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ON RECIPROCALS LEAP INDICES OF GRAPHS

AMMAR ALSINAI[1,*], ANWAR ALWARDI[2], N.D. SONER[1]

[1] *Department of Studies in Mathematics, University of Mysore, Manasagangotri,Mysuru - 570 006, India*
[2] *Department of Mathematics, University of Aden, Yemen*

[*] *Corresponding author:* aliiammar1985@gmail.com

Abstract. In the field of chemical graph theory, topological indices are calculated based on the molecular graph of a chemical compound. Topological indices are used in the development of Quantitative structure Activity/Propoerty Relations. To study the physico-chemical properties of molecules most commonly used are the Zagreb indices. In this paper, we introduce reciprocals leap indices as a modified version of leap Zagreb indices. The exact values of reciprocals leap indices of some well-known classes of graphs are calculated. Lower and upper bounds on the reciprocals leap indices of graphs are established. The relationship between reciprocals leap indices and leap Zagreb indices are presented.

1. Introduction

In last decade, graph theory has found a considerable use in the mathematical chemistry. In this area we can apply tools of graph theory to model the chemical phenomenon mathematically. This theory contributes a prominent in chemical science. A chemical structure of molecules can be represent by molecular graph, where vertices represent the atoms and edges represent the bonds between them. The graph theory based structure descriptors can be determined by considering graph vertices and edges. A simply arithmetic operators are carried out to get numerical indices. Topological indices are used in the development of Quantitative Structure Activity/Property Relations (QSAR/QSPR). A graph is a collection of points and

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lines connecting a subset of them. The points and lines of a graph are also called vertices and edges of the graph, respectively. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Let $|V(G)| = n$ and $|E(G)| = m$, if two vertices u and v of the graph G are adjacent, then the edge connecting them will be denoted by uv. If $u, v \in V(G)$ then the distance $d_G(u, v)$ between u and v is defined as the length of a shortest path in G connecting them. The diameter of a connected graph G is the length of any longest geodesic, denoted by $diam(G)$. In a graph G, the degree (first degree) of a vertex v, denoted $d(v)$, is the number of first neighbors (the number of edges incident with v) and the second degree of v, denoted $d_2(v)$, is the number of second neighbors. The maximum and minimum degrees among the vertices of G, are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. The join graph $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G)\, and\, v \in V(H)\}$. A wheel $W_{1,n}$ and a friendship graphs are defined as $W_{1,n} = K_1 + C_n$ and $F_n = K_1 + \frac{n-1}{2}K_2$, respactively. A graph G is called F-free graph if no induced subgraph of G is isomorphic to F. In this paper, we only conceder with a simple connected graphs. Any undefined term or notation in this paper can be found in ([2] [7]).

One of the oldest and most commonly used to study the physico-chemical properties of molecules topological index are Zagreb indices introduced by Gutman and Trinajstic on based degree of vertices of G. The first and second Zagreb indices of a graph G are defined as:

$$M_1(G) = \sum_{v\in V(G)} d(v)^2$$

$$M_2(G) = \sum_{uv\in E(G)} d(u)d(v).$$

The quantity $M_1(G)$ was first time considered in 1972 [4], whereas $M_2(G)$ in 1975 [5]. For more information on Zagreb and beyond topological indices, readers are referred to the survey [6], and the references therein.

In 2017, Naji et al. [9] have introduced a new distance-degree-based topological indices conceived depending on the second degrees of vertices, and are so-called leap Zagreb indices of a graph G and are defined as:

$$LM_1(G) = \sum_{v\in V(G)} d_2(v)^2$$

$$LM_2(G) = \sum_{uv\in E(G)} d_2(u)d_2(v)$$

$$LM_3(G) = \sum_{v\in V(G)} d(v)d_2(v).$$

The leap Zagreb indices have several chemical applications. Surprisingly, the first leap Zagreb index has very good correlation with physical properties of chemical compound, like bolling point, entropy, DHVAP,

HVAP and eccentric factor [1]. Consequently, the new class of graphs, that so called leap graphs was defined and studied in [10], and was defined as, A graph G is said to be a leap graph, if and only if for every vertex $v \in V(G)$, $d(v) = d_2(v)$.

The inverse degree index of a graph the first was introduced in 2005 [11], and was defined by

$$ID(G) = \sum_{v \in V(G)} \frac{1}{d(v)}.$$

The inverse degree has attracted attention through a conjecture generating computer Graffiti [3]. The modified first Zagreb index ${}^mM_1(G)$ was first introduced in [8], and defined as

$${}^mM_1(G) = \sum_{v \in V(G)} \frac{1}{d(v)^2}.$$

Motivated by the inverse degree index, we herewith define the inverse second degree index of a graph, as following

$$ID_2(G) = \sum_{v \in V(G)} \frac{1}{d_2(v) + 1}.$$

Note that,we added one to the second degree of a vertex, because there are infinity graph with some vertex whose $d(v) = n - 1$ and so $d_2(v) = 0$. Thus, the $ID_2(G)$ defined here is well-defined for every graph.

Lemma 1.1. *Let G be the connected graph with $\delta \geq 1$. Then*

$$\frac{n}{n - \delta} \leq ID_2(G) \leq n.$$

Motivated by the modified Zagreb and leap Zagreb indices of graphs and the huge applications of them, we in this work define the first, second and third reciprocals leap indices of a graph as a modified version of leap Zagreb indices. The exact values of some well-known graphs are computed. Some upper and lower bounds on reciprocals leap indices of a graph are established. Finally, we investigate and present the relationship between reciprocals leap indices and leap Zagreb indices of graphs.

We need the following fundamental results to prove our main results.

Lemma 1.2. *(A): Let G be a connected graph with n vertices and m edges. Then for every vertex $v \in V(G)$,*

$$d_2(v) \leq \Big(\sum_{u \in N(v)} d(u)\Big) - d(v).$$

Equality holds if and only if G is a (C_3, C_4) free.

Lemma 1.3. *(B): Let G be a connected graph with n vertex. Then for every vertex $v \in V(G)$,*

$$d_2(v) \leq n - 1 - d(v).$$

Equality holds if and only if G having diameter of most two.

Lemma 1.4. *(C): Let G be k-regular (C_3, C_4)-free graph. Then for every vertex $v \in V(G)$*

$$d_2(v) = k(k-1).$$

2. Reciprocals Leap Indices of Graphs

In this section, we present the definitions of first, second and third reciprocals leap indices of a graph and explore how we can calculation them for a connected simple graph.

Definition 2.1.

For a connected graph G, the first, second and third reciprocals leap indices are defined as:

$$\begin{aligned} RL_1 = RL_1(G) &= \sum_{v \in V(G)} \frac{1}{(d_2(v)+1)^2} \\ RL_2 = RL_2(G) &= \sum_{uv \in E(G)} \frac{1}{(d_2(u)+1)(d_2(v)+1)} \\ RL_3 = RL_3(G) &= \sum_{v \in V(G)} \frac{1}{d(v)(d_2(v)+1)}. \end{aligned}$$

To illustrate these invariants, we compute them for a graph G shown in Figure 1.

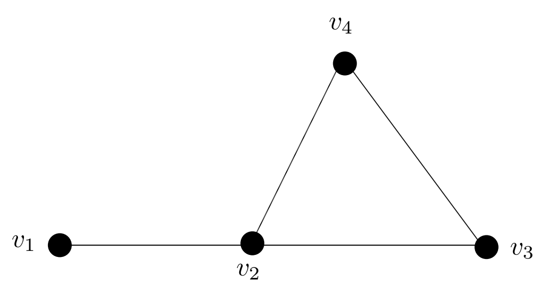

Figure 1.

It is easy to show that $d_2(v_1) = 2$, $d_2(v_2) = 0$, $d_2(v_3) = d_2(v_4) = 1$. Hence,

$$\begin{aligned} RL_1(G) &= \frac{1}{(2+1)^2} + \frac{1}{(0+1)^2} + \frac{1}{(1+1)^2} + \frac{1}{(1+1)^2} \\ &= \frac{1}{4} + \frac{1}{1} + \frac{1}{4} + \frac{1}{4} = \frac{29}{18} \end{aligned}$$

$$
\begin{aligned}
RL_2(G) &= \frac{1}{(2+1)(0+1)} + \frac{1}{(0+1)(1+1)} + \frac{1}{(0+1)(1+1)} + \frac{1}{(1+1)(1+1)} \\
&= \frac{1}{3} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{19}{12}, \\
RL_3(G) &= \frac{1}{1(2+1)} + \frac{1}{3(0+1)} + \frac{1}{2(1+1)} + \frac{1}{2(1+1)} \\
&= \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{7}{6}.
\end{aligned}
$$

3. Reciprocals Leap Indices for some Families of Graphs

In this section, we establish the formulaes of the exact values of reciprocals leap indices for some well-known graph classes.

Proposition 3.1.

For a positive integer $n \geq 2$,

(1) *For the complete graph K_n,*

- $RL_1(K_n) = n$,
- $RL_2(K_n) = \frac{n(n-1)}{2}$,
- $RL_3(K_n) = \frac{n}{(n-1)}$.

(2) *For the path P_n,*

- $RL_1(P_n) = \begin{cases} 2, & \text{if } n = 2 \text{ ;} \\ \frac{3}{2}, & \text{if } n = 3 \text{ ;} \\ \frac{n+5}{9}, & \text{otherwise.} \end{cases}$
- $RL_2(P_n) = \begin{cases} 1, & \text{if } n = 2,3 \text{ ;} \\ \frac{3}{4}, & \text{if } n = 4 \text{ ;} \\ \frac{2n+5}{18}, & \text{otherwise.} \end{cases}$
- $RL_3(P_n) = \begin{cases} 1, & \text{if } n = 2 \text{ ;} \\ \frac{3}{2}, & \text{if } n = 3 \text{ ;} \\ \frac{n+5}{6}, & \text{otherwise.} \end{cases}$

(3) *For the cycle C_n,*

- $RL_1(C_n) = RL_2(C_n) = \begin{cases} 3, & \text{if } n = 3 \text{ ;} \\ 1, & \text{if } n = 4 \text{ ;} \\ \frac{n}{9}, & \text{otherwise.} \end{cases}$
- $RL_3(C_n) = \begin{cases} \frac{3}{2}, & \text{if } n = 3 \text{ ;} \\ 1, & \text{if } n = 4 \text{ ;} \\ \frac{n}{6}, & \text{otherwise.} \end{cases}$

(4) *For the complete bipartite graph* $K_{r,s}$, $1 \leq r \leq s \leq n-1$,

- $RL_1(K_{r,s}) = RL_3(K_{r,s}) = \frac{r+s}{rs}$.
- $RL_2(K_{r,s}) = 1$.

(5) *For the star graph* $K_{1,n-1}$,

- $RL_1(K_{1,n-1}) = RL_3(K_{1,n-1}) = \frac{n}{n-1}$
- $RL_2(K_{1,n-1}) = 1$.

(6) *For the wheel graph,* $W_{1,n}, n \geq 3$,

- $RL_1(W_{1,n}) = \begin{cases} 4, \ \textit{if } n = 3 \ ; \\ \frac{n^2-3n+4}{(n-2)^2}, \ \textit{otherwise.} \end{cases}$
- $RL_2(W_{1,n}) = \begin{cases} 6, \ \textit{if } n = 3 \ ; \\ \frac{n(n-1)}{(n-2)^2}, \ \textit{otherwise.} \end{cases}$
- $RL_3(W_{1,n}) = \begin{cases} \frac{4}{3}, \ \textit{if } n = 3 \ ; \\ \frac{n^2+3n-6}{3n(n-2)}, \ \textit{otherwise.} \end{cases}$

(7) *For the friendship graph* F_n, $n \geq 3$,

- $RL_1(F_n) = 1 + \frac{n-1}{(n-2)^2}$,
- $RL_2(F_n) = \frac{(n-1)(2n-3)}{2(n-2)^2}$,
- $RL_3(F_n) = \frac{(n^2-3)}{2(n-1)(n-2)}$.

Proposition 3.2. *Let G be a connected k-regular C_3, C_4-free graph. Then*

(1) $RL_1(G) = \frac{n}{(k^2-k+1)^2}$,
(2) $RL_2(G) = \frac{nk}{(k^2-k+1)^2}$,
(3) $RL_3(G) = \frac{n}{k(k^2-k+1)}$.

Proof. The proof is immediately consequence of Lemma 1.4 and the definitions of reciprocals leap indices of graphs.

□

4. Bounds on Reciprocals Leap Indices of graphs

In this section, we present some upper and lower bounds on reciprocals leap indices of a graph, in term of number of vertices, number of edges, minimum (maximum) degree, Inverse degree and second inverse degree indices of a graph.

Theorem 4.1. *Let G be a connected graph with $n \geq 2$ vertices. Then,*

$$RL_1(G) \leq n.$$

Equality holds if and only if G is a complete graph.

Proof. Let G be a connected graph with $n \geq 2$ vertices. Since for every $v \in V(G)$, $d_2(v) \geq 0$, which led to $d_2(v) + 1 \geq 1$. Thus, $\frac{1}{(d_2(v)+1)^2} \leq 1$, for every $v \in V(G)$ and so,

$$\begin{aligned} RL_1(G) &= \sum_{v \in V(G)} \frac{1}{(d_2(v)+1)^2} \\ &\leq \sum_{v \in V(G)} 1 = n. \end{aligned}$$

To prove the equality, we assume, on the contrary, that $G \neq K_n$, for $n \geq 2$. Then there are at least two vertices $u, v \in V(G)$ such that $d_2(u) \geq 1$ and $d_2(v) \geq 1$. Thus,

$$\begin{aligned} RL_1(G) &= \frac{1}{(d_2(u)+1)^2} + \frac{1}{(d_2(v)+1)^2} + \sum_{w \in V(G)-\{u,v\}} \frac{1}{(d_2(w)+1)^2} \\ &\leq \frac{1}{(2)^2} + \frac{1}{(2)^2} + \sum_{w \in V(G)-\{u,v\}} \frac{1}{(0+1)^2} \\ &= \frac{1}{4} + \frac{1}{4} + (n-2) \\ &= \frac{4(n-2)+2}{4} = \frac{4n-6}{4} \\ &= n - \frac{3}{2} < n. \end{aligned}$$

which is a contradiction.

Conversely, if G is a complete graph, then it is immediate the result follow from Proposition 3.1. □

Corollary 4.1. *For any connected graph G, with n vertices, $RL_1(G) = n$ if and only if G is a complete graph.*

Theorem 4.2. *Let G be a connected graph with $m \geq 1$ edegs. Then,*

$$RL_2(G) \leq m.$$

Equality holds if and only if G is a complete graph.

Proof. The proof is similar to the proof of Theorem 4.1, then we omit it. □

Corollary 4.2. *For any connected graph G, with m edges, $RL_2(G) = m$, if and only if G is a complete graph.*

Theorem 4.3. *Let G be a connected graph with $n \geq 2$ vertices. Then,*

$$RL_3(G) \leq n.$$

Equality holds if and only if $G = K_2$.

Proof. Let G be a connected graph with $n \geq 2$ vertices. Since $d(v) \geq 1$ and $d_2(v) \geq 0$, for every $v \in V(G)$. So $d(v)(d_2(v)+1) \geq 1$, for every $v \in V(G)$. Hence,

$$\begin{aligned} RL_3(G) &= \sum_{v \in V(G)} \frac{1}{d(v)(d_2(v)+1)} \\ &\leq \sum_{v \in V(G)} \frac{1}{1} = n. \end{aligned}$$

Suppose the equality $RL_3(G) = n$ holds. Then $d(v)(d_2(v)+1) = 1$, for every vertex $v \in V(G)$. Since $d(v)$ and $d_2(v)$ are a positive integers numbers. Then $d(v)(d_2(v)+1) = 1$, if and only if $d(v) = 1$ and $d_2(v) = 0$. Since for any graph G and any vertex $v \in V(G)$, $d_2(v) = 0$ if and only if $d(v) = n-1$.

Then, $d_2(v) = 0$ and $d(v) = 1$, if and only if $n = 2$ and G is a complete graph. Therefore, $RL_3(G) = n$, if and only if $G = K_2$. □

Corollary 4.3. *For any connected graph G, with n vertices, $RL_3(G) = n$ if and only if $n = 2$.*

Theorem 4.4. *Let G be a connected graph with n vertices. Then,*

$$\frac{n}{\Delta} \leq RL_3(G) \leq \frac{n}{\delta}.$$

The lower bound attains on $K_{1,n-1}$ and K_n, whereas the upper bound attains on K_n.

Proof. Let G be a connected graph with n vertices. Since for every vertex $v \in V(G)$, $\delta \leq d(v) \leq \Delta$ and $d_2(v) \geq 0$. Then, $\delta \leq d(v)(d_2(v)+1) \leq \Delta$. So

$$\begin{aligned} RL_3(G) &= \sum_{v \in V(G)} \frac{1}{d(v)(d_2(v)+1)} \\ &\geq \sum_{v \in V(G)} \frac{1}{\Delta(1)} = \frac{n}{\Delta}. \\ &= \sum_{v \in V(G)} \frac{1}{d(v)(d_2(v)+1)} \\ &\leq \sum_{v \in V(G)} \frac{1}{\delta(1)} = \frac{n}{\delta}. \end{aligned}$$

□

Theorem 4.5. *For any connected graph G with n vertices,*

$$RL_1(G) \geq \frac{n}{(n-\delta)^2}.$$

Equality holds if and only if G is a regular graph with diameter at most two.

Proof. Since for any connected graph $d(v) \geq \delta$, for every $v \in V(G)$ and by using Lemma 1.3, $d_2(v) \leq n-1-d(v)$, for every $v \in V(G)$. Then,

$d_2(v)+1 \leq n-d(v) \leq n-\delta$, for every $v \in V(G)$. Hence,

$$\begin{aligned} RL_1(G) &= \sum_{v\in V(G)} \frac{1}{(d_2(v)+1)^2} \\ &\geq \sum_{v\in V(G)} \frac{1}{(n-\delta)^2} \\ &= \frac{n}{(n-\delta)^2}. \end{aligned}$$

Suppose the equality $RL_1(G) = \frac{n}{(n-\delta)^2}$ is holding. Then by Lemma 1.3, equality $d_2(v) = n-1-d(v)$ holds for every vertex $v \in V(G)$, if and only if $diam(G) \leq 2$. Thus, $d_2(v)+1 = n-d(v) = n-\delta$, for every $v \in V(G)$ if and only if $diam(G) \leq 2$ and G is δ-regular graph. Therefore, equality $RL_1(G) = \frac{n}{(n-\delta)^2}$ hold if and only if G is a regular graph with $diam(G) \leq 2$. □

The following results immediately follow from the facts that, $\delta(G) \geq 1$ for any connected graph G and for every vertex $v \in V(G)$ and if G is a k-regular graph, then $\delta(G) = k$ for every vertex $v \in V(G)$.

Corollary 4.4. *For any connected graph G,*

$$RL_1(G) \geq \frac{n}{(n-1)^2}.$$

Equality holds if and only if $G = K_2$.

Corollary 4.5. *For any k-regular graph,*

$$RL_1(G) \geq \frac{n}{(n-k)^2}.$$

Equality holds if and only if $diam(G) \leq 2$.

Theorem 4.6. *For any connected graph G,*

$$RL_2(G) \geq \frac{m}{(n-\delta)^2}.$$

Equality holds if and only if G is a regular graph with diameter at most two.

Proof. The proof is similar to the proof of Theorem 4.5. □

Corollary 4.6. *For any connecte k-regular graph,*

$$RL_2(G) \geq \frac{nk}{(n-k)^2}.$$

Equality holds if and only if G having diameter at most two.

Theorem 4.7. *For any connected graph G,*

$$RL_3(G) \geq \frac{n}{\Delta(n-\delta)}.$$

Equality holds if and only if G is regular graph with diameter at most two.

Proof. Let G be a connected graph with minimum and maximum degrees $1 \leq \delta \leq \Delta$. Then by Lemma 1.3, $d_2(v) + 1 \leq n - d(v)$. Since $\delta \leq d(v) \leq \Delta$, then $d(v)(d_2(v) + 1) \leq \Delta(n - \delta)$. Hence,

$$RL_3(G) \geq \sum_{v \in V(G)} \frac{1}{\Delta(n-\delta)} = \frac{n}{\Delta(n-\delta)}.$$

Suppose the equality $RL_3(G) = \frac{n}{\Delta(n-\delta)}$ holds. Then by Lemma 1.3, $d_2(v)+1 = n-d(v)$, for every $v \in V(G)$, if and only if $diam(G) \leq 2$ and $d(v)(d_2(v) + 1) = \Delta(n - \delta)$, if and only if $diam(G) \leq 2$ and $d(v) = \delta = \Delta$. This complete the proof. □

Since, for every vertex v in a graph G, $\delta \leq \Delta$ which implies that $\delta^2 \leq \delta\Delta$ and $\Delta n - \delta^2 \geq \Delta n - \delta\Delta$. Then the following results strieghtforward.

Proposition 4.1. *For any connected graph G,*

$$RL_3(G) \geq \frac{n}{\Delta n - \delta^2}.$$

Equality holds if and only if G is regular graph with diameter at most two.

Corollary 4.7. *For any $k-$regular graph,*

$$RL_3(G) \geq \frac{n}{k(n-k)}.$$

Equality holds if and only if $diam(G) \leq 2$.

Theorem 4.8. *For any connected graph G,*

$$RL_1(G) \leq ID_2(G).$$

Equality holds if and only if G is a complete.

Proof. Since $d_2(v)$ is a positive integer number for every $v \in V(G)$. Then $(d_2(v) + 1)^2 \geq d_2(v) + 1$ and hence $\frac{1}{(d_2(v)+1)^2} \leq \frac{1}{d_2(v)+1}$. Therefore,

$$\begin{aligned} RL_1(G) &= \sum_{v \in V(G)} \frac{1}{(d_2(v)+1)^2} \\ &\leq \sum_{v \in V(G)} \frac{1}{d_2(v)+1} \\ &= ID_2(G). \end{aligned}$$

Suppose the equality $RL_1(G) = ID_2(G)$ holds. Then $(d_2(v)+1)^2 = d_2(v)+1$, if and only if $d_2(v)+1 = 1$, if and only if $d_2(v) = 0$, for every $v \in V(G)$, if and only if $G = K_n$. □

Theorem 4.9. *For any connected graph G,*

$$RL_3(G) \leq ID(G).$$

Equality holds if and only if $G = K_n$.

Proof. Let G be a connected graph. Since $d_2(v)+1 \geq 1$, for every $v \in V(G)$. Then $\frac{1}{d(v)(d_2(v)+1)} \leq \frac{1}{d(v)}$ and hence,

$$\begin{aligned} RL_3(G) &= \sum_{v \in V(G)} \frac{1}{d(v)(d_2(v)+1)} \\ &\leq \sum_{v \in V(G)} \frac{1}{d(v)} = ID(G). \end{aligned}$$

If the equality $RL_3(G) = ID(G)$ holds, then $d(v)(d_2(v)+1) = d(v)$, for every $v \in V(G)$, that implies that $d_2(v) = 0$ for every $v \in V(G)$ and hence $G = K_n$.

Conversely, it is easy to check that if $G = K_n$, then $RL_3(G) = ID(G)$. □

Theorem 4.10. *Let G be a connected graph of order n and with maximum degree $\Delta \geq 1$,*

$$RL_3(G) \geq \frac{ID_2(G)}{\Delta}.$$

The equality holds if and only if G is regular.

Proof. Let G be a connected graph with n vertices and $\Delta \geq 1$. The Chebyshers sum inequality stat that if $a_1 \geq a_2 \geq ... \geq a_n$ and $b_1 \geq b_2 \geq ... \geq b_n$, then

$$n \sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i.$$

Hence, by put $a_i = \frac{1}{d(v_i)}$ and $b_i = \frac{1}{d_2(v_i)+1}$, for every $i = 1, 2, .., n$. We get

$$\begin{aligned} nRL_3(G) &\geq (\sum_{i=1}^{n} \frac{1}{d(v_i)})(\sum_{i=1}^{n} \frac{1}{(d_2(v_i)+1)}) \\ &\geq (\sum_{i=1}^{n} \frac{1}{\Delta})(ID_2(G)) \\ &= \frac{n}{\Delta} ID_2(G). \end{aligned}$$

Therefore, $RL_3(G) \geq \frac{1}{\Delta} ID_2(G)$. Suppose the equality $RL_3(G) = \frac{1}{\Delta} ID_2(G)$ holds. Then $d(v) = \Delta$ for every $v \in V(G)$, if and only if G is a Δ-regular graph.

Conversely, let G be a k-regular graph. Then

$$RL_3(G) = \sum_{v \in V(G)} \frac{1}{k(d_2(v)+1)} = \frac{1}{k} \sum_{v \in V(G)} \frac{1}{d_2(v)+1} = \frac{ID_2(G)}{k}.$$ □

Theorem 4.11. *Let G be a connected graph of order n. Then*

(1) $RL_1(G) \leq n[1 - 2\ln(n-\delta)]$.

(2) $RL_2(G) \leq m[1 - 2\ln(n-\delta)]$.

(3) $RL_3(G) \leq n[1 - \ln(\Delta n - \delta^2)]$.

The equality holds in (1),(2) *and* (3) *if and only if G is a complete.*

Proof. Let G be a connected graph with n vertices, m edges and minimum (maximum) degrees $\delta \geq 1$ ($\Delta \leq n-1$). We prove an inequality for $RL_1(G)$, and the proof of the inequalities (2) and (3) are similar. Assume the function $f(x) = x - \ln x - 1$. Easy calculating gives us that $f(x) \geq 0$, for every positive real number. Thus, for every $v \in V(G)$, we get

$$\frac{1}{(d_2(v)+1)^2} - \ln(\frac{1}{(d_2(v)+1)^2}) - 1 \geq 0.$$

Hence,

$$\frac{1}{(d_2(v)+1)^2} \geq 1 + \ln(\frac{1}{(d_2(v)+1)^2}).$$

By taking the summation on two sides of the inequality over the vertex set of the graph, we get

$$\begin{aligned} RL_1(G) &= \sum_{v\in V(G)} \frac{1}{(d_2(v)+1)^2} \\ &\geq \sum_{v\in V(G)} 1 + \sum_{v\in V(G)} \ln(\frac{1}{(d_2(v)+1)^2}) \\ &= n + \ln\left[\prod_{j=1}^{n}(\frac{1}{(d_2(v)+1)^2})\right]. \end{aligned}$$

By using the fact, for every $v \in V(G)$, $d_2(v)+1 \leq n-\delta$, we get,

$$\begin{aligned} RL_1(G) &\geq n + \ln\left[\prod_{j=1}^{n}(\frac{1}{(n-\delta)^2})\right] \\ &= n + \ln\left[\frac{1}{(n-\delta)^{2n}}\right] \\ &= n - 2n\ln(n-\delta) \\ &= n[1 - 2\ln(n-\delta)]. \end{aligned}$$

The equality holds if and only if $f(\frac{1}{d_2(v)+1}) = 0$, if and only if $\frac{1}{d_2(v)+1} = 1$, if and only if $d_2(v) = 0$, for every $v \in V(G)$, if and only if $G = K_n$, $n \geq 2$. □

Theorem 4.12. *Let G be a leap graph of order n and minimum and maximum degrees $\delta \geq 1$ and $\Delta \leq n-1$. Then*

(1) $\frac{n}{(\Delta+1)^2} \leq RL_1(G) \leq \frac{n}{(\delta+1)^2}$.

(2) $\frac{m}{(\Delta+1)^2} \leq RL_2(G) \leq \frac{m}{(\delta+1)^2}$.

(3) $\frac{n}{\Delta(\Delta+1)} \leq RL_3(G) \leq \frac{n}{\delta(\delta+1)}$.

Equalities holds in $(1), (2)$ *and* (3) *if and only if* G *is a regular.*

Proof. Let G be a connected leap graph with $n \geq 4$ vertex, minimum degree $\delta \geq 1$ and maximum degree $\Delta \leq n-1$. We prove part (1)only, and the proofs of (2) and (3) are similar. Since G is a leap graph and $\delta \leq d(v) \leq \Delta$, for every $v \in V(G)$.Then $\delta \leq d_2(v) \leq \Delta$, for every $v \in V(G)$. Hence,

$$\frac{1}{(\Delta+1)^2} \leq \frac{1}{(d_2(v)+1)^2} \leq \frac{1}{(\delta+1)^2},$$

by taken the summation over vertex set of G, we get

$$\sum_{v \in V(G)} \frac{1}{(\Delta+1)^2} \leq RL_1(G) \leq \sum_{v \in V(G)} \frac{1}{(\delta+1)^2}.$$

So,

$$\frac{n}{(\Delta+1)^2} \leq RL_1(G) \leq \frac{n}{(\delta+1)^2}.$$

Suppose the equality holds in the lower bound. Then $d(v)+1 = \Delta+1$, for every $v \in V(G)$. That mean G is a Δ-regular graph. Similarly, for the upper bound, which let G is δ-regular graph.
Conversely, let G be a k-regular leap graph. Then $\Delta = \delta = k$, and hence

$$RL_1(G) = \sum_{v \in V(G)} \frac{1}{(d(v)+1)^2} = \sum_{v \in V(G)} \frac{1}{(k+1)^2} = \frac{n}{(k+1)^2}.$$

□

5. Relationship Between Reciprocals Leap Indices and Leap Zagreb Indices of graphs

In this section, we investigate the relationship between reciprocals leap indices and leap Zagreb indices of graphs. Also, the relation between first and third reciprocals leap indices of a graph are presented.

Theorem 5.1. *For any connected graph* G *with* n *vertices and* m *edges.*

(1) $RL_1(G) + LM_1(G) \geq n + 4m - 2M_1(G)$.

(2) $RL_2(G) + LM_2(G) \geq m - LM_3(G)$.

(3) $RL_3(G) + LM_3(G) \geq 2(n-m)$.

equalities hold in (1) and (3) if and only if $G = K_2$*, whereas in (2) if and only if* G *is complete.*

Proof. Let G be a connected graph of order n and size m. Assume the function $f(x) = x + \frac{1}{x} - 2$. Easy calculation gives $f(x) \geq 0$, for every positive real number x. So, if we put $x = \frac{1}{(d_2(v)+1)^2}$, then for every $v \in V(G)$,

$$\frac{1}{(d_2(v)+1)^2} + (d_2(v)+1)^2 - 2 \geq 0$$

and hence $\frac{1}{(d_2(v)+1)^2} \geq 2-(d_2(v)+1)^2$. Since, by Lemma 1.2,
$\sum_{v\in V(G)} d_2(v) \leq M_1(G)-2m$. Then

$$\begin{aligned} RL_1(G) &= \sum_{v\in V(G)} \frac{1}{(d_2(v)+1)^2} \\ &\geq 2n - \sum_{v\in V(G)} \left(d_2^2(v)+2d_2(v)+1\right) \\ &= 2n-(LM_1(G)+2M_1(G)-4m-n)\,. \end{aligned}$$

Therefore,

$$RL_1(G)+LM_1(G) \geq n+4m-2M_1(G).$$

Since the equality $f(x)=0$, if and only if $x=1$. Then the equality holds in (1), if and only if $\frac{1}{(d_2(v)+1)^2}=1$, if and only if $d_2(v)=0$, for every $v\in V(G)$, if and only if $G+K_n$, and by Lemma 1.2, G is C_3, C_4-free graph. Thus $G=K_2$.

To prove the inequality (2), put $x=\frac{1}{(d_2(v)+1)(d_2(v)+1)}$. So for every $uv\in E(G)$, we get
$\frac{1}{(d_2(v)+1)(d_2(v)+1)} \geq 2-(d_2(u)+1)(d_2(v)+1)$. By taken the summation over edge set of graph G, we get

$$\begin{aligned} RL_2(G) &\geq 2m - \sum_{uv\in E(G)} (d_2(u)+1)(d_2(v)+1)) \\ &= 2m - \sum_{uv\in E(G)} [d_2(u)d_2(v)+d_2(u)+d_2(v)+1] \\ &= 2m-[LM_2(G)+LM_3(G)+m] \\ &= m-LM_2(G)-LM_3(G). \end{aligned}$$

Therefore, $RL_2(G)+LM_2(G) \geq m-LM_3(G)$.

The proof of an equality in (2) is very similar to the proof of part (1).

Now, by put $x=\frac{1}{d(v)(d_2(v)+1)}$, we get for every $v\in V(G)$

$$\frac{1}{d(v)(d_2(v)+1)} \geq 2-d(v)(d_2(v)+1)$$

and hence, by taken the summation over vertex set of G,

$$\begin{aligned} RL_3(G) &\geq 2n - \sum_{v\in V(G)} (d(v)d_2(v)+d(v)) \\ &= 2n-LM_3(G)-2m. \end{aligned}$$

Therefore, $RL_3(G)+LM_3(G) \geq 2(n-m)$.

The proof of equality in (3) is similar to the proof of part (1). □

Theorem 5.2. *Let G be a connected graph of order n and size m. Then*

(1) $\frac{1}{RL_1(G)} \leq \frac{LM_1(G)+n(2n-1)-4m}{n^2}$.

(2) $\frac{1}{RL_2(G)} \leq \frac{LM_2(G)+LM_3(G)+m}{m^2}$.

(3) $\frac{1}{RL_3(G)} \leq \frac{LM_3(G)+2m}{n^2}$.

The equality holds in (1) and (2), if and only if $G = K_n$, whereas in (3) if and only if $G = K_2$.

Proof. Let G be a connected graph of order n and size m. Cauchy-Schwartz state, for every real numbers a_i and b_i, $i = 1, 2, ..., n$, that

$$(\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$$

If we put $a_i = \frac{1}{d_2(v)+1}$ and $b_i = d_2(v) + 1$, for $i = 1, 2, ..., n$, and for every vertex $v \in V(G)$, then

$$\begin{aligned}
n^2 = (\sum_{i=1}^{n} 1)^2 &= \left(\sum_{v\in V(G)} \frac{1}{(d_2(v)+1)}(d_2(v)+1)\right)^2 \\
&\leq (\sum_{v\in V(G)} \frac{1}{(d_2(v)+1)^2})(\sum_{v\in V(G)} (d_2(v)+1)^2) \\
&\leq RL_1(G) \sum_{v\in V(G)} (d_2^2(v) + 2d_2(v) + 1)) \\
&= RL_1(G) \left[LM_1(G) + 2(n(n-1)) - 4m + n\right] \\
&= RL_1(G) \left[LM_1(G) + n(2n-1) - 4m\right].
\end{aligned}$$

Therefore, $\frac{1}{RL_1(G)} \leq \frac{LM_1(G)+n(2n-1)-4m}{n^2}$.

Since the equality hold in Cauchy-Schwartiz, if and only if $a_i = b_i$, for every $i = 1, 2, ..., n$. Then the equality holds in (1), if and only if $\frac{1}{d_2(v)+1} = d_2(v) + 1$, if and only if $d_2(v) = 0$, for every $v \in V(G)$, if and only if G is a complete graph.

Now, to prove an inequality (2), we put $a_i = \frac{1}{\sqrt{(d_2(u)+1)(d_2(v)+1)}}$ and $b_i = \sqrt{(d_2(u)+1)(d_2(v)+1)}$ for every edge $uv \in E(G)$, we get

$$\begin{aligned}
m^2 = (\sum_{uv\in E} 1)^2 &= \left(\sum_{uv\in E(G)} \frac{1}{\sqrt{(d_2(u)+1)(d_2(v)+1)}} \cdot \sqrt{(d_2(u)+1)(d_2(v)+1)}\right)^2 \\
&\leq \left(\sum_{uv\in E(G)} \frac{1}{(d_2(u)+1)(d_2(v)+1)}\right)\left(\sum_{uv\in E(G)} (d_2(u)+1)(d_2(v)+1)\right) \\
&= RL_2(G) \left[\sum_{uv\in E(G)} (d_2(u)d_2(v) + (d_2(u) + d_2(v)) + 1)\right] \\
&= RL_2(G) \left[LM_2(G) + LM_3(G) + m\right]
\end{aligned}$$

Therefore, $\frac{1}{RL_2(G)} \leq \frac{LM_2(G)+LM_3(G)+m}{m^2}$.

Finally, by putting $a_i = \frac{1}{\sqrt{(d(v))(d_2(v)+1)}}$ and $b_i = \sqrt{(d(v))(d_2(v)+1)}$ in Cauchy-Schwartz inequality. Then for every $v \in V(G)$,

$$\begin{aligned} n^2 = (\sum_{v\in V(G)} 1)^2 &= \left(\sum_{v\in V(G)} \frac{1}{\sqrt{d(v)(d_2(v)+1)}} \cdot \sqrt{d(v)(d_2(v)+1)}\right)^2 \\ &\leq \left(\sum_{v\in V(G)} \frac{1}{d(v)(d_2(v)+1)}\right)\left(\sum_{v\in V(G)} d(v)(d_2(v)+1)\right) \\ &= RL_3(G)\left[\sum_{v\in V(G)} (d(v)d_2(v)+d(v))\right] \\ &= RL_3(G)\left[LM_3(G)+2m\right]. \end{aligned}$$

Therefore, $\frac{1}{RL_3(G)} \leq \frac{LM_3(G)+2m}{n^2}$. □

Theorem 5.3. *Let G be a connected graph of order n. Then*

$$RL_3(G) \leq \frac{1}{\delta}\sqrt{nRL_1(G)}.$$

Equality holds if and only if G is an $\frac{n}{2}$-regular graph.

Proof. Let G be a connected graph of order n. Put $a_i = \frac{1}{d(v)}$ and $b_i = \frac{1}{d_2(v)+1}$ in Cauchy Schwartz inequality. Then

$$\begin{aligned} (RL_3(G))^2 &= (\sum_{v\in V(G)} \frac{1}{d(v)(d_2(v)+1)})^2 \\ &\leq \left(\sum_{v\in V(G)} \frac{1}{d(v)}\right)\left(\sum_{v\in V(G)} \frac{1}{(d_2(v)+1)^2}\right) \\ &\leq \left(\sum_{v\in V(G)} \frac{1}{\delta^2}\right)(RL_1(G)) \\ &= \frac{n}{\delta^2}RL_1(G). \end{aligned}$$

Therefore $RL_3(G) \leq \frac{1}{\delta}\sqrt{nRL_1(G)}$. The equality $RL_3(G) = \frac{1}{\delta}\sqrt{nRL_1(G)}$ holds if and only if $\frac{1}{d(v)} = \frac{1}{d_2(v)+1}$, if and only if $d(v) = d_2(v)+1$, if and only if $d_2(v) = d(v)-1$, if and only if $d_2(v) = \delta - 1$, for every vertex $v \in V(G)$. So G is a δ-regular graph with $d_2(v) = \delta - 1$, for every $v \in V(G)$. Hence by this conclusion and apply Lemma 1.3, we get , $\delta = \frac{n}{2}$. Therefore the equality $RL_3(G) = \frac{1}{\delta}\sqrt{nRL_1(G)}$ holds if and only if G is an $\frac{n}{2}$-regular graph. □

Corollary 5.1. *Let G be a connected graph. Then*

$$RL_3(G) \leq ({}^{m}M_1(G))(RL_1(G)).$$

Corollary 5.2. *Let G be a connected graph. Then*

$$RL_3(G) \leq n({}^m M_1(G)).$$

Theorem 5.4. *Let G be a connected graph of order n. Then*

(1) $RL_1(G) \leq \frac{(n-\delta+1)ID_2(G)-n}{n-\delta}$.

(2) $RL_2(G) \leq m - \frac{LM_2(G)+LM_3(G)}{n-\delta}$.

(3) $RL_3(G) \leq \frac{n}{\delta} + \frac{n\delta - LM_3(G)-2m}{\delta\Delta(n-\delta)}$.

Equality holds in (1), (2) *and* (3) *if and only if G is regular graph with $diam(G) \leq 2$.*

Proof. Let G be a connected graph of order n, size m and $\delta \geq 1$, $\Delta \leq n-1$. Diaz-Metcalf inequality stat that if $a_i \neq 0$ and b_i for $i = 0, 1, 2, ..., n$ satisfy $t \leq \frac{b_i}{a_i} \leq T$, then

$$\sum_{i=1}^{n} b_i^2 + tT\sum_{i=1}^{n} a_i^2 \leq (T+t)\sum_{i=1}^{n} a_i b_i.$$

Equality holds if and only if $b_i = ta_i$ or $b_i = Ta_i$, for $i = 1, 2, ..., n$.

1) By Diaz-Matcalf inequality, we prove inequality (1). Put $b_i = 1$ and $a_i = \frac{1}{d_2(v_i)+1}$ for $i = 1, 2, ..., n$. Since $d_2(v) + 1 \leq n - \delta$, for every $v \in V(G)$. Then $1 \leq \frac{b_i}{a_i} \leq n - \delta$ and hence $t = 1$ and $T = n - \delta$. Thus,

$$\sum_{i=1}^{n} 1 + (n-\delta)\sum_{i=1}^{n} \frac{1}{(d_2(v_i)+1)^2} \leq (n-\delta+1)\sum_{i=1}^{n} \frac{1}{d_2(v_i)+1}.$$

So, $n + (n-\delta)RL_1(G) \leq (n-\delta+1)ID_2(G)$. Therefore, $RL_1(G) \leq \frac{(n-\delta+1)ID_2(G)-n}{(n-\delta)}$. Suppose the equality holds in (1). Then by Diaz-Metcalf inequality $1 = \frac{1}{d_2(v)+1}$, which implies that $d_2(v) + 1 = 1$, so $d_2(v) = 0$ for every $v \in V(G)$ or $1 = \frac{n-\delta}{d_1(v)+1}$ which implies that $d_2(v) + 1 = n - \delta$, so $d_2(v) = n - 1 - \delta$, which implies, by Lemma 1.2, that $d(v) = \delta$ for every $v \in V(G)$ and $diam(G) \leq 2$.

Conversely, let G be a k-regular graph with $diam(G) \leq 2$. Then by Lemma 1.2, $d_2(v) + 1 = n - k$, for every $v \in V(G)$ and hence,

$RL_1(G) = \sum_{i=1}^{n} \frac{1}{(n-k)^2} = \frac{n}{(n-k)^2}$, and $ID_2(G) = \sum_{i=1}^{n} \frac{1}{(n-k)} = \frac{n}{(n-k)}$.

On the other hand,

$$\begin{aligned} \frac{(n-\delta+1)ID_2(G)-n}{(n-\delta)} &= \frac{(n-k+1)(\frac{n}{n-k})-n}{n-k} \\ &= \frac{n(n-k-n)-n(n-k)}{(n-k)^2} \\ &= \frac{n}{(n-k)^2} = RL_1(G). \end{aligned}$$

Therefore, the equality holds.

2) To prove inequality (2), put in Diaz-Matcalf inequality $b_i = \sqrt{(d_1(u)+1)(d_2(v)+1)}$ and $a_i =$

$\frac{1}{\sqrt{(d_2(u)+1)(d_2(v)+1}}$ for every $uv \in E(G)$. Hence, $1 \leq \frac{b_i}{a_i} \leq (n-\delta)^2$ for every $i = 1, 2, ..., n$. Thus, $t = 1$ and $T = n - \delta$, and hence,

$$\sum_{uv\in E(G)} (d_2(u)+1)(d_2(v)+1) + (n-\delta)^2 \sum_{uv\in E(G)} \frac{1}{(d_1(u)+1)(d_2(v)+1)} \leq [(n-\delta)^2] \sum_{uv\in E(G)} 1$$

$$\sum_{uv\in E(G)} [d_2(u)d_2(v) + d_2(u) + d_2(v) + 1] + (n-\delta)^2 RL_2(G) \leq m(n-\delta)^2 + 1$$

$$LM_2(G) + LM_3(G) + m + (n-\delta)^2 RL_2(G) \leq m(n-\delta)^2 + m.$$

Therefore,

$$\begin{aligned} RL_2(G) &\leq \frac{m(n-\delta)^2 + m - LM_2(G) - LM_3(G) - m}{(n-\delta)^2} \\ &= m - \frac{LM_2(G) + LM_3(G)}{(n-\delta)^2}. \end{aligned}$$

The proof of the equality is similar to the proof of inequality (1), so we left it.

3) To prove the inequality (3), we put $b_i = \sqrt{(d(v)(d_2(v)+1)}$ and $a_i = \frac{1}{\sqrt{(d(v))(d_2(v)+1)}}$ for every $v \in V(G)$.Thus, $\delta \leq \frac{b_i}{a_i} \leq \Delta(n-\delta)$, for every $i = 1, 2, ..., n$. Hence, $t = \delta$ and $T = \Delta(n-\delta)$. So,

$$\sum_{v\in V(G)} d(v)(d_2(v)+1) + \delta\Delta(n-\delta) \sum_{v\in V(G)} \frac{1}{d(v)(d_2(v)+1)} \leq (\Delta(n-\delta)+\delta) \sum_{v\in V(G)} 1$$

$$LM_3(G) + 2m + \delta\Delta(n-\delta) RL_3(G) \leq \delta n(n-\delta) + \delta n.$$

Thus,

$$\begin{aligned} RL_3(G) &\leq \frac{\Delta n(n-\delta) + \delta n - LM_3(G) - 2m}{\delta\Delta(n-\delta)} \\ &= \frac{n}{\delta} + \frac{\delta n - LM_3(G) - 2m}{\delta\Delta(n-\delta)}. \end{aligned}$$

The proof of the equality is similar to the proof of inequality (1). □

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