



## FUZZINESS AND ROUGHNESS IN HYPERQUANTALES

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**ABSTRACT.** Theories of fuzzy set and rough set are powerful mathematical tools for modelling various types of uncertainty. In this paper, we introduce the notions of bi-hyperideal, fuzzy bi-hyperideals of hyperquantales and their related properties is given. Furthermore we introduce the notion of generalized rough fuzzy bi-hyperideals. Moreover, we will describe the set-valued homomorphism and strong set-valued homomorphism of hyperquantales and some related properties will be study.

### 1. INTRODUCTION

The theory of rough sets was introduced by Pawlak [15,16], to deal with uncertain knowledge in information systems. The rough set theory has been emerged as another major mathematical approach for managing uncertainty that arises from inexact, noisy or incomplete information. It has turned out to be fundamentally important in artificial intelligence and cognitive sciences, especially in fields such as machine learning, knowledge acquisition, decision analysis, expert systems, pattern recognition. With the development of rough set theory, possible connections between rough sets and various algebraic systems were considered by many authors. Inspired by the construction of Pawlak rough set algebras and the investigation in algebraic properties

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of rough sets in [17,18]. As a combination of algebraic structures and partially ordered structures, the theory of quantales was initiated by Mulvey [20] to study the spectrum of C-algebras and the foundations of quantum mechanics. Wang and Zhao [22,23] proposed the concepts of ideals and prime ideals of quantales. Yang and Xu [24], considered quantales as universal sets and introduced the notions of rough (prime, semi-prime, primary) ideals and prime radicals of upper rough ideals of quantales. Wang in [21] studied prime radical theorem in quantales. The concept of fuzzy sets was introduced by Zadeh [19] in 1965. The theory of fuzzy sets has been developed fast and has many applications in many branches of sciences. Luo and Wang in [25], studied roughness and fuzziness in quantales. Davvaz et al. in [6] applied Atanassov's intuitionistic fuzzy set theory to quantales. In [29,30], Saqib and Shabir studied relationship between generalized rough sets and quantale by using fuzzy ideals of quantale.

Algebraic hyperstructures represent a natural extension of classical algebraic structures and they were originally proposed in 1934 by a French mathematician Marty [1], at the 8<sup>th</sup> Congress of Scandinavian Mathematicians. One of the main reason which attracts researches towards hyperstructures is its unique property that in hyperstructures composition of two elements is a set, while in classical algebraic structures the composition of two elements is an element. Thus algebraic hyperstructures are natural extension of classical algebraic structures. Since then, hyperstructures are widely investigated from the theoretical point of view and for their applications to many branches of pure and applied mathematics (see [2-5,8]). Since then, there appeared many components of hyperalgebras such as hypergroups in [9], hyperrings etc in [10,11]. Konstantinidou and Mittas have introduced the concept of hyperlattices in [12], fuzzy ideal of hyperlattices have been introduced in [7]. The notion of hyperlattices is a generalization of the notion of lattices and there are some intimate connections between hyperlattices and lattices. In particular, Rasouli and Davvaz further studied the theory of hyperlattices and obtained some interesting results [13,14], which enrich the theory of hyperlattices. In [28], Estaji and Bayati studied rough Sets in terms of Hyperlattices. In [26], Khan et al. introduced the notions of hyperideals and fuzzy hyperideals of hyperquantales.

In this paper, we introduce the notions of bi-hyperideal and fuzzy bi-hyperideals of hyperquantales and give several characterizations. In addition, we will introduce the notions of generalized rough fuzzy bi-hyperideal in hyperquantales and some new properties will be obtain.

## 2. PRELIMINARIES

A map  $*$  :  $S \times S \rightarrow \mathcal{P}^*(S)$  is called *hyperoperation* or *join operation* on the set  $S$ , where  $S$  is a non-empty set and  $\mathcal{P}^*(S) = \mathcal{P}(S) \setminus \{\emptyset\}$  denotes the set of all non-empty subsets of  $S$ .

A *hyperstructure* is called the pair  $(S, *)$  where  $*$  is a hyperoperation on the set  $S$ .

**Definition 2.1.** (see [26]). A hyperquantale is a complete hyperlattice  $Q$  with an associative binary operation  $*$  satisfying  $x * (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x * y_i)$ ,  $(\bigvee_{i \in I} x_i) * y = \bigvee_{i \in I} (x_i * y)$  for all  $x, y, x_i, y_i \in Q$  ( $i \in I$ ) where  $I$  is an index set.

A hyperquantale  $Q$  is called commutative if  $x * y = y * x$  for all  $x, y \in Q$ .

Throughout this paper, we denote the least and greatest elements of a hyperquantale denoted by  $\perp$  and  $\top$  respectively.

**Definition 2.2.** (see [26]). Let  $Q$  be a hyperquantale. A non-empty subset  $A$  of  $Q$  is called a left (resp. right) hyperideal of  $Q$  if it satisfies the following conditions:

- (1)  $x, y \in A$  implies  $x \vee y \subseteq A$ .
- (2)  $(\forall x, y \in Q) x \in A$  and  $y \leq x$  imply  $y \in A$ .
- (3)  $\forall x \in Q$  and  $a \in A$ , we have  $x * a \subseteq A$  (resp.  $a * x \subseteq A$ ).

A non empty subset  $A$  of  $Q$  is called a two sided hyperideal or simply a hyperideal of  $Q$  if it is both a left hyperideal and right hyperideal of  $Q$ .

**Definition 2.3.** Let  $Q$  be a hyperquantale. A non-empty subset  $B$  of  $Q$  is called a bi-hyperideal of  $Q$  if it satisfies the following conditions:

- (1)  $x, y \in B$  implies  $x \vee y \subseteq B$ .
- (2)  $x, y \in B$  implies  $x * y \subseteq B$ .
- (3)  $(\forall x, y \in Q) x \in B$  and  $y \leq x$  imply  $y \in B$ .
- (4)  $\forall y \in Q$  and  $x, z \in B$ , we have  $x * y * z \subseteq B$ .

**Example 2.1.** Let  $Q = \{\perp, e_1, e_2, e_3, \top\}$  and define  $*$  and  $\vee$  by the following Cayley tables:

$*$	$\perp$	$e_1$	$e_2$	$e_3$	$\top$
$\perp$	$\{\perp\}$	$\{\perp\}$	$\{\perp\}$	$\{\perp\}$	$\{\perp\}$
$e_1$	$\{\perp\}$	$\{e_1\}$	$\{e_1\}$	$\{e_1\}$	$\{e_1\}$
$e_2$	$\{\perp\}$	$\{e_1\}$	$\{e_2\}$	$\{e_1\}$	$\{e_2\}$
$e_3$	$\{\perp\}$	$\{e_1\}$	$\{e_1\}$	$\{e_3\}$	$\{e_3\}$
$\top$	$\{\perp\}$	$\{e_1\}$	$\{e_2\}$	$\{e_3\}$	$\{\top\}$

and

$\vee$	$\perp$	$e_1$	$e_2$	$e_3$	$\top$
$\perp$	$\{\perp\}$	$\{e_1\}$	$\{e_2\}$	$\{e_3\}$	$\{\top\}$
$e_1$	$\{e_1\}$	$\{\perp, e_1\}$	$\{\perp, e_2\}$	$\{\perp, e_3\}$	$\{\perp, \top\}$
$e_2$	$\{e_2\}$	$\{\perp, e_2\}$	$\{\perp, e_1, e_2\}$	$\{\perp, \top\}$	$\{\perp, e_3, \top\}$
$e_3$	$\{e_3\}$	$\{\perp, e_3\}$	$\{\perp, \top\}$	$\{\perp, e_1, e_3\}$	$\{\perp, e_2, \top\}$
$\top$	$\{\top\}$	$\{\perp, \top\}$	$\{\perp, e_3, \top\}$	$\{\perp, e_2, \top\}$	$\{\perp, e_1, e_2, e_3, \top\}$

Thus all bi-hyperideals of  $Q$  are  $\{\perp\}, \{\perp, e_1\}, \{\perp, e_1, e_2\}, \{\perp, e_1, e_3\}$  and  $Q$ .

For  $A, B \subseteq Q$ , we have  $A * B := \bigcup\{a * b : a \in A, b \in B\}$  and

$A \vee B := \bigcup\{a \vee b : a \in A, b \in B\}$ .

For  $A \subseteq Q$ , we denote  $(A) := \{a \in Q : a \leq b \text{ for some } b \in A\}$ .

### 3. FUZZY HYPERIDEALS OF HYPERQUANTALE

Let  $Q$  be a hyperquantale. A function  $f$  from a nonempty set  $X$  to the unit interval  $[0, 1]$  is called a fuzzy subset of  $Q$ .

Let  $Q$  be a hyperquantale and  $f$  be a fuzzy subset of  $Q$ . Then for every  $t \in [0, 1]$  the set

$$U(f; t) = \{x \mid x \in Q, f(x) \geq t\}.$$

For  $x \in Q$ , we define  $A_x = \{(y, z) \in Q \times Q \mid x \leq y * z\}$ .

**Definition 3.1.** (see [26]). Let  $Q$  be a hyperquantale and  $f, g$  are any two fuzzy subsets of  $Q$ . We define the product  $f \circ g$  of  $f$  and  $g$  as follows:

$$(f \circ g)(x) = \begin{cases} \bigvee_{(y,z) \in A_x} \{f(y) \wedge g(z)\}, & \text{if } A_x \neq \emptyset \\ 0, & \text{if } A_x = \emptyset \end{cases}.$$

For two functions  $f$  and  $g$  then  $f \subseteq g$  if and only if  $f(x) \leq g(x)$ .

Let  $Q$  be a hyperquantale and  $\emptyset \neq A \subseteq Q$ . Then the characteristic function  $\chi_A$  of  $A$  is defined as:

$$\chi_A : Q \longrightarrow [0, 1], \longrightarrow \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

**Definition 3.2.** (see [26]). Let  $Q$  be a hyperquantale. A fuzzy subset  $f$  of  $Q$  is called a fuzzy subhyperquantale of  $Q$  if it satisfies the following conditions:

- (1)  $(\forall x, y \in Q) \bigwedge_{\alpha \in x * y} f(\alpha) \geq f(x) \wedge f(y)$ .
- (2)  $(\forall x, y \in Q) \bigwedge_{\beta \in x \vee y} f(\beta) \geq f(x) \wedge f(y)$ .

**Definition 3.3.** (see [26]). Let  $Q$  be a hyperquantale. A fuzzy subset  $f$  of  $Q$  is called a fuzzy left (resp. right) hyperideal of  $Q$  if it satisfies the following conditions:

- (1)  $(\forall x, y \in Q) \bigwedge_{\alpha \in x*y} f(\alpha) \geq f(y)$  (resp.  $\bigwedge_{\alpha \in x*y} f(\alpha) \geq f(x)$ ).
- (2)  $(\forall x, y \in Q) \bigwedge_{\beta \in x \vee y} f(\beta) \geq f(x) \wedge f(y)$ .
- (3)  $(\forall x, y \in Q) x \leq y$  then  $f(x) \geq f(y)$ .

#### 4. FUZZY BI-HYPERIDEAL OF HYPERQUANTALE

In this section, we introduce the notion of fuzzy bi-hyperideal of hyperquantale and investigate some related properties.

**Definition 4.1.** Let  $Q$  be a hyperquantale. A fuzzy subset  $f$  of  $Q$  is called a fuzzy bi-hyperideal of  $Q$  if it satisfies the following conditions:

- (1)  $(\forall x, y \in Q) \bigwedge_{\alpha \in x*y} f(\alpha) \geq f(x) \wedge f(y)$ .
- (2)  $(\forall x, y \in Q) \bigwedge_{\beta \in x \vee y} f(\beta) \geq f(x) \wedge f(y)$ .
- (3)  $(\forall x, y, z \in Q) \bigwedge_{\gamma \in (x*y*z)} f(\gamma) \geq f(x) \wedge f(z)$ .
- (4)  $(\forall x, y \in Q) x \leq y$  then  $f(x) \geq f(y)$ .

**Example 4.1.** Let  $Q = \{\perp, e_1, e_2, \top\}$  and define  $*$  and  $\vee$  by the following Cayley tables:

$*$	$\perp$	$e_1$	$e_2$	$\top$
$\perp$	$\{\perp\}$	$\{\perp\}$	$\{\perp\}$	$\{\perp\}$
$e_1$	$\{\perp\}$	$\{e_1\}$	$\{\perp\}$	$\{e_1\}$
$e_2$	$\{\perp\}$	$\{\perp\}$	$\{e_2\}$	$\{e_2\}$
$\top$	$\{\perp\}$	$\{e_1\}$	$\{e_2\}$	$\{\top\}$

and

$\vee$	$\perp$	$e_1$	$e_2$	$\top$
$\perp$	$\{\perp\}$	$\{e_1\}$	$\{e_2\}$	$\{\top\}$
$e_1$	$\{e_1\}$	$\{\perp, e_1\}$	$\{\top\}$	$\{e_2, \top\}$
$e_2$	$\{e_2\}$	$\{\top\}$	$\{\perp, e_2\}$	$\{e_1, \top\}$
$\top$	$\{\top\}$	$\{e_2, \top\}$	$\{e_1, \top\}$	$Q$

Let us define a fuzzy subset  $f : Q \rightarrow [0, 1]$  as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = \perp \\ 0.4 & \text{if } x \in \{e_1, e_2, \top\} \end{cases}$$

Then it is easy to verify that  $f$  is a fuzzy bi-hyperideal of  $Q$ .

**Theorem 4.1.** *Let  $B$  be a non empty subset of a hyperquantale  $Q$ . Then  $B$  is a bi-hyperideal of  $Q$  if and only if  $\chi_B$  is a fuzzy hyperideal of  $Q$ .*

*Proof.* Suppose that  $B$  is a hyperideal of  $Q$ . Let  $x, y \in Q$ . If  $x, y \in B$  then  $x \vee y \subseteq B$  and  $x * y \subseteq B$ . Since  $x, y \in B$ , we have  $\chi_B(x) = \chi_B(y) = 1$ , for any  $\alpha \in x \vee y \subseteq B$ , we have  $\bigwedge_{\alpha \in x \vee y} \chi_B(\alpha) = 1 = \chi_B(x) \wedge \chi_B(y)$ . Also for any  $\beta \in x * y \subseteq B$ , we have  $\bigwedge_{\beta \in x * y} \chi_B(\beta) = 1 = \chi_B(x) \wedge \chi_B(y)$ . If  $x \notin B$  or  $y \notin B$ . Then  $x \vee y \subseteq B$  or  $x \vee y \not\subseteq B$  and  $x * y \subseteq B$  or  $x * y \not\subseteq B$ . In all the cases we have  $\chi_B(x) \wedge \chi_B(y) = 0 \leq \bigwedge_{\alpha \in x \vee y} \chi_B(\alpha)$  and  $\chi_B(x) \wedge \chi_B(y) = 0 \leq \bigwedge_{\beta \in x * y} \chi_B(\beta)$ . Let now  $x, y \in Q, x \leq y$ . Then  $\chi_B(x) \geq \chi_B(y)$ . In fact, if  $y \in B$ , then  $\chi_B(y) = 1$ . Since  $Q \ni x \leq y \in B$ , by hypothesis we have  $x \in B$ , then  $\chi_B(x) = 1$ . Thus  $\chi_B(x) \geq \chi_B(y)$ . If  $y \notin B$ , then  $\chi_B(y) = 0$ . Since  $x \in Q$ , we have  $\chi_B(x) \geq 0 = \chi_B(y)$ . Let  $x, y$  and  $z$  be any elements of  $Q$ . If  $x, z \in B$ , then  $\chi_B(x) = \chi_B(z) = 1$  and since for every  $\alpha \in x * y * z \subseteq B$ , we have  $\chi_B(\alpha) = 1 = \chi_B(x) \wedge \chi_B(z)$ . Thus  $\bigwedge_{\alpha \in x * y * z} \chi_B(\alpha) = 1 = \chi_B(x) \wedge \chi_B(z)$ . If  $x \notin B$  or  $z \notin B$ , then  $\chi_B(x) = 0$  or  $\chi_B(z) = 0$ , and so we have  $\chi_B(\alpha) \geq 0 = \chi_B(x) \wedge \chi_B(z)$ . Thus  $\bigwedge_{\alpha \in x * y * z} \chi_B(\alpha) \geq \chi_B(x) \wedge \chi_B(z)$ . Therefore  $\chi_B$  is a fuzzy bi-hyperideal of  $Q$ .

Conversely, assume that  $\chi_B$  is a fuzzy bi-hyperideal of  $Q$ . Let  $x, y \in B$ . Then  $\bigwedge_{\alpha \in x \vee y} \chi_B(\alpha) = \chi_B(x) \wedge \chi_B(y) = 1$ , and thus  $\alpha \in x \vee y \subseteq B$ . Since  $x, y \in B$ . Then for any  $z \in x * y$ , we have  $\bigwedge_{z \in x * y} \chi_A(z) \geq \chi_A(x) \wedge \chi_A(y) = 1$ . Implies that  $\bigwedge_{z \in x * y} \chi_A(z) = 1$ . Thus  $x * y \subseteq A$ . If  $x \leq y$  and  $y \in B$ , then  $\chi_B(x) \geq \chi_B(y) = 1$ , implies that  $x \in B$ . Let  $x, z \in B$  and  $y \in S$  such that for any  $\alpha \in x * y * z$ , we have Since

$$\begin{aligned} \bigwedge_{\alpha \in x * y * z} \chi_B(\alpha) &\geq \chi_B(x) \wedge \chi_B(z) \\ &= 1 \wedge 1 \\ &= 1. \end{aligned}$$

Hence for each  $\alpha \in x * y * z$ , we have  $\chi_B(\alpha) = 1$ , and so  $\alpha \in B$ . Thus  $x * y * z \subseteq B$ . Thus  $B$  is a bi-hyperideal of  $Q$ . □

**Theorem 4.2.** *Let  $Q$  be a hyperquantale. A fuzzy subset  $f$  of  $Q$  is a fuzzy bi-hyperideal of  $Q$  if and only if for each  $t \in [0, 1], U(f; t) \neq \emptyset$  is a bi-hyperideal of  $Q$ .*

*Proof.* Assume that  $U(f; t)$  is a bi-hyperideal of  $Q$ . Let  $x, y \in Q$  such that  $x \leq y$ . If  $f(y) = 0$  then  $f(x) \geq f(y)$ . If  $f(y) = t$  then  $y \in U(f; t)$ . Since  $x \leq y$  and  $U(f; t)$  is a bi-hyperideal of  $Q$ , we have  $x \in U(f; t)$ . Then  $f(x) \geq t = f(y)$ . Since  $U(f; t) \neq \emptyset$  is a bi-hyperideal of  $Q$ . If  $\bigwedge_{\alpha \in x * y} f(\alpha) < f(x) \wedge f(y)$  for some  $x, y \in Q$ , then there exists  $t_0 \in [0, 1]$  such that  $\bigwedge_{\alpha \in x * y} f(\alpha) < t_0 \leq f(x) \wedge f(y)$ , which implies that  $x, y \in U(f; t)$  and  $x * y \not\subseteq U(f; t)$ . It contradicts the fact that  $U(f; t)$  is a bi-hyperideal of  $Q$ . Consequently,

$\bigwedge_{\alpha \in x * y} f(\alpha) \geq f(x) \wedge f(y)$  for all  $x, y \in Q$ . Next we show that  $\bigwedge_{B \in x \vee y} f(\alpha) \geq f(x) \wedge f(y)$  for all  $x, y \in Q$ . If there exist  $x, y \in Q$  and  $t_0 \in [0, 1]$  such that  $\bigwedge_{\beta \in x \vee y} f(\beta) < t_0 \leq f(x) \wedge f(y)$ . Then  $x, y \in U(f; t)$  and

$\beta \in x \vee y \not\subseteq U(f; t)$ . It is again contradicts the fact that  $U(f; t)$  is a bi-hyperideal of  $Q$ . Thus  $\bigwedge_{\beta \in x \vee y} f(\beta) \geq f(x) \wedge f(y)$ . Now let  $x, y, z \in U(f; t)$ . Then  $x * y * z \subseteq U(f; t)$ . Since  $x, z \in U(f; t)$ . Then  $f(x) \geq t$  and  $f(z) \geq t$ . So for any  $\alpha \in x * y * z$ , we have  $f(\alpha) \geq t$ . Thus  $f(x) \wedge f(z) = t \leq \bigwedge_{\alpha \in x * y * z} f(\alpha)$ . Therefore  $f$  is a fuzzy bi-hyperideal of  $Q$ .

Conversely, suppose that  $f$  be a fuzzy bi-hyperideal of  $Q$ . Let  $x, y \in U(f; t)$ . Then  $f(x) \geq t, f(y) \geq t$ . Since  $f$  is a fuzzy bi-hyperideal of  $Q$ , so we have  $\bigwedge_{\alpha \in x * y} f(\alpha) \geq f(x) \wedge f(y) = t$ . Hence  $f(\alpha) \geq t$  for all  $\alpha \in x * y$ , this implies  $\alpha \in U(f; t)$  that is  $x * y \subseteq U(f; t)$ . As  $f$  is a fuzzy bi-hyperideal of  $Q$ . Then  $\bigwedge_{w \in x \vee y} f(w) \geq f(x) \wedge f(y) \geq t$ . Hence  $f(w) \geq t$  for any  $w \in x \vee y$  implies that  $w \in U(f; t)$ . Thus  $x \vee y \subseteq U(f; t)$ . Now let  $x, y, z \in U(f; t)$ . Then  $f(x) \geq t, f(y) \geq t$  and  $f(z) \geq t$ . Since  $f$  is a fuzzy bi-hyperideal of  $Q$ , we have  $\bigwedge_{\beta \in x * y * z} f(\beta) \geq f(x) \wedge f(z) = t$ . So  $f(\beta) \geq t$ . Hence  $x * y * z \subseteq U(f; t)$ . Let  $x \in U(f; t)$  and  $y \in Q$  with  $y \leq x$ . Then  $t \leq f(x) \leq f_A(y)$ , we get  $y \in U(f; t)$ . Therefore  $U(f; t)$  is a bi-hyperideal of  $Q$ . □

**Theorem 4.3.** Let  $\{f_i \mid i \in I\}$  be a family of fuzzy bi-hyperideals of  $Q$ . Then  $f = \bigcap_{i \in I} f_i$  is a fuzzy bi-hyperideal of  $Q$  where  $(\bigcap_{i \in I} f_i)(x) = \bigwedge_{i \in I} (f_i(x))$ .

*Proof.* Let  $x, y \in Q$ . Then, since each  $f_i (i \in I)$  is a fuzzy bi-hyperideal of  $Q$ , so  $\bigwedge_{\alpha \in x \vee y} f_i(\alpha) \geq f_i(x) \wedge f_i(y)$ . Thus for any  $\alpha \in x \vee y, f_i(\alpha) \geq f_i(x) \wedge f_i(y)$ , and we have

$$\begin{aligned} f(\alpha) &= \left( \bigcap_{i \in I} f_i \right)(\alpha) \\ &= \bigwedge_{i \in I} (f_i(\alpha)) \\ &\geq \bigwedge_{i \in I} (f_i(x) \wedge f_i(y)) \\ &= \left( \bigwedge_{i \in I} (f_i(x)) \right) \wedge \left( \bigwedge_{i \in I} (f_i(y)) \right) \\ &= \left( \bigcap_{i \in I} f_i \right)(x) \wedge \left( \bigcap_{i \in I} f_i \right)(y) \\ &= f(x) \wedge f(y), \end{aligned}$$

which implies that  $\bigwedge_{\alpha \in x \vee y} f(\alpha) \geq f(x) \wedge f(y)$ . Let  $\beta \in x * y$  and  $\bigwedge_{\beta \in x * y} f_i(\beta) \geq f_i(x) \wedge f_i(y)$ . Thus for any  $\beta \in x * y, f_i(\beta) \geq f_i(x) \wedge f_i(y)$ . Then

$$\begin{aligned}
 f(\beta) &= \left(\bigcap_{i \in I} f_i\right)(\beta) \\
 &= \bigwedge_{i \in I} (f_i(\beta)) \\
 &\geq \bigwedge_{i \in I} (f_i(x) \wedge f_i(y)) \\
 &= \left(\bigcap_{i \in I} f_i\right)(x) \wedge \left(\bigcap_{i \in I} f_i\right)(y) \\
 &= f(x) \wedge f(y).
 \end{aligned}$$

Thus  $\bigwedge_{\beta \in x*y} f(\beta) \geq f(x) \wedge f(y)$ . Now let  $x, y, z \in Q$ . Then for any  $\gamma \in x*y*z$ , we have

$$\begin{aligned}
 f(\gamma) &= \left(\bigcap_{i \in I} f_i\right)(\gamma) \\
 &= \bigwedge_{i \in I} (f_i(\gamma)) \\
 &\geq \bigwedge_{i \in I} (f_i(x) \wedge f_i(z)) \\
 &= \left(\bigwedge_{i \in I} f_i(x)\right) \wedge \left(\bigwedge_{i \in I} f_i(z)\right) \\
 &= \left(\bigcap_{i \in I} f_i\right)(x) \wedge \left(\bigcap_{i \in I} f_i\right)(z) \\
 &= f(x) \wedge f(z).
 \end{aligned}$$

Thus  $\bigwedge_{\gamma \in x*y*z} f(\gamma) \geq f(x) \wedge f(z)$ .

Furthermore, if  $x \leq y$ , then  $f(x) \geq f(y)$ . Indeed: Since every  $f_i$  ( $i \in I$ ) is a fuzzy bi-hyperideal of  $Q$ , it can be obtained that  $f_i(x) \geq f_i(y)$  for all  $i \in I$ . Thus

$$\begin{aligned}
 f(x) &= \left(\bigcap_{i \in I} f_i\right)(x) \\
 &= \bigwedge_{i \in I} (f_i(x)) \\
 &\geq \bigwedge_{i \in I} (f_i(y)) \\
 &= \left(\bigcap_{i \in I} f_i\right)(y) \\
 &= f(y).
 \end{aligned}$$

Thus  $f = \bigcap_{i \in I} f_i$  is a fuzzy bi-hyperideal of  $Q$ . □

5. HOMOMORPHISM AND GENERALIZED ROUGH FUZZY BI-HYPERIDEALS OF HYPERQUANTALES

**Definition 5.1.** (see [27]). Let  $X$  and  $Y$  be two nonempty universes. Let  $F$  be a set-valued mapping given by  $F : X \rightarrow \mathcal{P}(Y)$ , where  $\mathcal{P}(Y)$  is the power set of  $Y$ . Then the triple  $(X, Y, F)$  is referred to as a generalized approximation space or generalized rough set. Any set-valued function from  $X$  to  $\mathcal{P}(Y)$  defines a binary relation from  $X$  to  $Y$  by setting  $\rho_F = \{(a, b) \mid b \in F(a)\}$ . Obviously, if  $\rho$  is an arbitrary relation from  $X$  to  $Y$ , then a set-valued mapping  $F_\rho : X \rightarrow \mathcal{P}(Y)$  can be defined by  $F_\rho(a) = \{b \in Y \mid (a, b) \in \rho\}$  where  $a \in X$ . For any set  $A \subseteq Y$ , the lower and upper approximations represented by  $F^-(A)$  and  $F^+(A)$  respectively, are defined as

$$F^-(A) = \{a \in X \mid F(a) \subseteq A\},$$

$$F^+(A) = \{a \in X \mid F(a) \cap A \neq \emptyset\}.$$

We call the pair  $(F^-(A), F^+(A))$  generalized rough set, and  $F^-, F^+$  are termed as lower and upper generalized approximation operators, respectively.

**Definition 5.2.** Let  $(Q_1, *_1)$  and  $(Q_2, *_2)$  be two hyperquantales. A set-valued mapping  $F : Q_1 \rightarrow \mathcal{P}^*(Q_2)$ , where  $\mathcal{P}^*(Q_2)$  represents the collection of all nonempty subsets of  $Q_2$  is called a set-valued homomorphism if, for all  $a_i, a, b \in Q_1 (i \in I)$ ,

- (1)  $F(a) *_2 F(b) \subseteq F(a *_1 b)$ .
- (2)  $\bigvee_{i \in I} F(a_i) \subseteq F\left(\bigvee_{i \in I} a_i\right)$ .

A set-valued mapping  $F : Q_1 \rightarrow \mathcal{P}^*(Q_2)$  is called a strong set-valued homomorphism if we replace  $\subseteq$  by  $=$  in (1) and (2).

**Definition 5.3.** Let  $(Q_1, *_1)$  and  $(Q_2, *_2)$  be two hyperquantales and let  $F$  be a set-valued homomorphism. Let  $f$  be any fuzzy subset of  $Q_2$ . Then for every  $x \in Q_1$ , we defines

$$F^-(f)(x) = \bigwedge_{y \in F(x)} f(y),$$

$$F^+(f)(x) = \bigvee_{y \in F(x)} f(y).$$

Here  $F^-(f)$  is the generalized lower approximation and  $F^+(f)$  is the generalized upper approximation of the fuzzy subset of  $f$ . The pair  $(F^-(f), F^+(f))$  is called generalized rough fuzzy subset of  $Q_1$ , if  $F^-(f) \neq F^+(f)$ .

**Definition 5.4.** Let  $F$  be a set-valued homomorphism. A fuzzy subset  $f$  of the hyperquantale  $Q_2$  is called a lower (resp. upper) generalized rough fuzzy bi-hyperideal of  $Q_2$  if  $F^-(f)$  (resp.  $F^+(f)$ ) is a fuzzy bi-hyperideal of  $Q_1$ . A fuzzy subset  $f$  of  $Q_2$ , which is both an upper and a lower generalized rough fuzzy bi-hyperideal of  $Q_2$ , is called generalized rough fuzzy bi-hyperideal of  $Q_2$ .

**Theorem 5.1.** *Let  $F$  be a strong set-valued homomorphism and let  $f$  be a fuzzy bi-hyperideal of  $Q_2$ . Then set  $F^-(f)$  is a fuzzy bi-hyperideal of  $Q_1$ .*

*Proof.* Assume that  $f$  is a fuzzy bi-hyperideal of  $Q_2$ , then we have  $\bigwedge_{\alpha \in x \vee y} f(\alpha) \geq f(x) \wedge f(y)$  imply that  $f(\alpha) \geq f(x) \wedge f(y) \forall x, y \in Q_2$  and  $\alpha \in x \vee y$ . Also  $F$  is a strong set-valued homomorphism, so  $F(x \vee y) = F(x) \vee F(y) \forall x, y \in Q_1$ . Therefore for any  $\alpha \in x \vee y$

$$F^-(f)(\alpha) = F^-(f)(x \vee y) = \bigwedge_{\alpha \in F(x \vee y)} f(\alpha) = \bigwedge_{\alpha \in F(x) \vee F(y)} f(\alpha).$$

Since  $\alpha \in F(x) \vee F(y)$ , there exist  $a \in F(x)$  and  $b \in F(y)$  such that  $\alpha \in a \vee b$ . Hence

$$\begin{aligned} F^-(f)(x \vee y) &= \bigwedge_{a \vee b \in F(x) \vee F(y)} f(a \vee b) \\ &\geq \bigwedge_{a \in F(x), b \in F(y)} (f(a) \wedge f(b)) \\ &= \left\{ \left( \bigwedge_{a \in F(x)} f(a) \right) \wedge \left( \bigwedge_{b \in F(y)} f(b) \right) \right\} \\ &= F^-(f)(x) \wedge F^-(f)(y). \end{aligned}$$

Hence  $\bigwedge_{\alpha \in x \vee y} F^-(f)(\alpha) \geq F^-(f)(x) \wedge F^-(f)(y) \forall x, y \in Q_1$ .

Again since  $F$  is a strong set-valued homomorphism, so we have  $F(x *_1 y) = F(x) *_2 F(y) \forall x, y \in Q_1$ .

Thus for any  $\beta \in x *_1 y$  we have,

$$F^-(f)(x *_1 y) = \bigwedge_{\beta \in F(x *_1 y)} f(\beta) = \bigwedge_{\beta \in F(x) *_2 F(y)} f(\beta).$$

Since  $\beta \in F(x) *_2 F(y)$ , there exist  $a \in F(x)$  and  $b \in F(y)$  such that  $\beta \in a *_2 b$ . Hence

$$\begin{aligned} F^-(f)(\beta) &= F^-(f)(x *_1 y) \\ &= \bigwedge_{a *_2 b \in F(x) *_2 F(y)} f(a *_2 b) \\ &\geq \bigwedge_{a \in F(x), b \in F(y)} (f(a) \wedge f(b)) \\ &= \left\{ \left( \bigwedge_{a \in F(x)} f(a) \right) \wedge \left( \bigwedge_{b \in F(y)} f(b) \right) \right\} \\ &= F^-(f)(x) \wedge F^-(f)(y). \end{aligned}$$

Hence  $\bigwedge_{\beta \in x *_1 y} F^-(f)(\beta) \geq F^-(f)(x) \wedge F^-(f)(y) \forall x, y \in Q_1$ . Again since  $f$  is a fuzzy bi-hyperideal of  $Q_2$ , so for any  $\gamma \in x *_1 y *_1 z$ , we have

$$F^-(f)(\gamma) = F^-(f)(x *_1 y *_1 z) = \bigwedge_{\gamma \in F(x *_1 y *_1 z)} f(\gamma) = \bigwedge_{\gamma \in (F(x) *_2 F(y) *_2 F(z))} f(\gamma).$$

Since  $\gamma \in F(x) *_2 F(y) *_2 F(z)$ , there exist  $a \in F(x)$  and  $b \in F(y)$  and  $c \in F(z)$  such that  $\gamma \in a *_2 b *_2 c$ .

Hence

$$\begin{aligned} F^-(f)(\gamma) &= F^-(f)(x *_1 y *_1 z) \\ &= \bigwedge_{a *_2 b *_2 c \in F(x) *_2 F(y) *_2 F(z)} f(a *_2 b *_2 c) \\ &\geq \bigwedge_{a \in F(x), c \in F(z)} (f(a) \wedge f(c)) \\ &= \left\{ \left( \bigwedge_{a \in F(x)} f(a) \right) \wedge \left( \bigwedge_{c \in F(z)} f(c) \right) \right\} \\ &= F^-(f)(x) \wedge F^-(f)(z). \end{aligned}$$

Hence  $\bigwedge_{\gamma \in (x *_1 y *_1 z)} F^-(f)(\gamma) \geq F^-(f)(x) \wedge F^-(f)(z) \forall x, y, z \in Q_1$ . □

**Theorem 5.2.** *Let  $F$  be a strong set-valued homomorphism and let  $f$  be a fuzzy bi-hyperideal of  $Q_2$ . Then  $F^+(f)$  is a fuzzy bi-hyperideal of  $Q_1$ .*

*Proof.* Assume that  $f$  is a fuzzy bi-hyperideal of  $Q_2$ , then we have  $\bigwedge_{\alpha \in x \vee y} f(\alpha) \geq f(x) \wedge f(y)$  imply that  $f(\alpha) \geq f(x) \wedge f(y) \forall x, y \in Q_2$  and  $\alpha \in x \vee y$ . Also  $F$  is a strong set-valued homomorphism, so  $F(x \vee y) = F(x) \vee F(y) \forall x, y \in Q_1$ . Therefore for any  $\alpha \in x \vee y$

$$F^+(f)(\alpha) = F^+(f)(x \vee y) = \bigvee_{\alpha \in F(x \vee y)} f(\alpha) = \bigvee_{\alpha \in F(x) \vee F(y)} f(\alpha).$$

Since  $\alpha \in F(x) \vee F(y)$ , there exist  $a \in F(x)$  and  $b \in F(y)$  such that  $\alpha \in a \vee b$ . Hence

$$\begin{aligned} F^+(f)(\alpha) &= F^+(f)(x \vee y) \\ &= \bigvee_{a \vee b \in F(x) \vee F(y)} f(a \vee b) \\ &\geq \bigvee_{a \in F(x), b \in F(y)} (f(a) \wedge f(b)) \\ &= \left\{ \left( \bigvee_{a \in F(x)} f(a) \right) \wedge \left( \bigvee_{b \in F(y)} f(b) \right) \right\} \\ &= F^+(f)(x) \wedge F^+(f)(y). \end{aligned}$$

Hence  $\bigwedge_{\alpha \in x \vee y} F^+(f)(\alpha) \geq F^+(f)(x) \wedge F^+(f)(y) \forall x, y \in Q_1$ .

Again since  $F$  is a strong set-valued homomorphism, so we have  $F(x *_1 y) = F(x) *_2 F(y) \forall x, y \in Q_1$ .

Thus for any  $\beta \in x *_1 y$  we have,

$$F^+(f)(\beta) = F^+(f)(x *_1 y) = \bigvee_{\beta \in F(x *_1 y)} f(\beta) = \bigvee_{\beta \in F(x) *_2 F(y)} f(\beta).$$

Since  $\beta \in F(x) *_2 F(y)$ , there exist  $a \in F(x)$  and  $b \in F(y)$  such that  $\beta \in a *_2 b$ . Hence

$$\begin{aligned} F^+(f)(\beta) &= F^+(f)(x *_1 y) \\ &= \bigvee_{a *_2 b \in F(x) *_2 F(y)} f(a *_2 b) \\ &\geq \bigvee_{a \in F(x), b \in F(y)} (f(a) \wedge f(b)) \\ &= \left\{ \left( \bigvee_{a \in F(x)} f(a) \right) \wedge \left( \bigvee_{b \in F(y)} f(b) \right) \right\} \\ &= F^+(f)(x) \wedge F^+(f)(y). \end{aligned}$$

Hence  $\bigwedge_{\beta \in x *_1 y} F^+(f)(\beta) \geq F^+(f)(x) \wedge F^+(f)(y) \forall x, y \in Q_1$ . Again since  $f$  is a fuzzy bi-hyperideal of  $Q_2$ , so for any  $\gamma \in x *_1 y *_1 z$ , we have

$$F^+(f)(\gamma) = F^+(f)(x *_1 y *_1 z) = \bigvee_{\gamma \in F(x *_1 y *_1 z)} f(\gamma) = \bigvee_{\gamma \in (F(x) *_2 F(y) *_2 F(z))} f(\gamma).$$

Since  $\gamma \in F(x) *_2 F(y) *_2 F(z)$ , there exist  $a \in F(x)$  and  $b \in F(y)$  and  $c \in F(z)$  such that  $\gamma \in a *_2 b *_2 c$ . Hence

$$\begin{aligned} F^+(f)(\gamma) &= F^+(f)(x *_1 y *_1 z) \\ &= \bigvee_{a *_2 b *_2 c \in (F(x) *_2 F(y) *_2 F(z))} f(a *_2 b *_2 c) \\ &\geq \bigvee_{a \in F(x), c \in F(z)} (f(a) \wedge f(c)) \\ &= \left\{ \left( \bigvee_{a \in F(x)} f(a) \right) \wedge \left( \bigvee_{c \in F(z)} f(c) \right) \right\} \\ &= F^+(f)(x) \wedge F^+(f)(z). \end{aligned}$$

Hence  $\bigwedge_{\gamma \in (x *_1 y *_1 z)} F^+(f)(\gamma) \geq F^+(f)(x) \wedge F^+(f)(z) \forall x, y, z \in Q_1$ . □

**Proposition 5.1.** *Let  $F$  be a strong set-valued homomorphism and let  $\{f_i\}_{i \in I}$  be a family of fuzzy bi-hyperideal of  $Q_2$ . Then  $F^-\left(\bigwedge_{i \in I} (f_i)\right)$  is a fuzzy bi-hyperideal of  $Q_1$ .*

*Proof.* Since every  $f_i$  is a fuzzy bi-hyperideals for every  $i \in I$ , and for every  $x, y \in Q_1$ ,

$$\begin{aligned}
 F^- \left( \bigwedge_{i \in I} (f_i) \right) (\alpha) &= F^- \left( \bigwedge_{i \in I} (f_i) \right) (x \vee y) \\
 &= \left( \bigwedge_{i \in I} F^- (f_i) \right) (x \vee y) \\
 &= \bigwedge_{i \in I} F^- (f_i) (x \vee y) \\
 &\geq \bigwedge_{i \in I} \left( F^- (f_i) (x) \wedge F^- (f_i) (y) \right) \\
 &= \left\{ \left( \bigwedge_{i \in I} F^- (f_i) \right) (x) \wedge \left( \bigwedge_{i \in I} F^- (f_i) \right) (y) \right\} \\
 &= F^- \left( \bigwedge_{i \in I} f_i \right) (x) \wedge F^- \left( \bigwedge_{i \in I} f_i \right) (y).
 \end{aligned}$$

Hence  $\bigwedge_{\alpha \in x \vee y} F^- \left( \bigwedge_{i \in I} f_i \right) (\alpha) \geq F^- \left( \bigwedge_{i \in I} f_i \right) (x) \wedge F^- \left( \bigwedge_{i \in I} f_i \right) (y) \forall x, y \in Q_1$ .

Now,

$$\begin{aligned}
 F^- \left( \bigwedge_{i \in I} (f_i) \right) (\beta) &= F^- \left( \bigwedge_{i \in I} (f_i) \right) (x *_1 y) \\
 &= \left( \bigwedge_{i \in I} F^- (f_i) \right) (x *_1 y) \\
 &= \bigwedge_{i \in I} F^- (f_i) (x *_1 y) \\
 &\geq \bigwedge_{i \in I} \left( F^- (f_i) (x) \wedge F^- (f_i) (y) \right) \\
 &= \left\{ \left( \bigwedge_{i \in I} F^- (f_i) \right) (x) \wedge \left( \bigwedge_{i \in I} F^- (f_i) \right) (y) \right\} \\
 &= F^- \left( \bigwedge_{i \in I} f_i \right) (x) \wedge F^- \left( \bigwedge_{i \in I} f_i \right) (y).
 \end{aligned}$$

Hence  $\bigwedge_{\beta \in x *_1 y} F^- \left( \bigwedge_{i \in I} f_i \right) (\beta) \geq F^- \left( \bigwedge_{i \in I} f_i \right) (x) \wedge F^- \left( \bigwedge_{i \in I} f_i \right) (y) \forall x, y \in Q_1$ .

Again since  $F$  is a strong set-valued homomorphism, and  $f$  is a fuzzy bi-hyperideal of  $Q_2$ , so for any  $\gamma \in x *_1 y *_1 z$ , we have,

$$\begin{aligned} F^- \left( \bigwedge_{i \in I} (f_i) \right) (\gamma) &= \left( F^- \bigwedge_{i \in I} (f_i) \right) (x *_1 y *_1 z) \\ &= \bigwedge_{i \in I} F^- (f_i) (x *_1 y *_1 z) \\ &\geq \bigwedge_{i \in I} \left( F^- (f_i) (x) \wedge F^- (f_i) (z) \right) \\ &= \left\{ \left( \bigwedge_{i \in I} F^- (f_i) \right) (x) \wedge \left( \bigwedge_{i \in I} F^- (f_i) \right) (z) \right\} \\ &= F^- \left( \bigwedge_{i \in I} f_i \right) (x) \wedge F^- \left( \bigwedge_{i \in I} f_i \right) (z). \end{aligned}$$

Hence  $\bigwedge_{\gamma \in x *_1 y *_1 z} F^- \left( \bigwedge_{i \in I} f_i \right) (\gamma) \geq F^- \left( \bigwedge_{i \in I} f_i \right) (x) \wedge F^- \left( \bigwedge_{i \in I} f_i \right) (z) \forall x, y, z \in Q_1$ . □

For the following Theorem we define the set  $f_\alpha$  where  $\alpha \in [0, 1]$  as following

$$f_\alpha = \{x \in Q \mid f(x) \geq \alpha\}.$$

**Theorem 5.3.** *Let  $F$  be a strong set-valued homomorphism and  $f$  be a fuzzy bi-hyperideal of  $Q_2$ . Then  $F^-(f)$  (resp.  $F^+(f)$ ) is a fuzzy bi-hyperideal of  $Q_1$  if and only if for each  $\alpha \in [0, 1]$ ,  $F^-(f_\alpha)$  (resp.  $F^+(f_\alpha)$ ), where  $f_\alpha \neq \emptyset$ , is a bi-hyperideal of  $Q_1$ .*

*Proof.* Assume that  $F^-(f)$  is a fuzzy bi-hyperideal of  $Q_1$ . We need to show that  $F^-(f_\alpha)$  is a bi-hyperideal of  $Q_1$ . Let  $x_1, x_2 \in F^-(f_\alpha)$ . Then  $F^-(f)(x_1) \geq \alpha$  and  $F^-(f)(x_2) \geq \alpha$ . But since  $F^-(f)$  is a fuzzy bi-hyperideal, so  $\bigwedge_{z \in x_1 \vee x_2} F^-(f)(z) \geq F^-(f)(x_1) \wedge F^-(f)(x_2) \geq \alpha$ . Implies that  $F^-(f)(z) \geq \alpha$ . Hence  $x_1 \vee x_2 \subseteq F^-(f_\alpha)$ . Let  $y \in F^-(f_\alpha)$ ,  $x \in Q_1$ , and  $x \leq y$ . Then  $F^-(f)(x) \geq F^-(f)(y) \geq \alpha$ . Hence  $F^-(f)(x) \geq \alpha$ . Hence  $x \in F^-(f_\alpha)$ . Let  $y_1, y_2 \in F^-(f_\alpha)$ , then  $F^-(f)(y_1) \geq \alpha$  and  $F^-(f)(y_2) \geq \alpha$ . Since  $F^-(f)$  is a fuzzy bi-hyperideal of  $Q_1$ , so we have  $\bigwedge_{z \in y_1 *_1 y_2} F^-(f)(z) \geq F^-(f)(y_1) \wedge F^-(f)(y_2) = \alpha$ . Hence  $F^-(f)(z) \geq \alpha$ , for all  $z \in y_1 *_1 y_2$ , this implies that  $z \in F^-(f_\alpha)$ . Hence  $y_1 *_1 y_2 \subseteq F^-(f_\alpha)$ . Now let  $u, v, w \in F^-(f_\alpha)$ . Then  $F^-(f)(u) \geq \alpha$ ,  $F^-(f)(v) \geq \alpha$  and  $F^-(f)(w) \geq \alpha$ . Again since  $F^-(f)$  is a fuzzy bi-hyperideal of  $Q_1$ , so we have  $\bigwedge_{\beta \in u *_1 v *_1 w} F^-(f)(\beta) \geq F^-(f)(u) \wedge F^-(f)(w) = \alpha$ . Hence  $F^-(f)(\beta) \geq \alpha$ . Thus  $u *_1 v *_1 w \subseteq F^-(f_\alpha)$ . Therefore  $F^-(f_\alpha)$  is a bi-hyperideal of  $Q_1$ .

Conversely, assume that  $F^-(f_\alpha)$  is a bi-hyperideal of  $Q_1$ . We shall show that  $F^-(f)$  is a fuzzy bi-hyperideal of  $Q_1$ . For any  $x, y \in Q_1$ , let  $\alpha = F^-(f)(x) \wedge F^-(f)(y) \in \text{range}(F^-(f))$ . Then  $F^-(f)(x) \geq \alpha$  and  $F^-(f)(y) \geq \alpha$ . So  $x, y \in F^-(f_\alpha)$ . Hence  $x \vee y \subseteq F^-(f_\alpha)$ .

Consider

$$F^-(f)(x \vee y) = \bigwedge_{z \in F(x \vee y)} f(z) = \bigwedge_{z \in F(x) \vee F(y)} f(z).$$

Since  $z \in F(x) \vee F(y)$ , there exist  $a \in F(x)$  and  $b \in F(y)$  such that  $z \in a \vee b$ . Hence

$$\begin{aligned} F^-(f)(x \vee y) &= \bigwedge_{a \vee b \in F(x) \vee F(y)} f(a \vee b) \\ &\geq \bigwedge_{a \in F(x), b \in F(y)} (f(a) \wedge f(b)) \\ &= \left\{ \left( \bigwedge_{a \in F(x)} f(a) \right) \wedge \left( \bigwedge_{b \in F(y)} f(b) \right) \right\} \\ &= F^-(f)(x) \wedge F^-(f)(y). \end{aligned}$$

Hence  $\bigwedge_{z \in x \vee y} F^-(f)(z) \geq F^-(f)(x) \wedge F^-(f)(y) \forall x, y \in Q_1$ .

Now for  $x, y \in F^-(f_\alpha)$ , we have  $x *_1 y \subseteq F^-(f_\alpha)$ . Hence for  $\beta \in x *_1 y$ , we have  $F^-(f)(\beta) \geq \alpha$ . Since  $x, y \in F^-(f_\alpha)$ , so  $F^-(f)(x) \geq \alpha$  and  $F^-(f)(y) \geq \alpha$ . Thus  $\bigwedge_{\beta \in x *_1 y} F^-(f)(\beta) \geq F^-(f)(x) \wedge F^-(f)(y)$ . Let  $x, y \in Q_1$  such that  $x \leq y$ . If  $F^-(f)(y) = 0$  then  $F^-(f)(x) \geq F^-(f)(y)$ . If  $F^-(f)(y) = \alpha$  then  $y \in F^-(f_\alpha)$ . Since  $x \leq y$  and  $F^-(f_\alpha)$  is a bi-hyperideal of  $Q_1$ , we have  $x \in F^-(f_\alpha)$ . Then  $F^-(f)(x) \geq \alpha = F^-(f)(y)$ . Now let  $x, y, z \in F^-(f_\alpha)$ . Then  $x *_1 y *_1 z \subseteq F^-(f_\alpha)$ . Since  $x, z \in F^-(f_\alpha)$ . Then  $F^-(f)(x) \geq \alpha$  and  $F^-(f)(z) \geq \alpha$ . So for any  $\gamma \in x *_1 y *_1 z$ , we have  $F^-(f)(\gamma) \geq \alpha$ . Thus  $F^-(f)(x) \wedge F^-(f)(z) = \alpha \leq \bigwedge_{\gamma \in (x *_1 y *_1 z)} F^-(f)(\gamma)$ . Therefore  $F^-(f)$  is a fuzzy bi-hyperideal of  $Q_1$ .  $\square$

## 6. CONCLUSION

In the present paper, we introduced the notion of bi-hyperideals of hyperquantales. Furthermore we introduced the notions of fuzzy bi-hyperideals and generalized rough fuzzy bi-hyperideals of hyperquantales and their related properties is provided. Finally we discussed the strong set-valued homomorphism and set-valued homomorphism of hyperquantales and generalized rough fuzzy bi-hyperideals and shown that how they are related. In our future study of hyperquantales, we will apply the above new idea to other algebraic structures for more applications.

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## REFERENCES

- [1] F. Marty, Sur une generalisation de la notion de group, in Proceedings of the 8th Congress des Mathematiciens Scandinaves, Stockholm, 1934, pp. 45–49.
- [2] J. Tang, B. Davvaz and Y. F. Luo, A study on fuzzy interior hyperideals in ordered semihypergroups, *Ital. J. Pure Appl. Math.* 36 (2016), 125-146.
- [3] J. Tang, A. Khan and Y. F. Luo, Characterization of semisimple ordered semihypergroups in terms of fuzzy hyperideals, *J. Intell. Fuzzy Syst.* 30 (2016), 1735-1753.
- [4] B. Davvaz, A. Khan and M. Farooq, Int-soft structures applied to ordered semihypergroups, *Le Matematiche*, LXXIII (2018), 235–259.
- [5] A. Khan, M. Farooq and H. U. Khan, Uni-soft hyperideals of ordered semihypergroups, *J. Intell. Fuzzy Syst.* 35 (2018), 4557–4571.
- [6] B. Davvaz, A. Khan and M. Khan, Atanassov’s intuitionistic fuzzy set theory applied to quantales, *Novi Sad J. Math.* 47 (2) (2017), 47-61.
- [7] B. B. N. Koguep, On fuzzy Ideals of hyperlattice, *Int. J. Algebra*, 2 (15) (2008), 739-750.
- [8] P. Corsini, V. Leoreanu, Applications of hyperstructure theory, Kluwer Academic Publishers, Dordrecht, 2003.
- [9] J. Zhan, B. Davvaz, P. K. Shun, Probability n-ary bypergroups. *Inform. Sci.* 180 (2010), 1159-1166.
- [10] J. Zhan, C. Irina,  $\Gamma$ -hypermdules: isomorphism theorems and regular relations. *U. P. B. Sci Bull. Ser. A.* 73 (2011), 71-78.
- [11] J. Zhan, B. Davvaz, P. K. Shun., A new view of fuzzy hyperrings. *Inform. Sci.* 178 (2008), 425-438.
- [12] M. Konstantinidou and J. Mittas, An introduction to the theory of hyperlattices, *Math. Balkanica*, 7 (1977), 187-193.
- [13] S. Rasouli and B. Davvaz, Lattices derived from hyperlattices. *Commun. Algebra*, 38 (8) (2010), 2720-2737.
- [14] S. Rasouli and B. Davvaz, Construction and spectral topology on hyperlattices, *Mediterr. J. Math.* 7 (2) (2010), 249-262.
- [15] Z. Pawlak, Information systems–theoretical foundations, *Inform. Syst.* 6 (1981), 205–218.
- [16] Z. Pawlak, Rough sets, *Int. J. Comput. Inform. Sci.* 11 (1982), 341–356.
- [17] J. Järvinen, Lattice Theory for Rough Sets, Transactions on Rough Sets VI, LNCS 4374, Springer-Verlag, Berlin Heidelberg, 2007 (pp. 400–498).
- [18] G.L. Liu and W. Zhu, The algebraic structures of generalized rough set theory, *Inform. Sci.* 178 (2008), 4105–4113.
- [19] L. A. Zadeh, Fuzzy sets, *Inform. Control*, 8 (1965), 338-353.
- [20] C. J. Mulvey, *Supplemento ai Rendiconti del Circolo Matematico di Palermo II* 12 (1986), 99–104.
- [21] K. Y. Wang, Prime radical theorem in quantales, *Fuzzy Syst. Math.* 25 (2) (2011), 60–64 (in Chinese).
- [22] S. Q. Wang and B. Zhao, Ideals of quantales, *J. Shaanxi Normal Univ. (Nat. Sci. Ed.)* 31 (4) (2003), 7–10 (in Chinese).
- [23] S. Q. Wang and B. Zhao, Prime ideal and weakly prime ideal of the quantale, *Fuzzy Syst. Math.* 19 (1) (2005), 78–81 (in Chinese).
- [24] L. Y. Yang, L.S. Xu, Roughness in quantales, *Inform. Sci.* 220 (2013), 568–579.
- [25] Q. Luo and G. Wang, Roughness and fuzziness in quantales, *Inform. Sci.* 271 (2014), 14–30.
- [26] M. Farooq, T. Mahmood, A. Khan, M. Izhar and B. Davvaz, Fuzzy hyperideals of hyperquantale, *J. Intell. Fuzzy Syst.* 36 (2019), 5605–5615.
- [27] S. Yamak, O. Kazancı, and B. Davvaz, Generalized lower and upper approximations in a ring, *Inform. Sci.* 180 (9) (2010), 1759–1768.
- [28] A. K. Estaji and F. Bayati, On Rough Sets and Hyperlattices, *Ratio Math.* 34 (2018), 15-33.

- 
- [29] S. M. Qurashi and M. Shabir, Generalized Rough Fuzzy Ideals in Quantales, *Discrete Dyn. Nat. Soc.* 2018 (2018), Article ID 1085201.
- [30] S. M. Qurashi and M. Shabir, Generalized approximations of  $(\in, \in \vee q)$ -fuzzy ideals in quantales, *Comput. Appl. Math.* 37 (2018), 6821–6837.