



SOME MULTISTEP ITERATIVE METHODS FOR NONLINEAR EQUATION USING QUADRATURE RULE

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ABSTRACT. We introduce a sequence of third and fourth order iterative schemes to determine the roots of nonlinear equations by applying quadrature formula and decomposition approach. We also examined the convergence of suggested iterative methods under varied constraint. Various numerical test examples are presented to exhibit the validity, efficiency and implementaion of our algorithms.

1. INTRODUCTION

An extensive range of problems that appeared in diverse directions of pure and applied sciences can be considered by mean of nonlinear equations in the framework of novel and inventive techniques. Many authors proposed, analyzed and modified diversity of numerical methods using various approaches such as Taylor series, decomposition methods, quadrature formulas, modified homotopy perturbation method and variational iterative method, and. A vast literature is available that highlight different methods used to solve nonlinear equations for example see, [1, 3-33] and references therein. In Adomian decomposition method solution is considered in terms of an infinite series, which converges to towards exact solution. Chun [6] and Abbasbandy [1] constructed and investigated different higher order iterative methods by applying the decomposition technique of Adomian [2]. Darvishi and Barati [9] also applied the adomian decomposition

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technique to develop Newton type methods that are cubically convergent for the solution of system of non-linear equation. Implementation of this Adomian decomposition technique required higher order derivatives evaluation, which is major pitfall of this method. To overcome this drawback, several new techniques have been suggested and analyzed by many researchers. Daftardar-Gejji and Jafari [10] used different modifications of adomian decomposition method and introduced simple technique which do not need derivative evaluation of the Adomian polynomial. Numerous authors used the decomposition technique [10] and derived different higher order iterative methods. Numerical computation of univariate integral is one of the most notable problem in numerical analysis. Weerakon and Fernando cite32 improved the convergence of Newton method using quadrature rule. Later on, Ozban [27] investigated some new variant forms of Newton method by using the concept of harmonic mean and mid-point rule. Frontini and Sormani [11] modified the P -dimensional case using quadrature formula and developed new iterative methods for system of non-linear equations that are better than classical methods. Noor [25] developed fifth order convergent iterative method using Gaussian quadrature formula and investigated its efficacy as compared to the methods already existed in literature. Ali et al.[3] introduced a family of iterative method using the quadrature formula as well as the fundamental theorem of calculus and decomposition technique and checked the validity and performance of these methods by examining two mathematical models. Also in [21, 29, 30], decomposition technique [10] is merged sophisticatedly with coupled system of equation to investigate various order iterative methods. Motivated and inspired by the on-going research activities in this direction in this work we consider the well-known fixed point iterative method in which we rewrite the nonlinear equation $f(u) = 0$ as $u = g(u)$. In section2, we use fundamental law of calculus, along with the functional equation $u = g(u)$ as coupled system of equation and applying the decomposition technique [10]. We present and introduce some new iterative methods. Section 3 consists of convergence analysis of proposed methods. Some numerical examples are presented to make a comparative study of newly constructed methods with known third and fourth order convergent iterative algorithms.

2. CREATION OF ITERATIVE METHODS

This section comprises of some new multistep third and fourth order convergent iterative methods in view of mid-point, Trapezoidal rule and decomposition technique [10].

2.1. Mid-Point Rule. Consider the nonlinear equation

$$f(u) = 0 \tag{2.1}$$

which is equivalent to

$$u = g(u). \tag{2.2}$$

Assume that α is simple root of nonlinear equation (2.1) and γ is initial guess sufficiently close to root. Using fundamental theorem of calculus and mid-point quadrature formula, we have

$$u = g(\gamma) + (u - \gamma)g'\left(\frac{u + \gamma}{2}\right). \quad (2.3)$$

Now using the technique of He [16], the nonlinear equation (2.1) can be written as an equivalent coupled system of equations

$$\begin{aligned} u &= g(\gamma) + (u - \gamma)g'\left(\frac{u + \gamma}{2}\right) + H(u) \\ H(u) &= g(u) - g(\gamma) - (u - \gamma)g'\left(\frac{u + \gamma}{2}\right) \\ &= u\left(1 - g'\left(\frac{u + \gamma}{2}\right)\right) + \gamma g'\left(\frac{u + \gamma}{2}\right) - g(\gamma), \end{aligned} \quad (2.4)$$

from which, it follows that

$$\begin{aligned} u &= \frac{H(u)}{1 - g'\left(\frac{u + \gamma}{2}\right)} + \frac{g(\gamma) - \gamma g'\left(\frac{u + \gamma}{2}\right)}{1 - g'\left(\frac{u + \gamma}{2}\right)} \\ &= c + M(u), \end{aligned} \quad (2.5)$$

where

$$c = \gamma \quad (2.6)$$

$$M(u) = \frac{H(u)}{1 - g'\left(\frac{u + \gamma}{2}\right)} + \frac{g(\gamma) - \gamma}{1 - g'\left(\frac{u + \gamma}{2}\right)}. \quad (2.7)$$

It is clear that $M(u)$ is nonlinear operator. Now we establish sequence of higher order iterative methods implementing decomposition technique presented by Daftardar-Gejji and Jafari [10]. In this technique, the solution of (2.1) can be represented as in terms of infinite series.

$$u = \sum_{i=0}^{\infty} u_i. \quad (2.8)$$

here the operator $M(u)$ can be decomposed as

$$M(u) = M(u_0) + \sum_{i=1}^{\infty} \left\{ M\left(\sum_{j=0}^i u_j\right) - M\left(\sum_{j=0}^{i-1} u_j\right) \right\}. \quad (2.9)$$

Thus from equation (2.5), (2.8) and (2.9), we have

$$\sum_{i=1}^{\infty} u_i = c + M(u_0) = \sum_{i=1}^{\infty} \left(M\left(\sum_{j=0}^i u_j\right) - M\left(\sum_{j=0}^{i-1} u_j\right) \right), \quad (2.10)$$

which generates the following iterative scheme

$$\begin{aligned}
 u_0 &= c \\
 u_1 &= M(u_0) \\
 u_2 &= M(u_0 + u_1) - M(u_0) \\
 &\cdot \\
 &\cdot \\
 &\cdot \\
 u_{n+1} &= M\left(\sum_{j=0}^n u_j\right) - M\left(\sum_{j=0}^{n-1} u_j\right) \quad n = 1, 2, \dots
 \end{aligned}
 \tag{2.11}$$

Consequently, it follows that

$$u_1 + u_2 + \dots + u_{n+1} = M(u_0 + u_1 + u_2 + \dots + u_n),$$

and

$$u = c + \sum_{i=1}^{\infty} u_i. \tag{2.12}$$

It is noted that u is approximated by

$$U_n = (u_0 + u_1 + u_2 + \dots + u_n),$$

and

$$\lim_{x \rightarrow \infty} U_n = u.$$

For $n=0$,

$$u \approx U_0 = u_0 = c = \gamma. \tag{2.13}$$

From (2.4), it can easily be computed as

$$H(u_0) = 0.$$

Using(2.7), we get

$$u_1 = M(u_0) = \frac{H(u_0)}{1 - g'(\frac{u_0 + \gamma}{2})} + \frac{g(\gamma) - \gamma}{1 - g'(\frac{u_0 + \gamma}{2})} = \frac{g(\gamma) - \gamma}{1 - g'(\frac{u_0 + \gamma}{2})}$$

For $n=1$, $u \approx U_1 = u_0 + u_1 = u_0 + M(u_0),$

$$u \approx U_1 = u_o + u_1 = \gamma + \frac{g(\gamma) - \gamma}{1 - g'(\frac{u_o + \gamma}{2})}.$$

Using(2.13), we have

$$u = \frac{g(\gamma) - \gamma g'(\gamma)}{1 - g'(\gamma)}. \quad (2.14)$$

This fixed point formulation is used to suggest the following algorithm.

Algorithm2A. For a given u_o (initial guess), approximate solution u_{n+1} is computed by the following iterative scheme

$$u_{n+1} = \frac{g(u_n) - u_n g'(u_n)}{1 - g'(u_n)}. \quad (2.15)$$

Kang et al. [28] developed this algorithm and proved that Algorithm 2A has quadratic convergence.

From(2.4) and (2.7),we have

$$H(u_o + u_1) = g(u_o + u_1) - g(\gamma) - (u_o + u_1 - \gamma)g'(\frac{u_o + u_1 + \gamma}{2}).$$

Thus

$$\begin{aligned} u_1 + u_2 &= M(u_o + u_1) = \frac{H(u_o + u_1)}{1 - g'(\frac{u_o + u_1 + \gamma}{2})} + \frac{g(\gamma) - \gamma}{1 - g'(\frac{u_o + u_1 + \gamma}{2})} \\ &= \frac{g(u_o + u_1) - g(\gamma) - (u_o + u_1 - \gamma)g'(\frac{u_o + u_1 + \gamma}{2})}{1 - g'(\frac{u_o + u_1 + \gamma}{2})} + \frac{g(\gamma) - \gamma}{1 - g'(\frac{u_o + u_1 + \gamma}{2})} \\ &= \frac{g(u_o + u_1)}{1 - g'(\frac{u_o + u_1 + \gamma}{2})} - \frac{(u_o + u_1 - \gamma)g'(\frac{u_o + u_1 + \gamma}{2})}{1 - g'(\frac{u_o + u_1 + \gamma}{2})} - \frac{\gamma}{1 - g'(\frac{u_o + u_1 + \gamma}{2})} \end{aligned}$$

For n=2, $u \approx U_2 = u_o + u_1 + u_2 = c + M(u_o + u_1).$

$$\begin{aligned} &= \gamma - \frac{\gamma}{1 - g'(\frac{u_o + u_1 + \gamma}{2})} + \frac{g(u_o + u_1)}{1 - g'(\frac{u_o + u_1 + \gamma}{2})} - \frac{(u_o + u_1 - \gamma)g'(\frac{u_o + u_1 + \gamma}{2})}{1 - g'(\frac{u_o + u_1 + \gamma}{2})} \\ &= \frac{g(u_o + u_1)}{1 - g'(\frac{u_o + u_1 + \gamma}{2})} - \frac{(u_o + u_1)g'(\frac{u_o + u_1 + \gamma}{2})}{1 - g'(\frac{u_o + u_1 + \gamma}{2})} \end{aligned}$$

Take

$$u_o + u_1 = v = \frac{g(\gamma) - \gamma g'(\gamma)}{1 - g'(\gamma)} = \frac{g(v)}{1 - g'(\frac{v + \gamma}{2})} - \frac{v g'(\frac{v + \gamma}{2})}{1 - g'(\frac{v + \gamma}{2})}.$$

This relation yields the following two step method for solving nonlinear equation

Algorithm 2B. For a given initial guess u_0 , u_{n+1} the approximated solution u_{n+1} can be computed by the following iterative schemes.

$$v_n = \frac{g(u_n) - u_n g'(u_n)}{1 - g'(u_n)} \tag{2.16}$$

$$u_{n+1} = \frac{g(v_n) - v_n g'(\frac{u_n+v_n}{2})}{1 - g'(\frac{u_n+v_n}{2})}, \quad n = 0, 1, 2, \dots \tag{2.17}$$

It is noted that

$$u_0 + u_1 + u_2 = w = \frac{g(u_o + u_1)}{1 - g'(\frac{u_o+u_1+\gamma}{2})} - \frac{(u_o + u_1)g'(\frac{u_o+u_1+\gamma}{2})}{1 - g'(\frac{u_o+u_1+\gamma}{2})}. \tag{2.18}$$

From (2.4) and (2.7), we can write

$$H(u_o + u_1 + u_2) = g(u_o + u_1 + u_2) - g(\gamma) - (u_o + u_1 + u_2 - \gamma)g'(\frac{u_o + u_1 + u_2 + \gamma}{2}).$$

and

$$\begin{aligned} u_1 + u_2 + u_3 &= M(u_o + u_1 + u_2) = \frac{H(u_o + u_1 + u_2)}{1 - g'(\frac{u_o+u_1+u_2+\gamma}{2})} + \frac{g(\gamma) - \gamma}{1 - g'(\frac{u_o+u_1+u_2+\gamma}{2})} \\ &= \frac{g(u_o + u_1 + u_2) - g(\gamma) - (u_o + u_1 + u_2 - \gamma)g'(\frac{u_o+u_1+u_2+\gamma}{2})}{1 - g'(\frac{u_o+u_1+u_2+\gamma}{2})} \\ &\quad + \frac{g(\gamma) - \gamma}{1 - g'(\frac{u_o+u_1+u_2+\gamma}{2})} \\ &= \frac{g(u_o + u_1 + u_2)}{1 - g'(\frac{u_o+u_1+u_2+\gamma}{2})} - \frac{(u_o + u_1 + u_2 - \gamma)g'(\frac{u_o+u_1+u_2+\gamma}{2})}{1 - g'(\frac{u_o+u_1+u_2+\gamma}{2})} - \frac{\gamma}{1 - g'(\frac{u_o+u_1+u_2+\gamma}{2})}. \end{aligned}$$

For n=3, $u \approx U_3 = u_o + u_1 + u_2 + u_3 = u_o + M(u_o + u_1 + u_2).$

$$= \gamma - \frac{\gamma}{1 - g'(\frac{u_o+u_1+u_2+\gamma}{2})} + \frac{g(u_o + u_1 + u_2)}{1 - g'(\frac{u_o+u_1+u_2+\gamma}{2})} - \frac{(u_o + u_1 + u_2 - \gamma)g'(\frac{u_o+u_1+u_2+\gamma}{2})}{1 - g'(\frac{u_o+u_1+u_2+\gamma}{2})}.$$

Using (2.18), we have

$$\begin{aligned} &= \gamma + \frac{g(w)}{1 - g'(\frac{w+\gamma}{2})} - \frac{(w - \gamma)g'(\frac{w+\gamma}{2})}{1 - g'(\frac{w+\gamma}{2})} - \frac{\gamma}{1 - g'(\frac{w+\gamma}{2})} \\ &= \frac{g(w)}{1 - g'(\frac{w+\gamma}{2})} - \frac{wg'(\frac{w+\gamma}{2})}{1 - g'(\frac{w+\gamma}{2})}. \end{aligned}$$

Using this relation, we suggest the following three step method for solving nonlinear equation (2.1).

Algorithm 2C. For a given initial guess u_0 , compute the approximate solution u_{n+1} by the following iterative scheme

$$v_n = \frac{g(u_n) - u_n g'(u_n)}{1 - g'(u_n)}$$

$$w_n = \frac{g(v_n) - v_n g'(\frac{u_n + v_n}{2})}{1 - g'(\frac{u_n + v_n}{2})} \tag{2.19}$$

$$u_{n+1} = \frac{g(w_n) - w_n g'(\frac{u_n + w_n}{2})}{1 - g'(\frac{u_n + w_n}{2})}, \quad n = 0, 1, 2, \dots \tag{2.20}$$

2.2. Trapezoidal rule. Again using the technique of He [16] and fundamental law of calculus along with Trapezoidal rule, we can obtain .

$$u = g(\gamma) + (u - \gamma) \left(\frac{g'(u) + g'(\gamma)}{2} \right) + H(u)$$

$$H(u) = g(u) - g(\gamma) - (u - \gamma) \left(\frac{g'(u) + g'(\gamma)}{2} \right)$$

$$= u \left[1 - \left(\frac{g'(u) + g'(\gamma)}{2} \right) \right] + \gamma \left(\frac{g'(u) + g'(\gamma)}{2} \right) - g(\gamma), \tag{2.21}$$

from which it follows that

$$u = \frac{H(u)}{1 - \left(\frac{g'(u) + g'(\gamma)}{2} \right)} + \frac{g(\gamma) - \gamma \left(\frac{g'(u) + g'(\gamma)}{2} \right)}{1 - \left(\frac{g'(u) + g'(\gamma)}{2} \right)}$$

$$= c + M(u), \tag{2.22}$$

where

$$c = \gamma \tag{2.23}$$

$$M(u) = \frac{H(u)}{1 - \left(\frac{g'(u) + g'(\gamma)}{2} \right)} + \frac{g(\gamma) - \gamma}{1 - \left(\frac{g'(u) + g'(\gamma)}{2} \right)}. \tag{2.24}$$

Now applying decomposition technique of Daftardar-Gejji and Jafari [10], we have

For $n=0$,

$$u \approx U_o = u_o = c = \gamma. \tag{2.25}$$

From (2.21), it can easily be computed as $H(u_o) = 0$.

Using(2.24), we get

$$u_1 = M(u_o) = \frac{H(u_o)}{1 - \left(\frac{g'(u_o) + g'(\gamma)}{2} \right)} + \frac{g(\gamma) - \gamma}{1 - \left(\frac{g'(u_o) + g'(\gamma)}{2} \right)} = \frac{g(\gamma) - \gamma}{1 - \left(\frac{g'(u_o) + g'(\gamma)}{2} \right)}.$$

For $n=1$,

$$u \approx U_1 = u_o + u_1 = u_o + M(u_o) \quad (2.26)$$

$$u \approx U_1 = u_o + u_1 = \gamma + \frac{g(\gamma) - \gamma}{1 - \left(\frac{g'(u_o) + g'(\gamma)}{2}\right)}.$$

Using (2.26), we have

$$u = \frac{g(\gamma) - \gamma g'(\gamma)}{1 - g'(\gamma)}. \quad (2.27)$$

This formulation determines the algorithm 2A.

From (2.21) and (2.24), we have

$$H(u_o + u_1) = g(u_o + u_1) - g(\gamma) - (u_o + u_1 - \gamma) \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2} \right).$$

and

$$\begin{aligned} u_1 + u_2 &= M(u_o + u_1) = \frac{H(u_o + u_1)}{1 - \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2}\right)} + \frac{g(\gamma) - \gamma}{1 - \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2}\right)} \\ &= \frac{g(u_o + u_1) - g(\gamma) - (u_o + u_1 - \gamma) \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2}\right)}{1 - \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2}\right)} + \frac{g(\gamma) - \gamma}{1 - \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2}\right)} \\ &= \frac{g(u_o + u_1)}{1 - \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2}\right)} - \frac{(u_o + u_1 - \gamma) \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2}\right)}{1 - \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2}\right)} \\ &\quad - \frac{\gamma}{1 - \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2}\right)}. \end{aligned}$$

For $n=2$, $u \approx U_2 = u_o + u_1 + u_2 = c + M(u_o + u_1)$.

$$= \gamma - \frac{\gamma}{1 - \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2}\right)} + \frac{g(u_o + u_1)}{1 - \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2}\right)} - \frac{(u_o + u_1 - \gamma) \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2}\right)}{1 - \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2}\right)}.$$

Take

$$\begin{aligned} u_o + u_1 &= v = \frac{g(\gamma) - \gamma g'(\gamma)}{1 - g'(\gamma)}. \\ &= \gamma - \frac{\gamma}{1 - \left(\frac{g'(v) + g'(\gamma)}{2}\right)} - \frac{g(v)}{1 - \left(\frac{g'(v) + g'(\gamma)}{2}\right)} - \frac{(v - \gamma) \left(\frac{g'(v) + g'(\gamma)}{2}\right)}{1 - \left(\frac{g'(v) + g'(\gamma)}{2}\right)} \\ &= \frac{g(v)}{1 - \left(\frac{g'(v) + g'(\gamma)}{2}\right)} - \frac{v \left(\frac{g'(v) + g'(\gamma)}{2}\right)}{1 - \left(\frac{g'(v) + g'(\gamma)}{2}\right)}. \end{aligned}$$

This relation yields the following two step method for solving nonlinear equation

Algorithm 2D. For a given initial guess u_0 , the approximate solution u_{n+1} can be computed by the following iterative schemes

$$v_n = \frac{g(u_n) - (u_n)g'(u_n)}{1 - g'(u_n)} \quad (2.28)$$

$$u_{n+1} = \frac{g(v_n) - v_n \left(\frac{g'(v_n) + g'(u_n)}{2} \right)}{1 - \left(\frac{g'(v_n) + g'(u_n)}{2} \right)} \quad n = 0, 1, 2, \dots \quad (2.29)$$

It is noted that

$$u_o + u_1 + u_2 = w = \frac{g(u_o + u_1)}{1 - \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2} \right)} - \frac{(u_o + u_1) \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2} \right)}{1 - \left(\frac{g'(u_o + u_1) + g'(\gamma)}{2} \right)}. \quad (2.30)$$

From (2.21) and (2.24), we can write

$$H(u_o + u_1 + u_2) = g(u_o + u_1 + u_2) - g(\gamma) - (u_o + u_1 + u_2 - \gamma) \left(\frac{g'(u_o + u_1 + u_2) + g'(\gamma)}{2} \right)$$

and

$$\begin{aligned} u_1 + u_2 + u_3 = M(u_o + u_1 + u_2) &= \frac{H(u_o + u_1 + u_2)}{1 - \left(\frac{g'(u_o + u_1 + u_2) + g'(\gamma)}{2} \right)} + \frac{g(\gamma) - \gamma}{1 - \left(\frac{g'(u_o + u_1 + u_2) + g'(\gamma)}{2} \right)} \\ &= \frac{g(u_o + u_1 + u_2) - g(\gamma) - (u_o + u_1 + u_2 - \gamma) \left(\frac{g'(u_o + u_1 + u_2) + g'(\gamma)}{2} \right)}{1 - \left(\frac{g'(u_o + u_1 + u_2) + g'(\gamma)}{2} \right)} \\ &\quad + \frac{g(\gamma) - \gamma}{\left(\frac{1 - g'(u_o + u_1 + u_2) + g'(\gamma)}{2} \right)} \\ &= \frac{g(u_o + u_1 + u_2)}{1 - \left(\frac{g'(u_o + u_1 + u_2) + g'(\gamma)}{2} \right)} - \frac{(u_o + u_1 + u_2 - \gamma) \left(\frac{g'(u_o + u_1 + u_2) + g'(\gamma)}{2} \right)}{1 - \left(\frac{g'(u_o + u_1 + u_2) + g'(\gamma)}{2} \right)} \\ &\quad - \frac{\gamma}{1 - \left(\frac{g'(u_o + u_1 + u_2) + g'(\gamma)}{2} \right)}. \end{aligned}$$

For n=3, $u \approx U_3 = u_o + u_1 + u_2 + u_3 = u_o + M(u_o + u_1 + u_2)$

$$\begin{aligned} &= \gamma - \frac{\gamma}{1 - \left(\frac{g'(u_o + u_1 + u_2) + g'(\gamma)}{2} \right)} + \frac{g(u_o + u_1 + u_2)}{1 - \left(\frac{g'(u_o + u_1 + u_2) + g'(\gamma)}{2} \right)} \\ &\quad - \frac{(u_o + u_1 + u_2 - \gamma) \left(\frac{g'(u_o + u_1 + u_2) + g'(\gamma)}{2} \right)}{1 - \left(\frac{g'(u_o + u_1 + u_2) + g'(\gamma)}{2} \right)}. \end{aligned}$$

Using (2.29), we obtain

$$\begin{aligned} &= \gamma - \frac{\gamma}{1 - \left(\frac{g'(w)+g'(\gamma)}{2}\right)} - \frac{g(w)}{1 - \left(\frac{g'(w)+g'(\gamma)}{2}\right)} - \frac{(w - \gamma)\left(\frac{g'(w)+g'(\gamma)}{2}\right)}{1 - \left(\frac{g'(w)+g'(\gamma)}{2}\right)} \\ &= \frac{g(w)}{1 - \left(\frac{g'(w)+g'(\gamma)}{2}\right)} - \frac{w\left(\frac{g'(w)+g'(\gamma)}{2}\right)}{1 - \left(\frac{g'(w)+g'(\gamma)}{2}\right)}. \end{aligned}$$

This formulation yields the following three step method for solving nonlinear equation(2.1).

Algorithm 2E. For a given initial guess u_0 , the approximated solution u_{n+1} is computed by the following iterative schemes.

$$\begin{aligned} v_n &= \frac{g(u_n) - u_n g'(u_n)}{1 - g'(u_n)} \\ w_n &= \frac{g(v_n) - v_n \left(\frac{g'(v_n)+g'(u_n)}{2}\right)}{1 - \left(\frac{g'(v_n)+g'(u_n)}{2}\right)} \end{aligned} \tag{2.31}$$

$$u_{n+1} = \frac{g(w_n) - w_n \left(\frac{g'(w_n)+g'(u_n)}{2}\right)}{1 - \left(\frac{g'(w_n)+g'(u_n)}{2}\right)}, \quad n = 0, 1, 2, \dots \tag{2.32}$$

3. CONVERGENCE ANALYSIS OF PROPOSED ITERATIVE METHODS

This section comprises of convergence analysis of proposed algorithms 2B, 2D, 2C and 2E and it shown that these methods are third and fourth order convergent respectively.

Theorem 3.1. Let $I \subset R$ be an open interval and $f : I \rightarrow R$ is differential function. If $\beta \in I$ be root of $f(u) = 0$ and u_0 is sufficiently close to β then multistep method defined by Algorithm 2B and Algorithm 2C has at least third and fourth order of convergence and satisfies the error equation.

Poof. Let β be the root of nonlinear equation $f(u) = 0$ or equivalently $u = g(u)$. Let e_n and e_{n+1} be the errors at nth and (n+1) iterations respectively.

Now expanding $g(u)$ and $g'(u)$ by using taylor series about α , we have

$$\begin{aligned} g(u) &= \beta + e_n g'(\beta) + \frac{e_n^2}{2} g''(\beta) + \frac{e_n^3}{6} g'''(\beta) + O(e_n^4) \\ g'(u) &= g'(\beta) + e_n g''(\beta) + \frac{e_n^2}{2} g'''(\beta) + \frac{e_n^3}{6} g^{iv}(\beta) + O(e_n^4) \\ g(u_n) - u_n g(u_n) &= \beta - \beta g'(\beta) - \beta g''(\beta) e_n - \frac{1}{2} \left(g''(\beta) + \beta g'''(\beta)\right) e_n^2 + \frac{1}{6} \left(2g'''(\beta) + \beta g^{iv}(\beta)\right) e_n^3 + O(e_n^4) \\ 1 - g'(u_n) &= 1 - g'(\beta) - g''(\beta) e_n - \frac{e_n^2}{2} g'''(\beta) - \frac{e_n^3}{6} g^{iv}(\beta) + O(e_n^4) \\ \frac{g(u_n) - u_n g(u_n)}{1 - g'(u_n)} &= \beta + \frac{g''(\beta)}{2(-1 + g'(\beta))} e_n^2 - \frac{2g'''(\beta) - 2g'''(\beta)g'(\beta) + 3g''^2(\beta)}{6(-1 + g'(\beta))^2} e_n^3 + O(e_n^4). \end{aligned}$$

From (2.16), we obtain

$$\begin{aligned}
 v_n &= \beta + \frac{g''(\beta)}{2(-1 + g'(\beta))} e_n^2 + \left(\frac{-2g'''(\beta) + 2g'''(\beta)g'(\beta) - 3g''^2(\beta)}{6(-1 + g'(\beta))^2} \right) e_n^3 \\
 &+ \frac{1}{12(-1 + g'(\beta))^3} \left(2g^{iv}(\beta) - 4g^{iv}(\beta)g'(\beta) + 2g^{iv}(\beta)g'^2(\beta) + 7g''(\beta)g'''(\beta) - 7g''(\beta)g'''(\beta)g'(\beta) \right. \\
 &\quad \left. + 6g''^3(\beta) \right) e_n^4 + O(e_n^5). \\
 g(v_n) &= \beta + \frac{g'(\beta)g''(\beta)}{2(-1 + g'(\beta))} e_n^2 + \frac{\left(g'(\beta) \left(-2g'''(\beta) + 2g'''(\beta)g'(\beta) - 3g''^2(\beta) \right) \right)}{6(-1 + g'(\beta))^2} e_n^3 \\
 &+ \frac{1}{24(-1 + g'(\beta))^3} \left(4g^{iv}(\beta)g'(\beta) - 8g^{iv}(\beta)g'^2(\beta) + 4g^{iv}(\beta)g'^3(\beta) - 14g''(\beta)g'''(\beta)g'^2(\beta) \right. \\
 &\quad \left. + 14g''(\beta)g'''(\beta)g'(\beta) + 15g''^3(\beta)g'(\beta) - 3g''^3(\beta) \right) e_n^4 + O(e_n^5). \tag{3.1}
 \end{aligned}$$

Expanding $g'\left(\frac{v_n+u_n}{2}\right)$ in terms of Taylors series about α , we have

$$\begin{aligned}
 g'\left(\frac{v_n + u_n}{2}\right) &= g'(\beta) + \frac{g''(\beta)}{2} e_n + \frac{\left(2g''^2(\beta) + g'''(\beta)g'(\beta) - g'''(\beta) \right)}{8(-1 + g'(\beta))} e_n^2 \\
 &+ \left(\frac{-14g''(\beta)g'''(\beta) + 14g''(\beta)g'''(\beta)g'(\beta) + g^{iv}(\beta) - 12g''^3(\beta) - 2g^{iv}(\beta)g'(\beta) + g^{iv}(\beta)g'^2(\beta)}{48(-1 + g'(\beta))^2} \right) e_n^3 \\
 &+ \frac{1}{96(-1 + g'(\beta))^3} \left(11g^{iv}(\beta)g''(\beta) - 22g^{iv}(\beta)g'(\beta)g''(\beta) + 11g^{iv}(\beta)g'^2(\beta)g''(\beta) + 37g''^2(\beta)g'''(\beta) \right. \\
 &\quad \left. - 37g''^2(\beta)g'''(\beta)g'(\beta) + 24g''^4(\beta) + 8g'''^2(\beta) - 16g'''^3(\beta)g'(\beta) + 8g'^2(\beta)g'''^2(\beta) \right) + O(e_n^5). \tag{3.2}
 \end{aligned}$$

$$\begin{aligned}
 g(v_n) - v_n g'\left(\frac{v_n + u_n}{2}\right) &= -\beta(-1 + g'(\beta)) - \frac{1}{2}g''(\beta)e_n - \frac{1}{8(-1 + g'(\beta))} \left(\beta \left(2g''^2(\beta) - g'''(\beta) \right) \right. \\
 &\quad \left. - g'''(\beta)g'(\beta) \right) e_n^2 - \frac{1}{48(-1 + g'(\beta))^2} \left(-14\beta g''(\beta)g'''(\beta) + 14\beta g'(\beta)g''(\beta)g'''(\beta) - 12\beta g''^2(\beta) \right. \\
 &\quad \left. + \beta g^{iv}(\beta) - 2(\beta)g^{iv}(\beta)g'(\beta) + \beta g^{iv}(\beta)g'^2(\beta) - 12g''^2(\beta) + 12g'(\beta)g''^2(\beta) \right) e_n^3 \\
 &+ \frac{1}{96(-1 + g'(\beta))^3} \left(-44g''^2(\beta)g'''(\beta)g'(\beta) + 22g'''(\beta)g''(\beta)g'^2(\beta) + 8\beta g'''^2(\beta) + 24g''^3(\beta) \right. \\
 &\quad \left. - 24g'(\beta)g''^3(\beta) + 22g'''(\beta)g''(\beta) + 11\beta g''(\beta)g^{iv}(\beta) + 37\beta g''^2(\beta)g'''(\beta) - 16\beta g'''^2(\beta)g'(\beta) \right. \\
 &\quad \left. + 8g'''^2(\beta)g'^2(\beta) + 24\beta g''^4(\beta) - 22\beta g'(\beta)g''(\beta)g^{iv}(\beta) + 11\beta g'^2(\beta)g''(\beta)g^{iv}(\beta) \right. \\
 &\quad \left. - 37\beta g'(\beta)g'''(\beta)g''^2(\beta) \right) e_n^4 + O(e_n^5). \tag{3.3}
 \end{aligned}$$

$$\begin{aligned}
 1 - g'\left(\frac{v_n + u_n}{2}\right) &= 1 - g'(\beta) - \frac{g''(\beta)}{2}e_n + \frac{(2g''^2(\beta) + g'''(\beta)g'(\beta) - g'''(\beta))}{8(-1 + g'(\beta))}e_n^2 \\
 - \left(\frac{-14g''(\beta)g'''(\beta) + 14g''(\beta)g'''(\beta)g'(\beta) + g^{iv}(\beta) - 12g'''^3(\beta) - 2g^{iv}(\beta)g'(\beta) + g^{iv}(\beta)g'^2(\beta)}{48(-1 + g'(\beta))^2}\right)e_n^3 & \quad (3.4) \\
 - \frac{1}{96(-1 + g'(\beta))^3} \left(11g^{iv}(\beta)g''(\beta) - 22g^{iv}(\beta)g'(\beta)g''(\beta) + 11g^{iv}(\beta)g'^2(\beta)g''(\beta) + 37g''^2(\beta)g'''(\beta) \right. \\
 \left. - 37g''^2(\beta)g'''(\beta)g'(\beta) + 24g''^4(\beta) + 8g'''^2(\beta) - 16g'''^3(\beta)g'(\beta) + 8g'^2(\beta)g'''^2(\beta)\right)e_n^4 + O(e_n^5)
 \end{aligned}$$

Dividing (3.3),(3.4) and from (2.19), we have

$$\begin{aligned}
 w_n &= \frac{g(v_n) - v_n g'\left(\frac{v_n + u_n}{2}\right)}{1 - g'\left(\frac{v_n + u_n}{2}\right)} \\
 &= \beta + \left(\frac{g''^2(\beta)}{4(-1 + g'(\beta))^2}\right)e_n^3 + \frac{1}{48(-1 + g'(\beta))^3} \left(g'''(\beta) \left(11g'''(\beta)g'(\beta) - 11g'''(\beta) - 18g''^2(\beta)\right)\right)e_n^4 \\
 &+ O(e_n^5).
 \end{aligned}$$

Expanding $g(w_n)$ in terms of Taylor series about α , we obtain

$$\begin{aligned}
 g(w_n) &= \beta + \left(\frac{g'(\beta)g''^2(\beta)}{4(-1 + g'(\beta))^2}\right)e_n^3 \\
 - \frac{1}{48(-1 + g'(\beta))^3} \left(g'(\beta)g''(\beta) \left(-11g'''(\beta)g'(\beta) + 11g'''(\beta) + 18g''^2(\beta)\right)\right)e_n^4 + O(e_n^5). & \quad (3.5)
 \end{aligned}$$

Expanding $g\left(\frac{w_n + u_n}{2}\right)$ in terms of Taylor series about α , we get

$$\begin{aligned}
 g'\left(\frac{w_n + u_n}{2}\right) &= g'(\beta) + \frac{g''(\beta)}{2}e_n + \frac{g'''(\beta)}{8}e_n^2 + \frac{1}{48(-1 + g'(\beta))^2} \left(6g'''^3(\beta) + g^{iv}(\beta) - 2g^{iv}(\beta)g'(\beta) \right. \\
 \left. + g^{iv}(\beta)g'^2(\beta)\right)e_n^3 - \frac{1}{96(-1 + g'(\beta))^3} \left(g''^2(\beta) - 17g'''(\beta)g'(\beta) + 17g'''(\beta) + 18g''^2(\beta)\right)e_n^4 + O(e_n^5). \\
 1 - g\left(\frac{w_n + u_n}{2}\right) &= 1 - g'(\beta) - \frac{g''(\beta)}{2}e_n - \frac{g'''(\beta)}{8}e_n^2 \\
 - \frac{(6g'''^3(\beta) + g^{iv}(\beta) - 2g^{iv}(\beta)g'(\beta) + g^{iv}(\beta)g'^2(\beta))}{48(-1 + g'(\beta))^2}e_n^3 & \quad (3.6) \\
 - \left(g''^2(\beta) \left(\frac{-17g''(\beta)g'(\beta) - 17g'''(\beta) + 18g''^2(\beta)}{96(-1 + g'(\beta))^3}\right)\right)e_n^4 + O(e_n^5).
 \end{aligned}$$

Now using (2.20)and dividing (3.5),(3.6), simplifying

$$\begin{aligned}
 u_{n+1} &= \frac{g(w_n) - w_n g'\left(\frac{w_n + u_n}{2}\right)}{1 - g'\left(\frac{w_n + u_n}{2}\right)} \\
 &= \beta + \left(\frac{g'''^3(\beta)}{8(-1 + g'(\beta))^3}\right)e_n^4 + O(e_n^5) \\
 e_{n+1} &= \beta + \left(\frac{g'''^3(\beta)}{8(-1 + g'(\beta))^3}\right)e_n^4 + O(e_n^5).
 \end{aligned}$$

Theorem 3.2. Let $f : I \rightarrow R$ is differential function and $I \subset R$ be an open interval . If $\beta \in I$ be root of $f(u) = 0$ and u_0 is sufficiently close to β then multistep method defined by Algorithm 2D and Algorithm 2E has at least third and fourth order of convergence and satisfies the error equation.

Proof. From equation (3.1), we have

$$\begin{aligned}
 g(v_n) &= \beta + \frac{g'(\beta)g''(\beta)}{2(-1 + g'(\beta))} e_n^2 + \frac{\left(g'(\beta)\left(-2g'''(\beta) + 2g'''(\beta)g'(\beta) - 3g''^2(\beta)\right)\right)}{6(-1 + g'(\beta))^2} e_n^3 \\
 &+ \frac{1}{24(-1 + g'(\beta))^3} \left(4g^{iv}(\beta)g'(\beta) - 8g^{iv}(\beta)g'^2(\beta) + 4g^{iv}(\beta)g'^3(\beta) - 14g''(\beta)g'''(\beta)g'^2(\beta) \right. \\
 &+ 14g''(\beta)g'''(\beta)g'(\beta) + 15g''^3(\beta)g'(\beta) - 3g''^3(\beta)\left. \right) e_n^4 + O(e_n^5). \\
 g'(v_n) &= g'(\beta) + \frac{g''(\beta)}{2(-1 + g'(\beta))} e_n^2 + \frac{\left(-2g'''(\beta) + 2g'''(\beta)g'(\beta) - 3g''^2(\beta)\right)}{6(-1 + g'(\beta))^2} e_n^3 \\
 &+ \frac{1}{24(-1 + g'(\beta))^3} \left(g''(\beta)\left(4g^{iv}(\beta) - 8g^{iv}(\beta)g'(\beta) + 4g^{iv}(\beta)g'^2(\beta) + 11g''(\beta)g'''(\beta) \right. \right. \\
 &\left. \left. - 11g''(\beta)g'''(\beta)g'(\beta) + 12g''^3(\beta)\right)\right) e_n^4 + O(e_n^5).
 \end{aligned}$$

Expanding $\left(\frac{g'(v_n) + g'(u_n)}{2}\right)$ in terms of Taylors series about α , we have

$$\begin{aligned}
 \left(\frac{g'(v_n) + g'(u_n)}{2}\right) &= g'(\beta) + \frac{g''(\beta)}{2} e_n + \frac{\left(g''^2(\beta) + g'''(\beta)g'(\beta) - g'''(\beta)\right)}{4(-1 + g'(\beta))} e_n^2 \\
 &+ \frac{1}{12(-1 + g'(\beta))^2} \left(-2g''(\beta)g'''(\beta) + 2g''(\beta)g'''(\beta)g'(\beta) + g^{iv}(\beta) - 3g''^3(\beta) - 2g^{iv}(\beta)g'(\beta) \right. \\
 &+ g^{iv}(\beta)g'^2(\beta)\left. \right) e_n^3 + \frac{1}{48(-1 + g'(\beta))^3} \left(g''(\beta)\left(4g^{iv}(\beta) - 8g^{iv}(\beta)g'(\beta) + 4g^{iv}(\beta)g'^2(\beta) \right. \right. \\
 &\left. \left. + 11g''(\beta)g'''(\beta) - 11g''(\beta)g'''(\beta)g'(\beta) + 12g''^3(\beta)\right)\right) + O(e_n^5).
 \end{aligned}$$

$$\begin{aligned}
 g(v_n) - v_n \left(\frac{g'(v_n) + g'(u_n)}{2}\right) &= -\beta(-1 + g'(\beta)) - \frac{1}{2}\beta g''(\beta)e_n - \frac{1}{4(-1 + g'(\beta))} \left(\beta\left(-g'''(\beta) - g''^2(\beta) \right. \right. \\
 &+ g'''(\beta)g'(\beta)\left. \right) e_n^2 - \frac{1}{12(-1 + g'(\beta))^2} \left(-2\beta g''(\beta)g'''(\beta) + 2\beta g'(\beta)g''(\beta)g'''(\beta) + \beta g^{iv}(\beta) - 2\beta g^{iv}(\beta)g'(\beta) \right. \\
 &+ \beta g^{iv}(\beta)g'^2(\beta) - 3(\beta)g''^2(\beta) - 3g''^2(\beta) + 3g'(\beta)g''^2(\beta)\left. \right) e_n^3 \\
 &- \frac{1}{48(-1 + g'(\beta))^3} \left(g''(\beta)\left(4\beta g^{iv}(\beta) + 11\beta g'''(\beta)g''(\beta) - 8\beta g^{iv}(\beta)g'(\beta) + 4\beta g^{iv}(\beta)g'^2(\beta) \right. \right. \\
 &- 11\beta g'(\beta)g''(\beta)g'''(\beta) - 28g'(\beta)g'''(\beta) + 14g''(\beta)g'^2(\beta) + 12g''^2(\beta) - 12g'(\beta)g''^2(\beta) \\
 &\left. \left. + 14g'''(\beta) + 12g''^3(\beta)\right)\right) e_n^4 + O(e_n^5).
 \end{aligned} \tag{3.7}$$

$$\begin{aligned}
 1 - \left(\frac{g'(v_n) + g'(u_n)}{2}\right) &= 1 - g'(\beta) - \frac{g''(\beta)}{2} e_n - \frac{\left(g''^2(\beta) + g'''(\beta)g'(\beta) - g'''(\beta)\right)}{4(-1 + g'(\beta))} e_n^2 \\
 &- \left(\frac{-2g''(\beta)g'''(\beta) + 2g''(\beta)g'''(\beta)g'(\beta) + g^{iv}(\beta) - 3g''^3(\beta) - 2g^{iv}(\beta)g'(\beta) + g^{iv}(\beta)g'^2(\beta)}{12(-1 + g'(\beta))^2}\right) e_n^3 \\
 &- \frac{1}{48(-1 + g'(\beta))^3} \left(g''(\beta)\left(4g^{iv}(\beta) - 8g^{iv}(\beta)g'(\beta) + 4g^{iv}(\beta)g'^2(\beta) + 11g''(\beta)g'''(\beta) \right. \right. \\
 &\left. \left. - 11g''(\beta)g'''(\beta)g'(\beta) + 12g''^3(\beta)\right)\right) + O(e_n^5).
 \end{aligned} \tag{3.8}$$

Substituting the values (3.7),(3.8) in (2.30) and simplifying, we obtain

$$\begin{aligned} w_n &= \frac{g(v_n) - v_n \left(\frac{g'(v_n) + g'(u_n)}{2} \right)}{1 - \left(\frac{g'(v_n) + g'(u_n)}{2} \right)} \\ &= \beta + \left(\frac{g''(\beta)}{4(-1 + g'(\beta))^2} \right) e_n^3 + \frac{1}{24(-1 + g'(\beta))^3} \left(g''(\beta) (7g'''(\beta) - 7g'''(\beta) - 9g''^2(\beta)) \right) e_n^4 \\ &\quad + O(e_n^5). \end{aligned}$$

Expanding $g(w_n)$ in terms of Taylor series about α , we obtain

$$\begin{aligned} g(w_n) &= \beta + \left(\frac{g'(\beta)g''(\beta)}{4(-1 + g'(\beta))^2} \right) e_n^3 + \frac{1}{24(-1 + g'(\beta))^3} \left(g''(\beta) (7g'''(\beta)g'(\beta) - 7g'''(\beta) - 9g''^2(\beta)) \right) e_n^4 \\ &\quad + O(e_n^5). \end{aligned} \tag{3.9}$$

Expanding $\left(\frac{g'(w_n) + g'(u_n)}{2} \right)$ in terms of Taylor series about α , we have

$$\begin{aligned} \left(\frac{g'(w_n) + g'(u_n)}{2} \right) &= g'(\beta) + \frac{g''(\beta)}{2} e_n + \frac{g'''(\beta)}{4} e_n^2 + \frac{1}{24(-1 + g'(\beta))^2} \left(2g^{iv}(\beta) - 4g^{iv}(\beta)g'(\beta) \right. \\ &\quad \left. + 2g^{iv}(\beta)g'^2(\beta) + 3g''^3(\beta) \right) e_n^3 + \left(\frac{g''^2(\beta) (7g'''(\beta)g'(\beta) - 7g'''(\beta) - 9g''^2(\beta))}{48(-1 + g'(\beta))^3} \right) e_n^4 + O(e_n^5). \end{aligned} \tag{3.10}$$

$$\begin{aligned} g(w_n) - w_n \left(\frac{g'(w_n) + g'(u_n)}{2} \right) &= -\beta(-1 + g'(\beta)) - \frac{1}{2}\beta g''(\beta)e_n - \frac{1}{4}\beta g'''(\beta)e_n^2 \\ &\quad - \frac{1}{24(-1 + g'(\beta))^2} \left(\beta (2g^{iv}(\beta) - 4g^{iv}(\beta)g'(\beta) + 2g^{iv}(\beta)g'^2(\beta) + 3g''^3(\beta)) \right) e_n^3 \\ &\quad - \frac{1}{48(-1 + g'(\beta))^3} \left(g''^2(\beta) (7\beta g'''(\beta)g'(\beta) - 9\beta g''^2(\beta) - 7\beta g'''(\beta) - 6g''(\beta) + 6g'(\beta)g''(\beta)) \right) e_n^4 \\ &\quad + O(e_n^5). \end{aligned} \tag{3.11}$$

$$\begin{aligned} 1 - \left(\frac{g'(w_n) + g'(u_n)}{2} \right) &= 1 - g'(\beta) - \frac{g''(\beta)}{2} e_n - \frac{g'''(\beta)}{4} e_n^2 - \frac{1}{24(-1 + g'(\beta))^2} \left(2g^{iv}(\beta) \right. \\ &\quad \left. - 4g^{iv}(\beta)g'(\beta) + 2g^{iv}(\beta)g'^2(\beta) + 3g''^3(\beta) \right) e_n^3 \\ &\quad - \frac{1}{48(-1 + g'(\beta))^3} \left(g''^2(\beta) (7g'''(\beta)g'(\beta) - 7g'''(\beta) - 9g''^2(\beta)) \right) e_n^4 \\ &\quad + O(e_n^5). \end{aligned} \tag{3.12}$$

Substituting the values from (3.11) and (3.12) in (2.31) and simplifying

$$\begin{aligned} u_{n+1} &= \frac{g(w_n) - w_n g' \left(\frac{g'(w_n) + g'(u_n)}{2} \right)}{1 - \left(\frac{g'(w_n) + g'(u_n)}{2} \right)} \\ &= \beta + \left(\frac{g''^2(\beta)}{8(-1 + g'(\beta))^3} \right) e_n^4 + O(e_n^5) \\ e_{n+1} &= \beta + \left(\frac{g''^2(\beta)}{8(-1 + g'(\beta))^3} \right) e_n^4 + O(e_n^5). \end{aligned}$$

This shows that Algorithm 2E has fourth order convergence. □

4. NUMERICAL RESULTS AND APPLICATIONS

This section elaborated the efficacy of algorithms introduced in this paper with the support of examples. We obtain estimated simple root rather than the exact based on the exactness ϵ of the computer. We utilize $\epsilon=10^{-15}$, For computational work we use the following stopping criteria

- (i). $|u_{n+1} - u_n| < \epsilon$ **and** (ii). $|f(u_{n+1})| < \epsilon$

We make a comparative representation of the Chun method (CM)[6], Newton method (NM), the Abbasbandy method (AM)[1], Halley method (HM) [12,13], Weerakoon and Fernando(WF) [30], with the Algorithm 2B (AG1), Algorithm 2C (AG2) and Algorithm 2D (AG3) and Algorithm 2E (AG4).As for the convergence criteria, it was required that the distance of two successive estimation for the zero was less than 10^{-15} . In Table 1 we also displayed, the number of iterations (IT), the approximate root u_n and the value $f(u_n)$

The examples are the same as in Chun [4]

$f_1(u) = u^3 - 10,$	$g(u) = \sqrt{\frac{10}{u}}$
$f_2(u) = \cos u - u,$	$g(u) = \cos u$
$f_3(u) = (u - 1)^3 - 1,$	$g(u) = 1 + \sqrt{\frac{1}{u - 1}}$
$f_4(u) = e^{u^2+7u-30} - 1,$	$g(u) = \frac{1}{7}(30 - u^2)$
$f_5(u) = \sin^2 u - u^2 + 1,$	$g(u) = \sin u + \frac{1}{\sin u + u}$
$f_6(u) = u^2 - e^u - 3u,$	$g(u) = \frac{u^2 - e^u + 2}{3}$

Table1.Numerical Examples

$f_1(u) = u^3 - 10, \quad g(u) = \sqrt{\frac{10}{u}}, \quad u_0=1.5$				
Methods	IT	u_n	$f(u_n)$	$\delta = u_n - u_{n-1} $
NM	6	2.1544346900318837	7.857615e-27	3.486728e-14
HM	6	2.1544346900318837	6.573851e-30	1.008516e-15
AM	4	2.1544346900318837	3.071853e-26	1.972938e-09
CM	9	2.1544346900318837	1.399647e-29	1.040560e-15
WF	4	2.1544346900318837	7.066989e-31	5.866631e-11
AG1	3	2.1544346900318837	8.936935e-27	3.625733e-09
AG2	3	2.1544346900318837	9.833722e-66	1.458067e-16
AG3	3	2.1544346900318837	4.856920e-26	6.374604e-09
AG4	3	2.1544346900318837	7.081450e-63	7.553158e-16
$f_2(u) = \cos u - u, \quad g(u) = \cos u, \quad u_0=1.7$				

NM	4	0.7390851332151609	3.924473e-16	3.258805e-08
HM	5	0.7390851332151606	2.373589e-27	8.014391e-14
AM	4	0.7390851332151606	8.935133e-33	2.845706e-11
CM	5	0.7390851332151606	7.035845e-25	9.756878e-13
WF	3	0.7390851332151606	1.735664e-21	4.084953e-07
AG1	3	0.7390851332151606	5.258162e-26	8.637471e-09
AG2	3	0.7390851332151606	3.458981e-60	3.722340e-15
AG3	3	0.7390851332151606	1.117701e-27	2.392676e-09
AG4	3	0.7390851332151606	2.655872e-66	1.101871e-16
$f_3(u) = (u - 1)^3 - 1, \quad g(u) = 1 + \sqrt{\frac{1}{u-1}}, \quad u_0=3.5$				
NM	7	2	2.484117e-21	2.877567e-11
HM	8	2	8.422670e-24	1.675577e-12
AM	6	2	2.626229e-20	6.615927e-11
CM	5	2	7.505295e-35	2.657272e-12
WF	5	2	9.839450e-37	6.550899e-13
AG1	3	2	1.956948e-17	4.708259e-06
AG2	3	2	1.845148e-41	1.408551e-10
AG3	3	2	2.746096e-20	5.271145e-07
AG4	3	2	9.745968e-50	1.200801e-12
$f_4(u) = e^{u^2+7u-30} - 1, \quad g(u) = \frac{1}{7}(30 - u^2), \quad u_0=3.5$				
NM	12	3	5.475744e-24	2.530687e-13
HM		Diverge		
AM	7	3	4.775960e-18	2.353603e-07
CM	8	3	9.665693e-21	7.518279e-12
WF	8	3	5.290016e-24	1.916154e-09
AG1	4	3	1.938304e-114	2.931715e-38
AG2	3	3	2.118685e-21	2.446184e-05
AG3	3	3	3.811229e-37	1.704784e-12
AG4	3	3	4.174513e-90	1.629758e-22

$f_5(u) = \sin^2 u - u^2 + 1, \quad g(u) = \sin u + \frac{1}{\sin u + u}, \quad u_0 = -1$				
NM	6	1.4044916482153412	1.819126e-25	3.058088e-13
HM	6	1.4044916482153412	9.561689e-28	2.217103e-14
CM	6	1.4044916482153412	1.669602e-16	6.551037e-09
AM	4	1.4044916482153412	6.138132e-21	1.329320e-07
WF	4	1.4044916482153412	9.413532e-30	1.793023e-10
AG1	4	1.4044916482153412	2.097175e-88	9.878333e-30
AG2	3	1.4044916482153412	7.391875e-68	3.273071e-17
AG3	3	1.4044916482153412	1.895915e-31	9.551650e-11
AG4	3	1.4044916482153412	3.473304e-75	4.818950e-19
$f_6(u) = u^2 - e^u - 3u, \quad g(u) = \frac{u^2 - e^u + 2}{3}, \quad u_0 = 2$				
NM	5	1.4044916482153412	3.439576e-27	9.869210e-14
HM	8	1.4044916482153412	1.163256e-29	5.739414e-15
CM	4	1.4044916482153412	7.664653e-31	1.280280e-10
AM	5	1.4044916482153412	9.348485e-20	3.638191e-10
WF	4	1.4044916482153412	6.103947e-34	1.630507e-11
AG1	3	1.4044916482153412	4.442861e-18	5.125169e-06
AG2	3	1.4044916482153412	7.862110e-56	7.105574e-14
AG3	4	1.4044916482153412	6.398407e-50	6.398407e-501
AG4	3	1.4044916482153412	5.810512e-38	2.083378e-09

5. CONCLUSION

We have developed the Algorithms 2A,2B,2C,2D,2E by approximating the definite integral $\int_{\gamma}^u g(u)du = g(u) - g(\gamma)$ by mean of quadrature rule along with writing the nonlinear equation as coupled system of equation and using the decomposition method. We have determined the convergence analysis of the newly suggested iterative schemes. With the help of test examples, computational comparison has been made with well known third and fourth order convergent iterative methods. Moreover it is observed that multistep algorithms suggested in this article requires only first derivative evaluation of function and our results can be considered as an alternative to already known third and fourth order convergent iterative method.

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