = \sqrt{|g(v, v)|}$. A curve in a manifold M is a smooth mapping $\alpha : I \rightarrow M$, where I is an open interval in the real line R . A curve α in a semi-Riemannian manifold M is spacelike if all of its velocity vectors $\alpha'(s)$ are spacelike, is null if all of its velocity vector $\alpha'(s)$ are null, timelike if all of its velocity vectors $\alpha'(s)$ are timelike [8].

In this study, the Darboux frames and formulas in the Minkowski space E_1^3 are given with metric

$$g = -dx_1^2 + dx_2^2 + dx_3^2.$$

Let S be an oriented surface in E_1^3 and let consider a non-null curve $\alpha(s)$ lying fully on S . Since the curve $\alpha(s)$ lies on the surface S there exists a frame along the curve $\alpha(s)$. This frame is called Darboux frame and denoted by $\{t, y, n\}$ which gives us an opportunity to investigate the properties of the curve according to the surface. In this frame t is the unit tangent of the curve, n is the unit normal of the surface S along curve $\alpha(s)$ and y is a unit vector given by $y = \mp n \times t$. According to the Lorentzian casual characters of the surface and the curve lying on surface, the derivative formulae of the Darboux frame can be changed as

follows:

i) If the surface is timelike, then the curve $\alpha(s)$ lying on surface can be spacelike or timelike. Thus, the derivative formulae of the Darboux frame is given by

$$\begin{bmatrix} t' \\ y' \\ n' \end{bmatrix} = \begin{bmatrix} 0 & k_g & -\varepsilon k_n \\ k_g & 0 & \varepsilon t_r \\ k_n & t_r & 0 \end{bmatrix} \begin{bmatrix} t \\ y \\ n \end{bmatrix}$$

$$\langle t, t \rangle = \varepsilon = \mp 1, \langle y, y \rangle = -\varepsilon, \langle n, n \rangle = 1$$

ii) If the surface is spacelike, then the curve $\alpha(s)$ lying on surface is spacelike. Thus, the derivative formulae of the Darboux frame is given by

$$\begin{bmatrix} t' \\ y' \\ n' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & t_r \\ k_n & t_r & 0 \end{bmatrix} \begin{bmatrix} t \\ y \\ n \end{bmatrix}$$

where

$$\langle t, t \rangle = 1, \langle y, y \rangle = 1, \langle n, n \rangle = -1$$

Here,

$$k_g(s) = \langle \alpha''(s), y(s) \rangle$$

is the geodesic curve, k_n is the normal curvature defined by equality

$$k_n(s) = \langle \alpha''(s), n(s) \rangle$$

and t_r is the geodesic torsion of $\alpha(s)$ defined by

$$t_r(s) = -\langle n'(s), y(s) \rangle$$

[2], [3], [4].

If $u, v \in E_1^3$, Lorentzian vector product of u and v is to the unique vector by $u \times v$ that satisfies

$$\langle u \times v, w \rangle = \det(u, v, w)$$

where $u \times v$ vector product is defined as follows

$$u \times v = \begin{vmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

[6].

The relations between geodesic curve, normal curvature, geodesic torsion and κ and τ are given, if both surface and curve are timelike or spacelike, then

$$k_g = \kappa \cos \theta$$

$$k_n = \kappa \sin \theta$$

if surface is timelike and curve is spacelike, then

$$k_g = \kappa \cosh \theta$$

$$k_n = \kappa \sinh \theta$$

[2], [3], [4].

Let \vec{x} and \vec{y} be future pointing (or past pointing) timelike vectors in R_1^3 . Then there is a unique real number $\theta > 0$ such that

$$\langle \vec{x}, \vec{y} \rangle = -\|\vec{x}\| \|\vec{y}\| \cos h\theta.$$

Let \vec{x} and \vec{y} be spacelike vectors in R_1^3 that span a timelike vector subspace. Then there is a unique real number $\theta > 0$ such that

$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \cos h\theta.$$

Let \vec{x} and \vec{y} be spacelike vectors in R_1^3 that span a spacelike vector subspace. Then there is a unique real number $\theta > 0$ such that

$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \cos \theta.$$

Let \vec{x} be a spacelike vector and \vec{y} be a timelike vector in R_1^3 . Then there is a unique real number $\theta > 0$, such that

$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \sinh \theta.$$

[9].

3. Ruled Surfaces with Constant Slope Ruling According to Darboux Frame in Minkowski 3-Space

Let M be a ruled surface whose generating lines are given by points on the curve X in Minkowski Space, while in all points they have the constant slope with respect to the tangent planes to the given surface. These surfaces will be called ruled surfaces with constant slope ruling with respect to the given surface in Minkowski Space. In this section, the developable conditions are investigated for the ruled surfaces with constant slope ruling with respect to the given surface with Darboux Frame $\{t, n, y\}$ and the striction lines of the surface are obtained. At the same time, various relations and special cases about developable conditions

and striction line of the surfaces are given.

3.1. Ruled Surfaces with Constant Slope Ruling with Respect to the (t, y) Planes. Generating lines of the surface M are given by points on the curve $X(s)$ and they have the constant slope σ with respect to the (t, y) planes to the curve at every point on the curve $X(s)$. The surface will be called the ruled surfaces with a constant slope ruling with respect to the (t, y) planes. We give the definition of a ruled surface with a constant slope ruling with respect to the (t, y) planes according to casual characters in following three cases in Minkowski space where

$$\langle X(s), n(s) \rangle = \sigma.$$

Case 3.1. If $\alpha(s)$ is a spacelike curve with the principal spacelike normal vector field $n(s)$, then surface is spacelike where $1 + \sigma^2 > 0$. Direction vector of generating line of the spacelike surface is given by

$$X(s) = \sin w(s).t(s) + \cos w(s).y(s) + \sigma n(s)$$

and the surface is parametrized by

$$M_1(s, v) = \alpha(s) + v(X(s)).$$

Case 3.2. If $\alpha(s)$ is a spacelike curve with the principal timelike normal vector field $n(s)$, then surfaces is spacelike where $\sigma^2 - 1 > 0$ and the surface is timelike where $\sigma^2 - 1 < 0$. Direction vector of generating line of the surface is given by

$$u(s) = \cosh w(s)t(s) + \sinh w(s)y(s) + \sigma n(s)$$

and the surface is parametrized by

$$M_2(s, v) = \alpha(s) + v(u(s)).$$

Case 3.3. If $\alpha(s)$ is a timelike curve with the principal spacelike normal vector field $n(s)$, then surface is timelike where $\sigma^2 + 1 > 0$ and direction vector is given as follows

$$\delta(s) = \sinh w(s)t(s) + \cosh w(s)y(s) + \sigma n(s)$$

so the timelike surface is parametrized by

$$M_3(s, v) = \alpha(s) + v(\delta(s))$$

where the vector t is the direction vector of a tangent to the curve X , n is the direction vector of a normal to the surface and $y = n \times t$ is the direction vector of intersection line of a tangent plane to the surface and the normal plane curve X at the point.

Theorem 3.1. *The spacelike surface $M_1(s, v)$ is developable if and only if*

$$\cos w(s) (\sin w(s) \cdot k_n + \cos w(s) \cdot t_r) = \sigma (\sin w(s) (k_g - w'(s)) + \sigma t_r).$$

Proof. The surface M_1 is developable if and only if $\det(t, X, X') = 0$. Thus derivative of the direction vector of generating lines of the surface is obtained as follows

$$\begin{aligned} X'(s) &= \left((w'(s) - k_g) \cos w(s) + \sigma k_n \right) t(s) + \left(- (w'(s) + k_g) \sin w + \sigma t_r \right) y(s) \\ &\quad + (k_n \sin w(s) + t_r \cos w(s)) \cdot n(s) \end{aligned}$$

$$\det(t, X, X') = \langle t \wedge X, X' \rangle.$$

And

$$\begin{aligned} &\langle t \wedge X, X' \rangle \\ &= \cos w(s) (\sin w(s) \cdot k_n + \cos w(s) \cdot t_r) - \sigma (\sin w(s) (k_g - w'(s)) + \sigma t_r) \end{aligned}$$

Thus developable condition for surface $M_1(s, v)$ is given by

$$\cos w(s) (\sin w(s) \cdot k_n + \cos w(s) \cdot t_r) = \sigma (\sin w(s) (k_g - w'(s)) + \sigma t_r).$$

□

Corollary 3.1. *Developable surface's generating line $X(s)$ is a asymptotic and Smarandache curve if and only if $\alpha(s)$ is a general helix with*

$$\frac{\tau}{\kappa} = \frac{\sigma x_1}{x_2^2 - \sigma^2}$$

where $\sin w(s) = x_1 = \text{const.}$ and $\cos w(s) = x_2 = \text{const.}$

Proof. If $X(s)$ is asymptotic curve, then $k_n = 0$

$$k_n = 0 \Rightarrow \cos^2 w(s) \cdot t_r = \sigma (\sin w(s) (k_g - w'(s)) + \sigma t_r)$$

and if surface and curve are the same character, then $\kappa^2 = k_n^2 + k_g^2$, so $k_g = \kappa$, $t_r = \tau$. If we replace the values in the last equation, we get the following equality

$$\frac{\tau}{\kappa} = \frac{\sigma x_1}{x_2^2 - \sigma^2}.$$

$M_1(s, v)$ developable surface's generating line $X(s)$ is a geodesic and Smarandache curve if and only if $\alpha(s)$

is a general helix, so that

$$\frac{\tau}{\kappa} = \frac{x_1 x_2}{\sigma^2 - x_2^2}$$

where $\sin w(s) = x_1 = \text{const.}$ and $\cos w(s) = x_2 = \text{const.}$

□

Corollary 3.2. $M_1(s, v)$ spacelike developable surface's generating line $X(s)$ is a line curvature

$$\frac{k_n}{k_g} = \frac{\sigma}{\cos w(s)}$$

where $w(s) = \text{const.}$

Theorem 3.2. The striction line on spacelike surface $M_1(s, v)$ is given by

$$\bar{\beta} = \alpha(s) - \frac{-\sigma k_n - (w'(s) - k_g) \cos w(s)}{(w'(s) - k_g) \begin{pmatrix} (w'(s) - k_g) (\sin^2 w(s) - \cos^2 w(s)) \\ -2\sigma (k_n \cos w(s) + t_r \sin w(s)) \end{pmatrix} + (k_n \sin w(s) + t_r \cos w(s))^2 + \sigma^2 (-k_n^2 + t_r^2)} X(s).$$

Remark 3.1. If $X(s)$ is a geodesic curve and $w(s) = -\sigma \int \kappa \sec w(s) d(s)$, then striction line of spacelike surface is equal to base curve.

Remark 3.2. If $X(s)$ is a asymptotic curve and $w(s) = \int \kappa d(s)$ where $\cos w(s) \neq 0$, then striction line of spacelike surface is equal to base curve.

Theorem 3.3. The spacelike surface $M_2(s, v)$ is developable if and only if

$$\sinh w(s) (\cosh w(s).k_n + \sinh w(s)t_r) - \sigma (\cosh w(s) (k_g + w'(s)) + \sigma t_r) = 0$$

where $\sigma^2 - 1 > 0$.

If $\sigma^2 - 1 < 0$, then the surface is timelike. So the timelike surface is developable if and only if

$$\varepsilon \sinh w(s) (\cosh w(s).k_n + \sinh w(s)t_r) - \sigma (\cosh w(s) (k_g + w'(s)) + \sigma t_r) = 0$$

where $\varepsilon = \langle t, t \rangle = \mp 1$.

Corollary 3.3. If $\sinh w(s) = x_3 = \text{const.}$ and $\cosh w(s) = x_4 = \text{const.}$, then generating lines of the surface $u(s)$ is a Smarandache curve. Let $M(s, v)$ be a spacelike developable surface and if $u(s)$ be a Smarandache and asymptotic curve, then $\alpha(s)$ is a general helix, so that

$$\frac{\tau}{\kappa} = \frac{\sigma x_4}{x_3^2 - \sigma^2}$$

if also $u(s)$ is a geodesic, then base curve is a general helix, so that

$$\frac{\tau}{\kappa} = \frac{x_3 x_4}{\sigma^2 - x_3^2}$$

Corollary 3.4. Let $M_2(s, v)$ be a spacelike developable surface and if $u(s)$ is a line and Smarandache curve,

$$\frac{k_n}{k_g} = \frac{\sigma}{x_3}$$

where $\sinh w(s) = x_3 = \text{const.}$ and $\cosh w(s) = x_4 = \text{const.}$

Corollary 3.5. If $\sinh w(s) = x_3 = \text{const.}$ and $\cosh w(s) = x_4 = \text{const.}$, generating lines of the surface $u(s)$ be a Smarandache curve. Let $M_2(s, v)$ be a timelike developable surface and $u(s)$ be a Smarandache curve, if at the same time $u(s)$ be a asymptotic, then base curve is a general helix, so that

$$\frac{\tau}{\kappa} = \frac{\sigma x_4}{\varepsilon x_3^2 - \sigma^2}$$

if also $u(s)$ is a geodesic, then base curve is a general helix, so that

$$\frac{\tau}{\kappa} = \frac{\varepsilon x_3 x_4}{\varepsilon x_3^2 - \sigma^2}$$

Proof. If $M_2(s, v)$ is a timelike developable surface and the base curve is spacelike, then $\kappa^2 = k_g^2 - k_n^2$. So we replace values $k_n = 0, k_g = \kappa, t_r = \tau$ in condition of developable equation

$$\varepsilon \sinh w(s) (\sinh w(s).t_r) = \sigma (\cosh w(s) (k_g) + \sigma t_r)$$

$$\varepsilon x_3 (x_3.\tau) = \sigma (x_4.\kappa + \sigma \tau)$$

$$\frac{\tau}{\kappa} = \frac{\sigma x_4}{\varepsilon x_3^2 - \sigma^2}$$

If $u(s)$ is a geodesic, then we write values $k_g = 0, k_n = -\kappa, t_r = \tau$ in condition of developable equation for the timelike surface, we obtained as follows

$$\varepsilon x_3 (x_4.(-\kappa) + x_3.\tau) = \sigma.(\sigma.\tau)$$

$$\frac{\tau}{\kappa} = \frac{\varepsilon x_3 x_4}{\varepsilon x_3^2 - \sigma^2}$$

□

Remark 3.3. Let $M_2(s, v)$ be a timelike developable surface and if $u(s)$ be a line and Smarandache curve,

$$\frac{k_n}{k_g} = \frac{\sigma}{\varepsilon x_3}$$

where $\sinh w(s) = x_3 = \text{const.}$ and $\cosh w(s) = x_4 = \text{const.}$

Theorem 3.4. *The striction line on spacelike surface $M_2(s, v)$ is given by*

$$\bar{\beta} = \alpha(s) - \frac{-\sigma k_n - (w'(s) - k_g) \sinh w(s)}{(w'(s) - k_g)^2 + \sigma^2 (-k_n^2 + t_r^2)} \cdot X(s) + (2\sigma (w'(s) - k_g) (-k_n \sinh w(s) + t_r \cosh w(s)) - (k_n \cosh w(s) + t_r \sinh w(s))^2)$$

where $\sigma^2 - 1 > 0$, and striction line on timelike surface is given by

$$\bar{\beta} = \alpha(s) - \frac{-\sigma k_n - (w'(s) - k_g) \sinh w(s)}{(w'(s) - k_g)^2 + \sigma^2 (-k_n^2 + t_r^2)} \cdot X(s) + (2\sigma (w'(s) - k_g) (-k_n \sinh w(s) + t_r \cosh w(s)) + \varepsilon^2 (k_n \cosh w(s) + t_r \sinh w(s))^2)$$

where $\sigma^2 - 1 < 0$, $\varepsilon = \langle t, t \rangle = \mp 1$.

Remark 3.4. *If $u(s)$ is a geodesic curve and*

$$w(s) = -\sigma \int \frac{\kappa}{\sinh w(s)} d(s)$$

where $\sigma^2 - 1 > 0$, then striction line of spacelike surface is equal to base curve.

Remark 3.5. *If $u(s)$ is a asymptotic curve and*

$$w(s) = \int \kappa d(s)$$

where $\sigma^2 - 1 > 0$ and $w(s) \neq 0$, then striction line of spacelike surface is equal to base curve.

Theorem 3.5. *The timelike surface $M_3(s, v)$ is developable if and only if*

$$\varepsilon \cosh w(s) (-\sinh w(s)k_n + \cosh w(s)t_r) - \sigma (\sinh w(s) (k_g + w'(s)) + \sigma t_r) = 0.$$

Remark 3.6. *If $\sinh w(s) = x_3 = \text{const.}$ and $\cosh w(s) = x_4 = \text{const.}$, generating lines of the surface $\delta(s)$ be a Smarandache curve. Let $M_3(s, v)$ be a developable timelike surface and $\delta(s)$ be a Smarandache curve, if at the same time $\delta(s)$ be a asymptotic, then the base curve is a general helix, so that*

$$\frac{\tau}{\kappa} = \frac{\sigma x_3}{\varepsilon x_4^2 - \sigma^2},$$

if also $\delta(s)$ be a geodesic, then the base curve is a general helix, so that

$$\frac{\tau}{\kappa} = \frac{-\varepsilon x_3 x_4}{\sigma^2 - \varepsilon x_4^2}.$$

Corollary 3.6. *If $u(s)$ be a line and Smarandache curve, then*

$$\frac{k_n}{k_g} = \frac{\sigma}{-\varepsilon x_4}$$

where $\sinh w(s) = x_3 = \text{const.}$ and $\cosh w(s) = x_4 = \text{const.}$

Theorem 3.6. *The striction line on timelike surface $M_3(s, v)$ is obtained as follows*

$$\begin{aligned} \bar{\beta} = \alpha - & \frac{-\sigma k_n - (w'(s) - k_g) \cdot \cosh w(s)}{- (w'(s) - k_g)^2 + \sigma^2 (-k_n^2 + t_r^2)} X(s) . \\ & + (2\sigma (w'(s) - k_g) (-k_n \cosh w(s) + t_r \sinh w(s))) \\ & + \varepsilon^2 (-k_n \sinh w(s) + t_r \cosh w(s))^2 \end{aligned}$$

Remark 3.7. *If $\delta(s)$ is a geodesic curve and*

$$w(s) = -\sigma \int \frac{\kappa}{\cosh w(s)} d(s),$$

then striction line of surface is equal to the base curve.

Remark 3.8. *If $\delta(s)$ is a asymptotic curve and*

$$w(s) = \int \kappa d(s),$$

then striction line of timelike surface is equal to the base curve.

3.2. Ruled Surfaces with Constant Slope Ruling with Respect to the (t, n) Planes. Generating lines of the surface \widetilde{M} are given by points on the curve $\widetilde{X}(s)$ and they have the constant slope σ with respect to the (t, n) planes to the curve at every point on the curve $\widetilde{X}(s)$. The surface will be called the ruled surfaces with constant slope ruling with respect to the (t, n) planes where

$$\langle \widetilde{X}(s), y(s) \rangle = \sigma.$$

Case 3.4. *If $\alpha(s)$ is a spacelike curve with the principal spacelike normal vector field $n(s)$, then surfaces is spacelike character where $1 + \sigma^2 > 0$. Direction vector of generating line of the spacelike surface is given by*

$$\widetilde{X}(s) = \sin w(s).t(s) + \cos w(s).n(s) + \sigma.y(s)$$

The surface with constant slope ruling \widetilde{M}_1 parametrization obtained by

$$\widetilde{M}_1(s, v) = \alpha(s) + v \left(\widetilde{X}(s) \right).$$

Case 3.5. If $\alpha(s)$ is a spacelike curve with the principal timelike normal vector field $n(s)$, then surfaces is spacelike character where $\sigma^2 - 1 > 0$ and the surfaces is timelike character where $\sigma^2 - 1 < 0$. Direction vector of generating line of the surface is given by

$$\tilde{u}(s) = \cosh w(s).t(s) + \sinh w(s).n(s) + \sigma.y(s)$$

and the surface is obtained by

$$\tilde{M}_2(s, v) = \alpha(s) + v(\tilde{u}(s)).$$

Case 3.6. If $\alpha(s)$ is a timelike curve with the principal spacelike normal vector field $n(s)$, then the surface is timelike where $\sigma^2 + 1 > 0$ and direction vector is given as follows

$$\tilde{\delta}(s) = \sinh w(s).t(s) + \cosh w(s).n(s) + \sigma.y(s).$$

The surface is parametrized by

$$\tilde{M}_3(s, v) = \alpha(s) + v(\tilde{\delta}(s)).$$

Theorem 3.7. The spacelike surface $\tilde{M}_1(s, v)$ is developable if and only if

$$\cos w(s) (\sin w(s)k_g + \cos w(s)t_r) - \sigma ((k_n - w'(s)) \sin w(s) + \sigma t_r) = 0.$$

Corollary 3.7. Developable surface's $\tilde{M}_1(s, v)$ generating line $\tilde{X}(s)$ is an asymptotic and Smarandache curve if and only if $\tilde{\alpha}(s)$ is a general helix with

$$\frac{\tau}{\kappa} = \frac{x_1 x_2}{\sigma^2 - x_2^2}$$

where $\sin w(s) = x_1 = \text{const.}$ and $\cos w(s) = x_2 = \text{const.}$

Corollary 3.8. $\tilde{M}_1(s, v)$ developable surface's generating line $\tilde{X}(s)$ is a geodesic and Smarandache curve if and only if $\tilde{\alpha}(s)$ is a general helix, so that

$$\frac{\tau}{\kappa} = \frac{\sigma x_1}{x_2^2 - \sigma^2}$$

where $\sin w(s) = x_1 = \text{const.}$ and $\cos w(s) = x_2 = \text{const.}$

Corollary 3.9. $\tilde{X}(s)$ is a line and Smarandache curve if and only if

$$\frac{k_n}{k_g} = \frac{\sigma}{x_2}$$

where $\cos w(s) = x_2 = \text{const.}$

Theorem 3.8. *The striction line on spacelike surface $\widetilde{M}_1(s, v)$ is given by*

$$\bar{\beta} = \alpha - \frac{\sigma k_g - (w'(s) + k_n) \cdot \cos w(s)}{-w'(s)^2 - 2w'(s) \cdot k_n + k_n^2 (\sin^2 w(s) - \cos^2 w(s)) + \sigma^2 (-k_g^2 + t_r^2) - 2\sigma (k_g \cdot (w'(s) + k_n) \cdot \cos w(s) - t_r \cdot \sin w(s) \cdot (-w'(s) + k_n)) + (\sin w(s) \cdot k_g + \cos w(s) \cdot t_r)^2} \cdot X(s).$$

Corollary 3.10. *If $\widetilde{X}(s)$ is an asymptotic curve and*

$$w(s) = \sigma \int \frac{\kappa}{\cos w(s)} d(s),$$

then striction line of surface is equal to the base curve.

Corollary 3.11. *If $\widetilde{X}(s)$ is a geodesic curve and*

$$w(s) = - \int \kappa d(s)$$

where $\cos w(s) \neq 0$, then striction line of the surface is equal to the base curve.

Theorem 3.9. *The spacelike surface $\widetilde{M}_2(s, v)$ is developable if and only if*

$$\sinh w(s) (\cosh w(s) \cdot k_g + \sinh w(s) t_r) - \sigma (\cosh w(s) (k_n + w'(s)) + \sigma t_r) = 0$$

where $\sigma^2 - 1 > 0$.

If $\sigma^2 - 1 < 0$, then the surface is timelike ruled surface. So the timelike surface is developable if and only if

$$\sinh w(s) \cdot (\cosh w(s) \cdot k_g + \sinh w(s) t_r) - \sigma (\cosh w(s) (-\varepsilon k_n + w'(s)) + \sigma \varepsilon t_r) = 0$$

where $\varepsilon = \langle t, t \rangle = \mp 1$.

Remark 3.9. *If $\sinh w(s) = x_3 = \text{const.}$ and $\cosh w(s) = x_4 = \text{const.}$, generating line of the surface $\widetilde{u}(s)$ be a Smarandache curve. Let $\widetilde{M}_2(s, v)$ be a developable spacelike surface and $\widetilde{u}(s)$ be a Smarandache curve, if at the same time $\widetilde{u}(s)$ be an asymptotic, then the base curve is a general helix with*

$$\frac{\tau}{\kappa} = \frac{x_3 x_4}{\sigma^2 - x_3^2},$$

if also $\widetilde{u}(s)$ be a geodesic, then the base curve is a general helix, so that

$$\frac{\tau}{\kappa} = \frac{\sigma}{x_3}.$$

Remark 3.10. If $\sinh w(s) = x_3 = \text{const.}$ and $\cosh w(s) = x_4 = \text{const.}$, generating lines of the surface $\tilde{u}(s)$ be a Smarandache curve. Let $\tilde{M}_2(s, v)$ be a developable timelike surface and $\tilde{u}(s)$ be a Smarandache curve, if at the same time $\tilde{u}(s)$ be an asymptotic, then the base curve is a general helix with

$$\frac{\tau}{\kappa} = \frac{x_3 x_4}{x_3^2 - \varepsilon \sigma^2}$$

if also $\tilde{u}(s)$ be a geodesic, then the base curve is a general helix with

$$\frac{\tau}{\kappa} = \frac{\varepsilon \sigma x_4}{\varepsilon \sigma^2 - x_3^2}$$

if $\tilde{u}(s)$ be a line curvature, then

$$\frac{k_n}{k_g} = \frac{-\varepsilon \sigma}{x_3}.$$

Theorem 3.10. The striction line on spacelike surface $\tilde{M}_3(s, v)$ is given by

$$\begin{aligned} \bar{\beta} = \alpha - & \frac{\sigma k_g - (w'(s) + k_n) \sinh w(s)}{(w'(s) + k_n)^2 + \sigma^2 (-k_g^2 + t_r^2)} X(s) \\ & + ((2\sigma (w'(s) + k_n) k_g \sinh w(s) + t_r \cosh w(s))) \\ & + (k_g \cosh w(s) + t_r \sinh w(s))^2 \end{aligned}$$

where $\sigma^2 - 1 > 0$, and striction line on timelike surface is given by

$$\begin{aligned} \bar{\beta} = \alpha - & \frac{-\sigma k_g - (w'(s) + k_n) \sinh w(s)}{w'(s)^2 + k_n^2 - 2.k_n.w'(s) (\sinh^2 w(s) - \varepsilon \cosh^2 w(s)) + \sigma^2 (k_g^2 + t_r^2)} X(s) \\ & + 2\sigma (k_g \sinh w(s) . (w'(s) + k_n) + \varepsilon.t_r \cosh w(s) . (w'(s) - \varepsilon k_n)) \\ & + (k_g \cosh w(s) + t_r \sinh w(s))^2 \end{aligned}$$

where $\sigma^2 - 1 < 0$, $\varepsilon = \langle t, t \rangle = \mp 1$.

Remark 3.11. If $\tilde{u}(s)$ is an asymptotic curve and

$$w(s) = \sigma \int \frac{\kappa}{\sinh w(s)} d(s)$$

where $\sigma^2 - 1 > 0$, then striction line of spacelike surface is equal to the base curve and

$$w(s) = -\sigma \int \frac{k_g}{\sinh w(s)} d(s)$$

where $\sigma^2 - 1 < 0$, then striction line of timelike surface is equal to the base curve.

Remark 3.12. If $\tilde{u}(s)$ is a geodesic curve and

$$w(s) = - \int \kappa d(s)$$

where $\sigma^2 - 1 > 0$ then striction line of spacelike surface is equal to base curve and

$$w(s) = \int k_n d(s)$$

where $\sigma^2 - 1 < 0$ and $\sinh w(s) \neq 0$, then striction line of timelike surface is equal to base curve.

Corollary 3.12. *The surface $\widetilde{M}_3(s, v)$ is torsal if and only if*

$$\cosh w(s) (\sinh w(s)k_g + \cosh w(s)t_r) - \sigma (\sinh w(s) (-\varepsilon k_n + w'(s)) + \varepsilon \sigma t_r) = 0$$

where $\varepsilon = \langle t, t \rangle = \mp 1$.

Corollary 3.13. *If $\sinh w(s) = x_3 = \text{const.}$ and $\cosh w(s) = x_4 = \text{const.}$, generating lines of the surface $\widetilde{\delta}(s)$ is a Smarandache curve. Let $\widetilde{M}_3(s, v)$ be a developable timelike surface and $\widetilde{\delta}(s)$ is a Smarandache curve, if at the same time $\widetilde{\delta}(s)$ be a asymptotic, then basic curve is a general helix, so that*

$$\frac{\tau}{\kappa} = \frac{x_3 x_4}{\sigma^2 - x_4^2}$$

if also $\widetilde{\delta}(s)$ be a geodesic, then base curve is a general helix, so that

$$\frac{\tau}{\kappa} = \frac{x_3}{\sigma x_4}$$

if $\widetilde{\delta}(s)$ be a line curvature, then

$$\frac{k_n}{k_g} = \frac{-\varepsilon \sigma}{x_4}$$

such that $\varepsilon = \langle t, t \rangle = \mp 1$.

Theorem 3.11. *The striction line on timelike surface $\widetilde{M}_3(s, v)$ is given by*

$$\bar{\beta} = \alpha - \frac{-\sigma k_g - (w'(s) + k_n) \cdot \cosh w(s)}{-\left(w'(s)^2 + k_n^2\right) - 2w'(s) \cdot k_n \cdot (\cosh^2 w(s) + \varepsilon \sinh^2 w(s)) + \sigma^2 (k_g^2 + t_r^2) + (\sinh w(s) \cdot k_g + \cosh w(s) \cdot t_r)^2} X(s).$$

Remark 3.13. *If generating line is a asymptotic curve on timelike surface $\widetilde{M}_3(s, v)$ and*

$$w(s) = -\sigma \int \frac{\kappa}{\cosh w(s)} d(s)$$

then striction line of surface is equal to base curve.

Remark 3.14. *If generating line is a geodesic curve on timelike surface $\widetilde{M}_3(s, v)$ and*

$$w(s) = - \int \kappa d(s)$$

then striction line of the surface is equal to the base curve.

3.3. Ruled Surfaces with Constant Slope Ruling with Respect to the (n, y) Planes. Surface \overline{M} is given by points on the curve $\overline{X}(s)$ and they have the constant slope σ with respect to the (n, y) planes. The surface will be defined the ruled surfaces with constant slope ruling with respect to the (n, y) planes to the curve where

$$\langle \overline{X}(s), t(s) \rangle = \sigma.$$

Case 3.7. *If $\alpha(s)$ is a spacelike curve with the principal spacelike normal vector field $n(s)$, then surface is spacelike where $1 + \sigma^2 > 0$. Direction vector of generating line of the spacelike surface is given by*

$$\overline{X}(s) = \sin w(s).n(s) + \cos w(s).y(s) + \sigma t(s).$$

The surface with constant slope \overline{M}_1 is parametrized by

$$\overline{M}_1(s, v) = \alpha(s) + v(\overline{X}(s)).$$

Case 3.8. *If $\alpha(s)$ is a spacelike curve with the principal timelike normal vector field $n(s)$, then surfaces is spacelike where $\sigma^2 - 1 > 0$ and the surface is timelike where $\sigma^2 - 1 < 0$. Direction vector of generating line of the surface is given by*

$$\overline{u}(s) = \cosh w(s).n(s) + \sinh w(s).y(s) + \sigma t(s)$$

and the surface is obtained as follows

$$\overline{M}_2(s, v) = \alpha(s) + v(\overline{u}(s)).$$

Case 3.9. *If $\alpha(s)$ is a timelike curve with the principal spacelike normal vector field $n(s)$, then the surface is timelike where $\sigma^2 + 1 > 0$ and direction vector is defined as follows*

$$\overline{\delta}(s) = \sinh w(s).n(s) + \cosh w(s).y(s) + \sigma t(s).$$

The surface parametrization is given as follows

$$\overline{M}_3(s, v) = \alpha(s) + v(\overline{\delta}(s)).$$

Theorem 3.12. *The spacelike surface $\overline{M}_1(s, v)$ is developable if and only if*

$$t_r(\sin^2 w(s) - \cos^2 w(s)) + \sigma(\cos w(s)k_n + \sin w(s)k_g) - w'(s) = 0.$$

Corollary 3.14. *Developable surface's generating line $\overline{X}(s)$ is a asymptotic and Smarandache curve if and only if $\alpha(s)$ is a general helix, so that*

$$\frac{\tau}{\kappa} = \frac{\sigma x_1}{x_2^2 - x_1^2}$$

where $\sin w(s) = x_1 = \text{const.}$ and $\cos w(s) = x_2 = \text{const.}$

Corollary 3.15. *$\overline{M}_1(s, v)$ developable surface's generating line $\overline{X}(s)$ is a geodesic and Smarandache curve if and only if $\alpha(s)$ is a general helix, so that*

$$\frac{\tau}{\kappa} = \frac{\sigma x_2}{x_2^2 - x_1^2}$$

where $\sin w(s) = x_1 = \text{const.}$ and $\cos w(s) = x_2 = \text{const.}$

Corollary 3.16. *$\overline{X}(s)$ is a line and Smarandache curve*

$$\frac{k_n}{k_g} = -\frac{x_2}{x_1}$$

where $\sin w(s) = x_1 = \text{const.}$ and $\cos w(s) = x_2 = \text{const.}$

Theorem 3.13. *The striction line on the spacelike surface $\overline{M}_1(s, v)$ is given by*

$$\overline{\beta} = \alpha - \frac{k_n \cdot \sin w(s) - k_g \cdot \cos w(s)}{(w'(s) - t_r^2) (\sin^2 w(s) - \cos^2 w(s)) - 2t_r \cdot w'(s) + \sigma^2 \cdot (k_n^2 + k_g^2) + (k_n \cdot \sin w(s) - k_g \cdot \cos w(s))^2} \cdot X(s).$$

Remark 3.15. *If*

$$\frac{k_n}{k_g} = -\cot w(s),$$

then the striction line on the surface $\overline{M}_1(s, v)$ equal to the base curve.

Theorem 3.14. *The spacelike ruled surface with constant slope $\overline{M}_2(s, v)$ is developable if and only if*

$$\sinh w(s) (\cosh w(s) \cdot k_n + \varepsilon \sinh w(s) k_g) - \sigma (\cosh w(s) (t_r + w'(s)) + \sigma k_g) = 0$$

$\varepsilon = \pm 1$. If $\sigma^2 - 1 > 0$, the surface is spacelike and $\varepsilon = -1$. If $\sigma^2 - 1 < 0$, the surface is timelike and $\varepsilon = +1$.

Corollary 3.17. *If $\sinh w(s) = x_3 = \text{const.}$ and $\cosh w(s) = x_4 = \text{const.}$, generating line of the surface $\bar{u}(s)$ is a Smarandache curve. Let $\bar{M}_2(s, v)$ be a developable surface and $\bar{u}(s)$ be a Smarandache curve, if at the same time $\bar{u}(s)$ is an asymptotic, then the base curve is a general helix with*

$$\frac{\tau}{\kappa} = \frac{\varepsilon x_3^2 - \sigma^2}{\sigma x_4},$$

if also $\bar{u}(s)$ is a geodesic, then the base curve is a general helix with

$$\frac{\tau}{\kappa} = \frac{\varepsilon (\sigma^2 - x_3 x_4)}{\sigma x_4}, \sigma^2 - 1 > 0$$

if $\bar{u}(s)$ is a line curvature, then

$$\frac{k_n}{k_g} = \frac{x_3 x_4}{\sigma^2 - \varepsilon x_4^2}.$$

Remark 3.16. *The striction line on the spacelike surface $\bar{M}_2(s, v)$ is given by*

$$\begin{aligned} \bar{\beta} = \alpha(s) + & \frac{k_n \cdot \cosh w(s) - k_g \cdot \sinh w(s)}{(w'(s) + t_r)^2 \cdot (\sinh^2 w(s) + \cosh^2 w(s)) + \sigma^2 (k_g^2 + k_n^2)} X(s) \\ & + (2\sigma (w'(s) + t_r) (k_n \cdot \sinh w(s) + k_g \cdot \cosh w(s))) \\ & - (k_n \cdot \cosh w(s) - k_g \cdot \sinh w(s))^2 \end{aligned}$$

where $\sigma^2 - 1 > 0$, and striction line on the timelike surface is given by

$$\begin{aligned} \bar{\beta} = \alpha(s) - & \frac{k_n \cdot \cosh w(s) + k_g \cdot \sinh w(s)}{(w'(s) + t_r)^2 + \sigma^2 (k_g^2 + k_n^2) + (k_n \cdot \cosh w(s) + k_g \cdot \sinh w(s))^2} X(s) \\ & + (2\sigma (w'(s) + t_r) (\varepsilon k_n \cdot \sinh w(s) + k_g \cdot \cosh w(s))) \end{aligned}$$

where $\sigma^2 - 1 < 0$, $\varepsilon = \langle t, t \rangle = \mp 1$.

Theorem 3.15. *The spacelike surface $\bar{M}_3(s, v)$ is developable if and only if*

$$\sinh w(s) (\sinh w(s) (t_r + w'(s)) + \sigma k_g) - \cosh w(s) (\cosh w(s) (\varepsilon t_r + w'(s)) - \varepsilon \sigma k_n)$$

where $\varepsilon = \langle t, t \rangle = \mp 1$.

Remark 3.17. *If $\sinh w(s) = x_3 = \text{const.}$ and $\cosh w(s) = x_4 = \text{const.}$, generating line of the surface $\bar{\delta}(s)$ is a Smarandache curve. Let $\bar{M}_3(s, v)$ be a developable timelike surface and $\bar{\delta}(s)$ be a Smarandache curve, if at the same time $\bar{\delta}(s)$ be an asymptotic, then the base curve is a general helix, so that*

$$\frac{\tau}{\kappa} = \frac{\sigma x_3}{\varepsilon x_4^2 - x_3^2}$$

for $\varepsilon = 1$,

$$\frac{\tau}{\kappa} = \sigma x_3$$

if also $\bar{\delta}(s)$ is a geodesic, then base curve is a general helix, so that

$$\frac{\tau}{\kappa} = \frac{-\varepsilon\sigma}{x_3^2 - \varepsilon x_4^2}$$

for $\varepsilon = -1$,

$$\frac{\tau}{\kappa} = \sigma$$

if $\bar{\delta}(s)$ is a line curvature, then

$$\frac{k_n}{k_g} = \frac{-\varepsilon x_4}{x_3}$$

where $\varepsilon = \langle t, t \rangle = \mp 1$.

Theorem 3.16. The striction line on timelike surface $\bar{M}_3(s, v)$ is given by

$$\bar{\beta} = \alpha(s) - \frac{k_n \cdot \sinh w(s) + k_g \cdot \cosh w(s)}{-(\cosh w(s) \cdot (\varepsilon \cdot t_r + w'(s)) - \varepsilon \sigma \cdot k_n)^2 + (\sinh w(s) \cdot (t_r + w'(s)) + \sigma \cdot k_g)^2 + (k_n \cdot \sinh w(s) + k_g \cdot \cosh w(s))^2} \cdot X(s).$$

4. Some Numerical Examples

In this section, we give examples of the surfaces with a constant slope ruling according to Darboux frame in Minkowski Space with respect to the given planes.

Example 4.1. The curve $\alpha(s)$ given by

$$\alpha(s) = \left(r \sin \frac{s}{r}, r \cos \frac{s}{r}, \frac{s}{r} \right)$$

and ruled surfaces with constant slope ruling is parametrized by

$$M_1(s, v) = \alpha(s) + v(\sin w(s)t(s) + \cos w(s)y(s) + \sigma n(s)).$$

Surface is visualized in following figure for $w(s) = \frac{\pi}{2}, r = 10, \sigma = 2$.

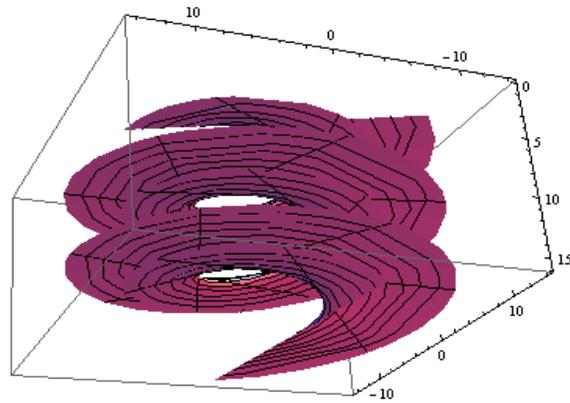


FIGURE 1.

Example 4.2. Ruled surfaces with constant slope ruling with respect to the $\{t, n\}$ planes

$$\widetilde{M}_1(s, v) = \alpha(s) + v(\sin w(s)t(s) + \cos w(s)n(s) + \sigma y(s)).$$

is shown following figure for $w(s) = \frac{s}{\pi}, r = 2, \sigma = \sqrt{3}$.

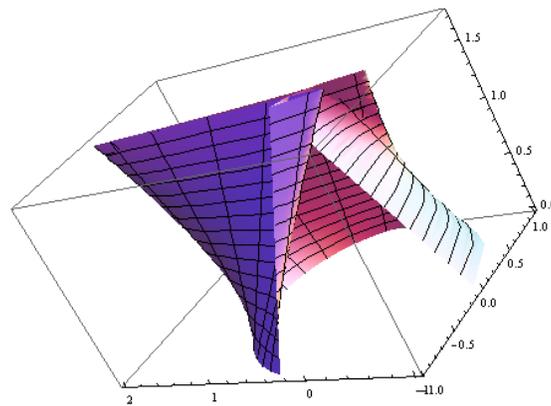


FIGURE 2.

Example 4.3. The curve $\beta(s)$ is given by

$$\beta(s) = \left(r \cosh \frac{s}{r}, r \sinh \frac{s}{r}, \frac{s}{r} \right)$$

and the timelike surface with a constant slope ruling is shown following figure for $w(s) = \frac{3\pi}{2}, r = 10, \sigma = \frac{1}{10}, s \in \left(-5, \frac{\pi^2}{4}\right), u \in (-1, 1)$.

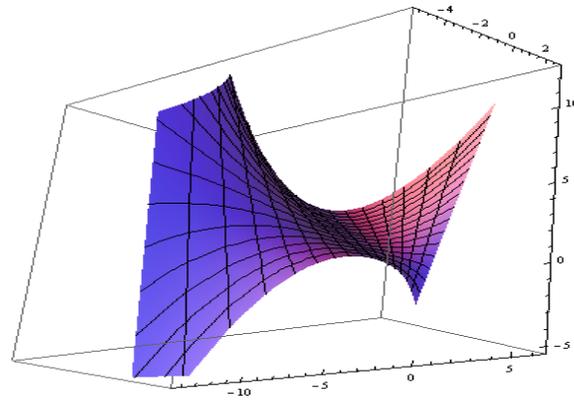


FIGURE 3.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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