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GENERALIZED ABSOLUTE RIESZ SUMMABILITY OF INFINITE SERIES AND FOURIER SERIES

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Abstract. In this paper, two known theorems dealing with $|\bar{N}, p_n|_k$ summability of infinite series and Fourier series have been generalized to $\varphi - |\bar{N}, p_n; \beta|_k$ summability.

1. Introduction

A sequence (A_n) is said to be δ-quasi-monotone if $A_n \to 0$, $A_n > 0$ ultimately and $\Delta A_n \geq -\delta_n$, where $\Delta A_n = A_n - A_{n+1}$ and $\delta = (\delta_n)$ is a sequence of positive numbers (see [1]). A sequence (g_n) is said to be of bounded variation, denoted by $(g_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta g_n| < \infty$. Let $\sum a_n$ be a given infinite series with the partial sums (s_n). Let (φ_n) be a sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |\bar{N}, p_n; \beta|_k$, $k \geq 1$ and $\beta \geq 0$, if (see [22])

$$\sum_{n=1}^{\infty} \varphi_n^{\beta k + k - 1} |u_n - u_{n-1}|^k < \infty$$

where (p_n) is a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1),$$

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and

$$u_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v.$$

For $\varphi_n = \frac{P_n}{p_n}$ and $\beta = 0$, $\varphi - |\bar{N}, p_n; \beta|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability (see [2]). Taking $\varphi_n = n$, $\beta = 0$ and $p_n = 1$ for all values of n, $\varphi - |\bar{N}, p_n; \beta|_k$ summability reduces to $|C, 1|_k$ summability (see [8]).

If we write $X_n = \sum_{v=1}^{n} p_v/P_v$, then (X_n) is a positive increasing sequence tending to infinity with n.

In [3], the following theorem on δ-quasi-monotone sequences has been proved.

Theorem 1.1. *Let $(\lambda_n) \to 0$ as $n \to \infty$ and (p_n) be a sequence of positive numbers such that $P_n = O(np_n)$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) which is δ-quasi-monotone with $\sum nX_n\delta_n < \infty$, $\sum A_n X_n$ is convergent, and $|\Delta\lambda_n| \leq |A_n|$ for all n. If the condition*

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad as \qquad m \to \infty \tag{1.1}$$

is satisfied, where (t_n) is the n-th $(C, 1)$ mean of the sequence (na_n), then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Lemma 1.1. *[3] Under the conditions of Theorem 1.1, we have that*

$$|\lambda_n| X_n = O(1) \quad as \quad n \to \infty, \tag{1.2}$$

$$nX_n A_n = O(1) \quad as \quad n \to \infty, \tag{1.3}$$

$$\sum_{n=1}^{\infty} nX_n |\Delta A_n| < \infty. \tag{1.4}$$

2. Main Result

There are some papers on absolute summability (see [4–6,9–12,16–18,23–25]). Now we generalize Theorem 1.1 as in the following form.

Theorem 2.1. *Let (φ_n) be a sequence of positive real numbers such that*

$$\varphi_n p_n = O(P_n), \tag{2.1}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}} = O\left(\varphi_v^{\beta k} \frac{1}{P_v}\right) \quad as \quad m \to \infty. \tag{2.2}$$

If all conditions of Theorem 1.1 are satisfied with the condition (1.1) replaced by

$$\sum_{n=1}^{m} \varphi_n^{\beta k-1} |t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{2.3}$$

then the series $\sum a_n \lambda_n$ *is summable* $\varphi - |\bar{N}, p_n; \beta|_k$, $k \geq 1$ *and* $0 \leq \beta < 1/k$.

3. Proof of Theorem 2.1

Let (I_n) indicates $(\bar{N}, p_n)$ mean of the series $\sum a_n \lambda_n$. Then, for $n \geq 1$, we obtain

$$\bar{\Delta} I_n = I_n - I_{n-1} \quad = \quad \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} \lambda_v}{v} v a_v.$$

Applying Abel's transformation, we get

$$\begin{aligned} \bar{\Delta} I_n \quad &= \quad \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{\lambda_{v+1}}{v} P_v t_v - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{v+1}{v} p_v \lambda_v t_v \\ &+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{v+1}{v} P_v t_v \Delta \lambda_v + \frac{(n+1)}{n P_n} p_n \lambda_n t_n \\ &= \quad I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}. \end{aligned}$$

For the proof of Theorem 2.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\beta k+k-1} \mid I_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4.$$

First,

$$\begin{aligned} \sum_{n=2}^{m+1} \varphi_n^{\beta k+k-1} \mid I_{n,1} \mid^k \quad &\leq \quad \sum_{n=2}^{m+1} \varphi_n^{\beta k+k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \left(\sum_{v=1}^{n-1} P_v \left| t_v \right| \frac{|\lambda_{v+1}|}{v} \right)^k \\ &= \quad \sum_{n=2}^{m+1} \varphi_n^{\beta k-1} \left(\frac{\varphi_n p_n}{P_n} \right)^k \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} P_v \left| t_v \right| \frac{|\lambda_{v+1}|}{v} \right)^k. \end{aligned}$$

Here (2.1) gives $\left(\frac{\varphi_n p_n}{P_n} \right)^k = O(1)$, also using Hölder's inequality, we obtain

$$\sum_{n=2}^{m+1} \varphi_n^{\beta k+k-1} \mid I_{n,1} \mid^k \quad = \quad O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}} \left(\sum_{v=1}^{n-1} P_v \left| t_v \right|^k \frac{|\lambda_{v+1}|^k}{v} \right) \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v}{v} \right)^{k-1}.$$

Now using the fact that $P_v = O(v p_v)$,

$$\sum_{n=2}^{m+1} \varphi_n^{\beta k+k-1} \mid I_{n,1} \mid^k \quad = \quad O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta k-1} \frac{1}{P_{n-1}} \left(\sum_{v=1}^{n-1} p_v \left| t_v \right|^k \left| \lambda_{v+1} \right|^k \right) \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right)^{k-1}.$$

Then, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}\mid I_{n,1}\mid^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\,|t_v|^k\,|\lambda_{v+1}|^k\\ &= O(1)\sum_{v=1}^{m}p_v|\lambda_{v+1}|^{k-1}|\lambda_{v+1}|\,|t_v|^k\sum_{n=v+1}^{m+1}\varphi_n^{\beta k-1}\frac{1}{P_{n-1}}.\end{aligned}$$

Here, by using (2.2) and (1.2),

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}\mid I_{n,1}\mid^k &= O(1)\sum_{v=1}^{m}\varphi_v^{\beta k}\frac{p_v}{P_v}|\lambda_{v+1}|\,|t_v|^k\\ &= O(1)\sum_{v=1}^{m}\varphi_v^{\beta k-1}\left(\frac{\varphi_v p_v}{P_v}\right)|\lambda_{v+1}|\,|t_v|^k.\end{aligned}$$

Again, from (2.1), we obtain

$$\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}\mid I_{n,1}\mid^k = O(1)\sum_{v=1}^{m}\varphi_v^{\beta k-1}|\lambda_{v+1}|\,|t_v|^k.$$

Hence, we get

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}\mid I_{n,1}\mid^k &= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_{v+1}|\sum_{r=1}^{v}\varphi_r^{\beta k-1}|t_r|^k+O(1)|\lambda_{m+1}|\sum_{v=1}^{m}\varphi_v^{\beta k-1}|t_v|^k\\ &= O(1)\sum_{v=1}^{m-1}|A_{v+1}|\,X_{v+1}+O(1)|\lambda_{m+1}|X_{m+1}=O(1)\quad as\quad m\to\infty,\end{aligned}$$

by using Abel's transformation, hypotheses of Theorem 2.1, and Lemma 1.1.

Now, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}\mid I_{n,2}\mid^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left(\sum_{v=1}^{n-1}p_v\,|\lambda_v|\,|t_v|\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta k-1}\left(\frac{\varphi_n p_n}{P_n}\right)^k\frac{1}{P_{n-1}^k}\left(\sum_{v=1}^{n-1}p_v\,|\lambda_v|\,|t_v|\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta k-1}\frac{1}{P_{n-1}^k}\left(\sum_{v=1}^{n-1}p_v\,|\lambda_v|\,|t_v|\right)^k.\end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}\mid I_{n,2}\mid^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta k-1}\frac{1}{P_{n-1}}\left(\sum_{v=1}^{n-1}p_v\,|\lambda_v|^k\,|t_v|^k\right)\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}\\ &= O(1)\sum_{v=1}^{m}p_v|\lambda_v|^k|t_v|^k\sum_{n=v+1}^{m+1}\varphi_n^{\beta k-1}\frac{1}{P_{n-1}}.\end{aligned}$$

By (2.2), (2.1) and (1.2), we get

$$\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}\mid I_{n,2}\mid^k = O(1)\sum_{v=1}^{m}\varphi_v^{\beta k-1}|\lambda_v||t_v|^k.$$

Here, using Abel's transformation as in $I_{n,1}$, we have

$$\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}\mid I_{n,2}\mid^k = O(1) \quad as \quad m\to\infty.$$

Again, using Hölder's inequality, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}\mid I_{n,3}\mid^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left(\sum_{v=1}^{n-1}P_v\,|t_v|\,|\Delta\lambda_v|\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta k-1}\left(\frac{\varphi_n p_n}{P_n}\right)^k\frac{1}{P_{n-1}^k}\left(\sum_{v=1}^{n-1}P_v\,|t_v|\,|A_v|\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta k-1}\frac{1}{P_{n-1}}\left(\sum_{v=1}^{n-1}p_v\,|t_v|^k\,(v\,|A_v|)^k\right)\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}\\ &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\,|t_v|^k\,(v\,|A_v|)^{k-1}(v\,|A_v|).\end{aligned}$$

Using (1.3), we get $(v\,|A_v|)^{k-1}=O(1)$, then

$$\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}\mid I_{n,3}\mid^k = O(1)\sum_{v=1}^{m}p_v|t_v|^k v\,|A_v|\sum_{n=v+1}^{m+1}\varphi_n^{\beta k-1}\frac{1}{P_{n-1}}.$$

Now using the conditions (2.2) and (2.1), we get

$$\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}\mid I_{n,3}\mid^k = O(1)\sum_{v=1}^{m}\varphi_v^{\beta k-1}|t_v|^k v\,|A_v|\,.$$

Then, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}\mid I_{n,3}\mid^k &= O(1)\sum_{v=1}^{m-1}\Delta(v\,|A_v|)\sum_{r=1}^{v}\varphi_r^{\beta k-1}|t_r|^k+O(1)m\,|A_m|\sum_{v=1}^{m}\varphi_v^{\beta k-1}|t_v|^k\\ &= O(1)\sum_{v=1}^{m-1}\Delta(v\,|A_v|)X_v+O(1)m\,|A_m|\,X_m\\ &= O(1)\sum_{v=1}^{m-1}vX_v|\Delta A_v|+O(1)\sum_{v=1}^{m-1}|A_{v+1}|\,X_{v+1}+O(1)m\,|A_m|\,X_m\\ &= O(1) \quad as \quad m\to\infty,\end{aligned}$$

by using Abel's transformation, hypotheses of Theorem 2.1, and Lemma 1.1.

Finally, we get

$$\begin{aligned}\sum_{n=1}^{m}\varphi_n^{\beta k+k-1}\mid I_{n,4}\mid^k &= O(1)\sum_{n=1}^{m}\varphi_n^{\beta k+k-1}\left(\frac{p_n}{P_n}\right)^k|\lambda_n|^{k-1}|\lambda_n||t_n|^k\\ &= O(1)\sum_{n=1}^{m}\varphi_n^{\beta k-1}|\lambda_n||t_n|^k.\end{aligned}$$

Here, as in $I_{n,1}$, we get

$$\sum_{n=1}^{m}\varphi_n^{\beta k+k-1}\mid I_{n,4}\mid^k = O(1) \quad as \quad m\to\infty.$$

Hence, the proof of Theorem 2.1 is completed.

4. Applications

There are some different papers dealing with applications of Fourier series (see [14, 15, 19–21]). Let f be a periodic function with period 2π and Lebesgue integrable over $(-\pi,\pi)$. The trigonometric Fourier series of f is defined as

$$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}C_n(x)$$

where

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)dx\quad\texttt{and}\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)dx.$$

Write

$$\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}$$

and

$$\phi_1(t)=\frac{1}{t}\int_0^t\phi(u)du.$$

If $\phi_1(t)\in\mathcal{BV}(0,\pi)$, then $t_n(x)=O(1)$, where $t_n(x)$ is the n-th $(C,1)$ mean of the sequence $(nC_n(x))$ (see [7]). By using this, the following theorem has been obtained in [3].

Theorem 4.1. *If $\phi_1(t)\in\mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n) and (X_n) satisfy the conditions of Theorem 1.1, then the series $\sum C_n(x)\lambda_n$ is summable $\mid\bar{N},p_n\mid_k$, $k\geq 1$.*

The following theorem gives a generalization of Theorem 4.1 for $\varphi-|\bar{N},p_n;\beta|_k$ summability.

Theorem 4.2. *If $\phi_1(t)\in\mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n), (A_n), (φ_n) and (X_n) satisfy the conditions of Theorem 2.1, then the series $\sum C_n(x)\lambda_n$ is summable $\varphi-|\bar{N},p_n;\beta|_k$, $k\geq 1$ and $0\leq\beta<1/k$.*

5. Conclusions

If we take $\varphi_n = \frac{P_n}{p_n}$ and $\beta = 0$ in Theorem 2.1, then the condition (2.3) reduces to the condition (1.1), and the conditions (2.1) and (2.2) are provided. Thus, Theorem 2.1 reduces to Theorem 1.1. If we take $\varphi_n = n$, $\beta = 0$ and $p_n = 1$ for all values of n, then we have a result for $|C, 1|_k$ summability of an infinite series (see [13]). Also, if we take $\varphi_n = \frac{P_n}{p_n}$ and $\beta = 0$ in Theorem 4.2, then we get Theorem 4.1.

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