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COUPLED COINCIDENCE POINT FOR $f(\psi, \varphi)-$CONTRACTIONS VIA GENERALIZED $\alpha-$ADMISSIBLE MAPPINGS WITH AN APPLICATION

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Abstract. The main objective of this manuscript is to discuss some coupled coincidence point (ccp) results for generalized $\alpha-$ admissible mappings which are $f(\psi, \varphi)-$ contractions in the context of $b-$metric spaces (b-ms). Also, an example to support the obtained theoretical theorems is derived. Ultimately, an analytical solution for nonlinear integral equation (nie) is discussed as an application.

1. Introduction and elementary discussions

Fixed point techniques plays an enormous role in many applications of mathematics. During the past thirty years various extension of a metric space have been discussed. The Banach contraction principle is a popular tool helps to solve problems in nonlinear analysis. A number of publications are interested to the study and solutions of many practical and theoretical problems by using this principle [1–8].

Bakhtin [9] in 1993 and Czerwik [10] in 1998 introduced the concept of (b-ms). Since then, several papers have been published on the fixed point theory of both classes of single-valued and multi-valued operators in (b-ms). [11], [12], [13–16].

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Definition 1.1. *[10] Let Γ be a nonempty set and $s \geq 1$ be a given real number. A function $\nu_b : \Gamma \times \Gamma \to [0, \infty)$ is a b-metric (b-m) iff, for all $e, r, \zeta \in \Upsilon$, the stipulations below are fulfilled:*

(b_1) $\nu_b(e, r) = 0 \Leftrightarrow e = r$;

(b_2) $\nu_b(e, r) = \nu_b(r, e)$;

(b_3) $\nu_b(e, r) \leq s[\nu_b(e, \zeta) + \nu_b(\zeta, r)]$.

The pair (Γ, ν_b) is called a (b-ms) with a constant $s \geq 1$.

Example 1.1. *Let $\Gamma = [0, \infty)$. Define the function $\nu_b : \Upsilon^2 \to [0, \infty)$ by $\nu_b(e, r) = (e - r)^2$. Then (Γ, ν_b) is a (b-ms) with a constant $s = 2$.*

Definition 1.2. *[10] Suppose that (Γ, ν_b) is a (b-ms). So a sequence $\{e_n\}$ in Γ is called:*

(i) convergent if there is $e \in \Gamma$ so that $\nu_b(e_n, e) \to 0$ as $n \to \infty$.

(ii) Cauchy sequence iff $\lim_{m,n\to\infty} \nu_b(e_m, e_n) \to 0$ as $n, m \to \infty$.

(iii) the pair (Γ, ν_b) is called a complete iff every Cauchy sequence $\{e_n\}$ in Γ converges to $e \in \Gamma$.

Lemma 1.1. *[11] Let (Γ, ν_b) be a (b-ms) with a coefficient $s \geq 1$, $\{e_n\}$ and $\{r_n\}$ be a convergent to points $e, r \in \Gamma$, respectively. Then we have*

$$\frac{1}{s^2}\nu_b(e, r) \leq \liminf_{n\to\infty} \nu_b(e_n, r_n) \leq \limsup_{n\to\infty} \nu_b(e_n, r_n) \leq s^2\nu_b(e, r).$$

In particular, if $e = r$, then $\lim_{n\to\infty} \nu_b(e_n, r_n) = 0$. Moreover, for each $\delta \in \Gamma$, we have

$$\frac{1}{s}\nu_b(e, \delta) \leq \liminf_{n\to\infty} \nu_b(e_n, \delta) \leq \limsup_{n\to\infty} \nu_b(e_n, \delta) \leq s\nu_b(e, \delta).$$

Lemma 1.2. *[12] Let $\{e_n\}$ be a sequence in a (b-ms) (Γ, ν_b) so that*

$$\nu_b(e_n, e_{n+1}) \leq \lambda\nu_b(e_{n-1}, e_n),$$

for some $\lambda, 0 < \lambda < \frac{1}{s}$, and for each $n \in \mathbb{N}$. Then $\{e_n\}$ is a Cauchy sequence in Γ.

The idea of coupled fixed point initiated and studied by Guo and Lakshmikantham [17]. After that, the monotone property is studied by Bhaskar and Lakshmikantham [18]. Many works are made to generalized this concept in various spaces under certain conditions, the reader can shed light on [19–23, 25, 26].

Definition 1.3. *[18] An element $(e, r) \in \Upsilon \times \Upsilon$ is called a (ccp) of the mappings $\Upsilon : \Gamma \times \Gamma \to \Gamma$ and $\Lambda : \Gamma \to \Gamma$ if $\Upsilon(e, r) = \Lambda e$ and $\Upsilon(r, e) = \Lambda r$.*

Definition 1.4. *[27] An element $(e, r) \in \Gamma \times \Gamma$ is called a (ccp) of mappings $\Upsilon, \Lambda : \Gamma \times \Gamma \to \Gamma$ if $\Upsilon(e, r) = \Lambda(e, r)$ and $\Upsilon(r, e) = \Lambda(r, e)$.*

Example 1.2. *Let $\Upsilon, \Lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $\Upsilon(e, r) = \Lambda r$ and $\Lambda(e, r) = \frac{2}{3}(e + r)$ for all $(e, r) \in \Gamma \times \Gamma$. Note that $(0, 0), (1, 2)$ and $(2, 1)$ are (ccp) of Υ and Λ .*

Definition 1.5. *[27] Let $\Upsilon, \Lambda : \Gamma \times \Gamma \to \Gamma$. We say that the pair (Υ, Λ) is generalized compatible if*

$$\begin{aligned} \nu_b(\Upsilon(\Lambda(e_n, r_n), \Lambda(r_n, e_n)), \Lambda(\Upsilon(e_n, r_n), \Upsilon(r_n, e_n))) &\to 0 \text{ as } n \to \infty, \\ \nu_b(\Upsilon(\Lambda(r_n, e_n), \Lambda(e_n, r_n)), \Lambda(\Upsilon(r_n, e_n), \Upsilon(e_n, r_n))) &\to 0 \text{ as } n \to \infty, \end{aligned}$$

whenever $\{e_n\}$ and $\{r_n\}$ are sequences in Γ such that for all $t_1, t_2 \in \Gamma$, we have

$$\begin{aligned} \lim_{n\to\infty} \Upsilon(e_n, r_n) &= \lim_{n\to\infty} \Lambda(e_n, r_n) = t_1, \\ \lim_{n\to\infty} \Upsilon(r_n, e_n) &= \lim_{n\to\infty} \Lambda(r_n, e_n) = t_2. \end{aligned}$$

Definition 1.6. *[28] Let $\Upsilon : \Gamma \to \Gamma$ and $\alpha : \Gamma \times \Gamma \to [0, +\infty)$. We say that Υ is an $\alpha-$ admissible mapping if $\alpha(e, r) \geq 1$ implies $\alpha(\Upsilon e, \Upsilon r) \geq 1$ for all $e, r \in \Upsilon$.*

Ansari [29] initiated the remarkable of $C-$class functions. This contribution covers a large class of contractive conditions. Here, we denote C-class functions as C.

Definition 1.7. *[29] A C-class function is a continuous mapping $f : [0, \infty)^2 \to \mathbb{R}$ which fulfills the stipulations below:*

(1) $f(e, r) \leq e$,

(2) $f(e, r) = e$ implies that either $e = 0$ or $r = 0$ for all $e, r \in [0, \infty)$.

Example 1.3. *The functions below $f : [0, \infty)^2 \to \mathbb{R}$ are elements of C, for all $e, r \in [0, \infty)$:*

(1) $f(e, r) = e - r$;

(2) $f(e, r) = \lambda e, 0 < \lambda < 1$;

(3) $f(e, r) = \frac{e}{(1+r)^\sigma}; \sigma \in (0, \infty)$;

(4) $f(e, r) = \frac{log(r+a^e)}{(1+r)}, a > 1$.

Here in this manuscript, we refers to:

• $\Psi = \{\psi : \psi : [0, \infty) \to [0, \infty)$ is a strictly nondecrasing and continuous function, $\psi(t) = 0 \Leftrightarrow t = 0\}$.

• An ultra altering distance function $\Phi = \{\varphi : \varphi : [0, \infty) \to [0, \infty)\}$ is a continuous, non-decreasing mapping such that $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) \geq 0$.

The goal of this paper is to obtain some new (ccp) results for a certain class of $f(\psi, \varphi)$- contractive via generalized $\alpha-$ admissible mappings in (b-ms). Ultimately, to support our work we present an example and application to find an analytical solution to the (nie).

2. Main results

We begin this part with the definition below:

Definition 2.1. *Let $\Upsilon, \Lambda : \Gamma^2 \to \Gamma$ and $\alpha : \Gamma^2 \times \Gamma^2 \to \mathbb{R}^+$ be given mappings. We say that Υ is a generalized α-admissible with respect to (w.r.t.) Λ if*

$$\alpha((\Lambda(e,r),\Lambda(r,e)),(\Lambda(\mu,\kappa),\Lambda(\kappa,\mu))) \geq 1 \text{ implies } \alpha((\Upsilon(e,r),\Upsilon(r,e)),(\Upsilon(\mu,\kappa),\Upsilon(\kappa,\mu))) \geq 1,$$

for all $(e,r),(\mu,\kappa) \in \Gamma^2$.

Now, we present the first main theorem:

Theorem 2.1. *Let (Γ, ν_b) be a complete (b-ms) (with parameter $s > 1$), and $\Upsilon, \Lambda : \Gamma^2 \to \Gamma$ be two generalized compatible mappings such that Υ is a generalized $\alpha-$admissible mapping w.r.t. Λ and Λ is continuous. Let there is $f \in C$, $\psi \in \Psi, \varphi \in \Phi$ so that the stipulation below holds*

$$\begin{aligned}
&(\psi\left(s^{\sigma}\nu_b(\Upsilon(e,r),\Upsilon(\mu,\kappa)\right) + \rho)^{\alpha\begin{pmatrix}(\Lambda(e,r),\Lambda(r,e)),\\(\Upsilon(e,r),\Upsilon(r,e))\end{pmatrix}\alpha\begin{pmatrix}(\Lambda(\mu,\kappa),\Lambda(\kappa,\mu)),\\(\Upsilon(\mu,\kappa),\Upsilon(\kappa,\mu))\end{pmatrix}} \\
\leq\ & f\begin{pmatrix}\psi(\frac{\nu_b(\Lambda(e,r),\Lambda(\mu,\kappa))+\nu_b(\Lambda(r,e),\Lambda(\kappa,\mu))}{2}),\\ \varphi(\frac{\nu_b(\Lambda(e,r),\Lambda(\mu,\kappa))+\nu_b(\Lambda(r,e),\Lambda(\kappa,\mu))}{2})\end{pmatrix} \\
&+\rho, \qquad (2.1)
\end{aligned}$$

for all $e, r, \mu, \kappa \in \Gamma, \sigma, \rho > 0$, and $\alpha : (\Gamma^2 \times \Gamma^2) \to [0, \infty)$. Assume that
(i) $\Upsilon(\Gamma^2) \subseteq \Lambda(\Gamma^2)$,
(ii) there is $e_0, r_0 \in \Gamma$ so that

$$\begin{aligned}
\alpha((\Lambda(e_0,r_0),\Lambda(r_0,e_0)),(\Upsilon(e_0,r_0),\Upsilon(r_0,e_0))) &\geq 1, \\
\alpha((\Lambda(r_0,e_0),\Lambda(e_0,r_0)),(\Upsilon(r_0,e_0),\Upsilon(e_0,r_0))) &\geq 1.
\end{aligned}$$

Also, suppose either
(iv) Υ is continuous, or
(v) $\{e_n\},\{r_n\}$ are two sequences in Γ so that

$$\alpha((e_{n+1},r_{n+1}),(e_n,r_n)) \geq 1 \text{ and } \alpha((r_n,e_n),(r_{n+1},e_{n+1})) \geq 1.$$

for all $n \in \mathbb{N} \cup \{0\}$, and $e_n \to e, r_n \to r$ as $n \to \infty$, $e, r \in \Gamma$, we have

$$\alpha((e,r),(e_n,r_n)) \geq 1 \text{ and } \alpha((r_n,e_n),(r,e)) \geq 1.$$

Then Υ and Λ have a (ccp).

Proof. Let $e_0, r_0 \in \Gamma$, so by condition (ii), we have

$$\begin{aligned}
\alpha((\Lambda(e_0,r_0),\Lambda(r_0,e_0)),(\Upsilon(e_0,r_0),\Upsilon(r_0,e_0))) &\geq 1, \\
\alpha((\Lambda(r_0,e_0),\Lambda(e_0,r_0)),(\Upsilon(r_0,e_0),\Upsilon(e_0,r_0))) &\geq 1.
\end{aligned}$$

According to (i), define two sequences $\{e_n\},\{r_n\}$ in Γ by

$$\Upsilon(e_n,r_n)=\Lambda(e_{n+1},r_{n+1}),\quad \Upsilon(r_n,e_n)=\Lambda(r_{n+1},e_{n+1}),\forall n=0,1,2,\ldots .$$

Since $\Upsilon(\Gamma^2)\subseteq\Lambda(\Gamma^2)$, then we can write

$$\begin{aligned}&\alpha((\Lambda(e_0,r_0),\Lambda(r_0,e_0)),(\Lambda(e_1,r_1),\Lambda(r_1,e_1)))\\ =\ &\alpha((\Lambda(e_0,r_0),\Lambda(r_0,e_0)),(\Upsilon(e_0,r_0),\Upsilon(r_0,e_0)))\geq 1,\end{aligned}$$

$$\begin{aligned}&\alpha((\Lambda(r_0,e_0),\Lambda(e_0,r_0)),(\Lambda(r_1,e_1),\Lambda(e_1,r_1)))\\ =\ &\alpha((\Lambda(r_0,e_0),\Lambda(e_0,r_0)),(\Upsilon(r_0,e_0),\Upsilon(e_0,r_0)))\geq 1.\end{aligned}$$

Again, since Υ is a generalized α-admissible mapping w.r.t. Λ, then we have that

$$\alpha((\Upsilon(e_0,r_0),\Upsilon(r_0,e_0)),(\Upsilon(e_1,r_1),\Upsilon(r_1,e_1)))\geq 1,$$

and

$$\alpha((\Upsilon(r_0,e_0),\Upsilon(e_0,r_0)),(\Upsilon(r_1,e_1),\Upsilon(e_1,r_1)))\geq 1.$$

By induction, we get for all $n\in\mathbb{N}\cup\{0\}$,

$$\begin{aligned}&\alpha((\Lambda(e_{n+1},r_{n+1}),\Lambda(r_{n+1},e_{n+1})),(\Upsilon(e_{n+1},r_{n+1}),\Upsilon(r_{n+1},e_{n+1})))\\ =\ &\alpha((\Upsilon(e_n,r_n),\Upsilon(r_n,e_n)),(\Upsilon(e_{n+1},r_{n+1}),\Upsilon(r_{n+1},e_{n+1})))\geq 1,\\ \text{and}\ &\alpha((\Lambda(r_{n+1},e_{n+1}),\Lambda(e_{n+1},r_{n+1})),(\Upsilon(r_{n+1},e_{n+1}),\Upsilon(e_{n+1},r_{n+1})))\\ =\ &\alpha((\Upsilon(r_n,e_n),\Upsilon(e_n,r_n)),(\Upsilon(r_{n+1},e_{n+1}),\Upsilon(e_{n+1},r_{n+1})))\geq 1.\end{aligned}$$

(2.2)

Denote

$$\lambda_n=\nu_b(\Lambda(e_n,r_n),\Lambda(e_{n+1},r_{n+1}))+\nu_b(\Lambda(r_n,e_n),\Lambda(r_{n+1},e_{n+1})),n\in\mathbb{N}\cup\{0\}.$$

We suppose that $\lambda_n>0,\forall n\in\mathbb{N}$ because if not, (e_n,r_n) will be a (ccp) and the proof is finished.

We claim that $\psi(s^\sigma\lambda_{n+1})\leq\psi(\lambda_n)$. Using (2.2) and letting $e=e_n, r=r_n, \mu=e_{n+1}$, and $\kappa=r_{n+1}$ in (2.1), we get

$$\begin{aligned}
& \psi\left(s^{\sigma}\nu_b(\Lambda(e_{n+1},r_{n+1}),\Lambda(e_{n+2},r_{n+2})\right)+\rho \\
= \ & \psi\left(s^{\sigma}\nu_b(\Upsilon(e_n,r_n),\Upsilon(e_{n+1},r_{n+1})\right)+\rho \\
\leq \ & \left(\psi\left(s^{\sigma}\nu_b(\Upsilon(e_n,r_n),\Upsilon(e_{n+1},r_{n+1})\right)+\rho\right)^{\mu_n}, \\
\leq \ & f\begin{pmatrix} \psi\left(\frac{\nu_b(\Lambda(e_n,r_n),\Lambda(e_{n+1},r_{n+1}))+\nu_b(\Lambda(r_n,e_n),\Lambda(r_{n+1},e_{n+1}))}{2}\right), \\ \varphi\left(\frac{\nu_b(\Lambda(e_n,r_n),\Lambda(e_{n+1},r_{n+1}))+\nu_b(\Lambda(r_n,e_n),\Lambda(r_{n+1},e_{n+1}))}{2}\right)\end{pmatrix}+\rho \\
= \ & f\left(\psi(\frac{\lambda_n}{2}),\varphi(\frac{\lambda_n}{2})\right)+\rho, \qquad (2.3)
\end{aligned}$$

where

$$\begin{aligned}
\mu_n = \ & \alpha((\Lambda(e_n,r_n),\Lambda(r_n,e_n)),(\Upsilon(e_n,r_n),\Upsilon(r_n,e_n)))\times \\
& \alpha((\Lambda(e_{n+1},r_{n+1}),\Lambda(r_{n+1},e_{n+1})),(\Upsilon(e_{n+1},r_{n+1}),\Upsilon(r_{n+1},e_{n+1}))).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \psi\left(s^{\sigma}\nu_b(\Lambda(r_{n+2},e_{n+2}),\Lambda(r_{n+1},e_{n+1})\right)+\rho \\
= \ & \psi\left(s^{\sigma}\nu_b(\Upsilon(r_{n+1},e_{n+1}),\Upsilon(r_n,e_n)\right)+\rho \\
\leq \ & \left(\psi\left(s^{\sigma}\nu_b(\Upsilon(r_{n+1},e_{n+1}),\Upsilon(r_n,e_n)\right)+\rho\right)^{\mu_n}, \\
\leq \ & f\begin{pmatrix} \psi\left(\frac{\nu_b(\Lambda(r_{n+1},e_{n+1}),\Lambda(r_n,e_n))+\nu_b(\Lambda(e_{n+1},r_{n+1}),\Lambda(e_n,r_n))}{2}\right), \\ \varphi\left(\frac{\nu_b(\Lambda(r_{n+1},e_{n+1}),\Lambda(r_n,e_n))+\nu_b(\Lambda(e_{n+1},r_{n+1}),\Lambda(e_n,r_n))}{2}\right)\end{pmatrix}+\rho \\
= \ & f\left(\psi(\frac{\lambda_n}{2}),\varphi(\frac{\lambda_n}{2})\right)+\rho, \qquad (2.4)
\end{aligned}$$

where

$$\begin{aligned}
\mu_n = \ & \alpha((\Lambda(r_{n+1},e_{n+1}),\Lambda(e_{n+1},r_{n+1})),(\Upsilon(r_{n+1},e_{n+1}),\Upsilon(e_{n+1},r_{n+1}))\times \\
& \alpha((\Lambda(r_n,e_n),\Lambda(e_n,r_n)),(\Upsilon(r_n,e_n),\Upsilon(e_n,r_n)).
\end{aligned}$$

Summing (2.3), (2.4), and since ψ is nondecreasing, we get

$$\psi\left(s^{\sigma}\lambda_{n+1}\right)\leq f\left(\psi(\frac{\lambda_n}{2}),\varphi(\frac{\lambda_n}{2})\right)\leq\psi(\frac{\lambda_n}{2}), \qquad (2.5)$$

since ψ is nondecreasing. By inequality (2.5), one can write

$$\lambda_{n+1}\leq\frac{1}{s^{\sigma}}\lambda_n.$$

Hence, by Lemma 1.2, the sequence λ_n is $b-$Cauchy in Γ and then $\{\Lambda(e_n,r_n)\}$ and $\{\Lambda(r_n,e_n)\}$ are also Cauchy sequences in Γ. By the completeness of Γ, there exist $e,r\in\Gamma$ so that

$$\lim_{n\to\infty}\Lambda(e_n,r_n)=\Upsilon(e_n,r_n)=\Lambda,\ \text{and}\ \lim_{n\to\infty}\Lambda(r_n,e_n)=\Upsilon(r_n,e_n)=r. \qquad (2.6)$$

Since the pair (Υ, Λ) satisfies the generalized compatible, by (2.6), we can write

$$\lim_{n\to\infty} \nu_b(\Upsilon(\Lambda(e_n, r_n), \Lambda(r_n, e_n)), \Lambda(\Upsilon(e_n, r_n), \Upsilon(r_n, e_n))) = 0, \tag{2.7}$$

$$\lim_{n\to\infty} \nu_b(\Upsilon(\Lambda(r_n, e_n), \Lambda(e_n, r_n)), \Lambda(\Upsilon(r_n, e_n), \Upsilon(e_n, r_n))) = 0. \tag{2.8}$$

Now, if (i) holds, that is Υ is continuous and Λ is already continuous from the hypothesis of the theorem, then in view of triangle inequality, we have

$$\begin{aligned} \nu_b(\Lambda(e, r), \Upsilon(\Lambda(e_n, r_n), \Lambda(r_n, e_n))) &\leq s[\nu_b(\Lambda(e, r), \Lambda(\Upsilon(e_n, r_n), \Upsilon(r_n, e_n))) \\ &+ \nu_b(\Lambda(\Upsilon(e_n, r_n), \Upsilon(r_n, e_n)), \Upsilon(\Lambda(e_n, r_n), \Lambda(r_n, e_n)))]. \end{aligned}$$

Passing $n \to \infty$ in the above inequality and using (2.6), (2.7), and the continuity of Υ and Λ, we get $\Lambda(e, r) = \Upsilon(e, r)$. Similarly, by (2.6), (2.8), we can show that $\Lambda(r, e) = \Upsilon(r, e)$.

Next, assume that (ii) holds. Since the pair Υ,Λ satisfies the generalized compatible and Λ is continuous, we have

$$\begin{aligned} \lim_{n\to\infty} \Lambda(\Lambda(e_n, r_n), \Lambda(r_n, e_n)) &= \Lambda(e, r) \\ &= \lim_{n\to\infty} \Lambda(\Upsilon(e_n, r_n), \Upsilon(r_n, e_n)) \\ &= \lim_{n\to\infty} \Upsilon(\Lambda(e_n, r_n), \Lambda(r_n, e_n)), \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \lim_{n\to\infty} \Lambda(\Lambda(r_n, e_n), \Lambda(e_n, r_n)) &= \Lambda(r, e) \\ &= \lim_{n\to\infty} \Lambda(\Upsilon(r_n, e_n), \Upsilon(e_n, r_n)) \\ &= \lim_{n\to\infty} \Upsilon(\Lambda(r_n, e_n), \Lambda(e_n, r_n)). \end{aligned} \tag{2.10}$$

Then, we have

$$\alpha((\Lambda(\Lambda(e_n, r_n), e(r_n, e_n)), \Lambda(\Lambda(r_n, e_n), \Lambda(e_n, r_n))), (\Lambda(e, r), \Lambda(r, e))) \geq 1,$$

and

$$\alpha((\Lambda(\Lambda(r_n, e_n), \Lambda(e_n, r_n)), \Lambda(\Lambda(e_n, r_n), \Lambda(r_n, e_n))), (\Lambda(r, e), \Lambda(e, r))) \geq 1.$$

Applying (2.1), we get

$$\begin{aligned}
&\psi(\nu_b(\Upsilon(e,r),e(e,r))) = \lim_{n\to\infty}\psi(\nu_b(\Upsilon(e,r),e(\Upsilon(e_n,r_n),\Upsilon(r_n,e_n))))\\
\leq\ & \lim_{n\to\infty}\psi(\nu_b(\Upsilon(e,r),\Upsilon(e(e_n,r_n),\Lambda(r_n,e_n))))+\rho\\
\leq\ & \lim_{n\to\infty}(\psi(s^{\sigma}\nu_b(\Upsilon(e,r),\Upsilon(e(e_n,r_n),\Lambda(r_n,e_n))))+\rho)^{\mu_n},\\
\leq\ & \lim_{n\to\infty} f\begin{pmatrix} \psi\left(\frac{\nu_b(\Lambda(e,r),\Lambda(\Lambda(e_n,r_n),\Lambda(r_n,e_n)))+\nu_b(\Lambda(r,e),\Lambda(\Lambda(r_n,e_n),\Lambda(e_nr_n)))}{2}\right), \\ \varphi\left(\frac{\nu_b(\Lambda(e,r),\Lambda(\Lambda(e_n,r_n),\Lambda(r_n,e_n)))+\nu_b(\Lambda(r,e),\Lambda(\Lambda(r_n,e_n),\Lambda(e_nr_n)))}{2}\right)\end{pmatrix}+\rho,
\end{aligned}$$

where

$$\begin{aligned}
\mu_n &= \alpha(\Lambda(e,r),\Lambda(r,e)),(\Upsilon(e,r),\Upsilon(r,e))\times\\
&\quad\alpha\begin{pmatrix}(\Lambda(\Lambda(r_n,e_n),\Lambda(e_n,r_n)),\Lambda(\Lambda(e_n,r_n),\Lambda(r_n,e_n))),\\ (\Upsilon(\Lambda(r_n,e_n)),\Lambda(e_n,r_n),\Upsilon(\Lambda(e_n,r_n)),\Lambda(r_n,e_n))\end{pmatrix}.
\end{aligned}$$

Using (2.9),(2.10), we get $\psi(\nu_b(\Upsilon(e,r),\Lambda(e,r)))=0$ implies that $\Upsilon(e,r)=\Lambda(e,r)$. Similarly, we can prove that $\Upsilon(r,e)=\Lambda(r,e)$. □

Theorem 2.2. *Let (Γ,ν_b) be a complete (b-ms) (with parameter $s>1$), and $\Upsilon,\Lambda:\Gamma^2\to\Gamma$ be two generalized compatible mappings so that Υ is a generalized $\alpha-$admissible mapping w.r.t. Λ and Λ is continuous. Let there is $f\in C$ and $\psi\in\Psi,\varphi\in\Phi$ so that the stipulation below holds:*

$$\begin{pmatrix}\alpha\begin{pmatrix}(\Lambda(e,r),\Lambda(r,e)),\\(\Upsilon(e,r),\Upsilon(r,e))\end{pmatrix}\alpha\begin{pmatrix}(\Lambda(\mu,\kappa),\Lambda(\kappa,\mu)),\\(\Upsilon(\mu,\kappa),\Upsilon(\kappa,\mu))\end{pmatrix}\\ +1\end{pmatrix}^{\psi(s^{\sigma}\nu_b(\Upsilon(e,r),\Upsilon(\mu,\kappa))}$$

$$\leq 2^{f\left(\psi\left(\frac{\nu_b(\Lambda(e,r),\Lambda(\mu,\kappa))+\nu_b(\Lambda(r,e),\Lambda(\kappa,\mu))}{2}\right),\varphi\left(\frac{\nu_b(\Lambda(e,r),\Lambda(\mu,\kappa))+\nu_b(\Lambda(r,e),\Lambda(\kappa,\mu))}{2}\right)\right)}, \tag{2.11}$$

for all $e,r,\mu,\kappa\in\Gamma,\rho,\sigma>0$, and $\alpha:(\Gamma^2\times\Gamma^2)\to[0,\infty)$. Assume that

(i) $\Upsilon(\Gamma^2)\subseteq\Lambda(\Gamma^2)$,

(ii) there is $e_0,r_0\in\Gamma$ so that

$$\begin{aligned}
\alpha((\Lambda(e_0,r_0),\Lambda(r_0,e_0)),(\Upsilon(e_0,r_0),\Upsilon(r_0,e_0))) &\geq 1,\\
\alpha((\Lambda(r_0,e_0),\Lambda(e_0,r_0)),(\Upsilon(r_0,e_0),\Upsilon(e_0,r_0))) &\geq 1.
\end{aligned}$$

Also, suppose either

(iv) Υ is continuous, or

(v) $\{e_n\},\{r_n\}$ are two sequences in Γ such that

$$\alpha((e_{n+1},r_{n+1}),(e_n,r_n))\geq 1 \text{ and } \alpha((r_n,e_n),(r_{n+1},e_{n+1}))\geq 1.$$

for all $n \in \mathbb{N} \cup \{0\}$*, and* $e_n \to e, r_n \to r$ *as* $n \to \infty$*,* $e, r \in \Gamma$*, we have*

$$\alpha((e,r),(e_n,r_n)) \geq 1 \text{ and } \alpha((r_n,e_n),(r,e)) \geq 1.$$

Then Υ *and* Λ *have a (ccp).*

Proof. As in Theorem (2.1), we can conclude that for all $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} & \alpha((\Lambda(e_{n+1},r_{n+1}),\Lambda(r_{n+1},e_{n+1})),(\Upsilon(e_{n+1},r_{n+1}),\Upsilon(r_{n+1},e_{n+1}))) \\ = \; & \alpha((\Upsilon(e_n,r_n),\Upsilon(r_n,e_n)),(\Upsilon(e_{n+1},r_{n+1}),\Upsilon(r_{n+1},e_{n+1}))) \geq 1, \\ & \text{and } \alpha((\Lambda(r_{n+1},e_{n+1}),\Lambda(e_{n+1},r_{n+1})),(\Upsilon(r_{n+1},e_{n+1}),\Upsilon(e_{n+1},r_{n+1}))) \\ = \; & \alpha((\Upsilon(r_n,e_n),\Upsilon(e_n,r_n)),(\Upsilon(r_{n+1},e_{n+1}),\Upsilon(e_{n+1},r_{n+1}))) \geq 1. \end{aligned} \tag{2.12}$$

Denote

$$\lambda_n = \nu_b(\Lambda(e_n,r_n),\Lambda(e_{n+1},r_{n+1})) + \nu_b(\Lambda(r_n,e_n),\Lambda(r_{n+1},e_{n+1})), n \in \mathbb{N} \cup \{0\}.$$

We suppose that $\lambda_n > 0, \forall n \in \mathbb{N}$ because if not, (e_n, r_n) will be a (ccp) and the proof is finished.
We claim that $\psi(s^\sigma \lambda_{n+1}) \leq \psi(\lambda_n)$. Using (2.12), letting $e = e_n, r = r_n, \mu = e_{n+1}$, and $\kappa = r_{n+1}$ in (2.11), we have

$$\begin{aligned} & 2^{\psi(s^\sigma \nu_b(\Lambda(e_{n+1},r_{n+1}),\Lambda(e_{n+2},r_{n+2}))} = 2^{\psi(s^\sigma \nu_b(\Upsilon(e_n,r_n),\Upsilon(e_{n+1},r_{n+1}))} \\ \leq \; & (\mu_n+1)^{\psi(s^\sigma \nu_b(\Upsilon(e_n,r_n),\Upsilon(e_{n+1},r_{n+1}))} \\ \leq \; & 2^{f\left(\begin{array}{c} \psi\left(\frac{\nu_b(\Lambda(e_n,r_n),\Lambda(e_{n+1},r_{n+1}))+\nu_b(\Lambda(r_n,e_n),\Lambda(r_{n+1},e_{n+1}))}{2}\right), \\ \varphi\left(\frac{\nu_b(\Lambda(e_n,r_n),\Lambda(e_{n+1},r_{n+1}))+\nu_b(\Lambda(r_n,e_n),\Lambda(r_{n+1},e_{n+1}))}{2}\right) \end{array} \right)} \\ = \; & 2^{f\left(\psi(\frac{\lambda_n}{2}),\varphi(\frac{\lambda_n}{2})\right)}, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} & \mu_n = \alpha((\Lambda(e_n,r_n),\Lambda(r_n,e_n)),(\Upsilon(e_n,r_n),\Upsilon(r_n,e_n))) \times \\ & \alpha((\Lambda(e_{n+1},r_{n+1}),\Lambda(r_{n+1},e_{n+1})),(\Upsilon(e_{n+1},r_{n+1}),\Upsilon(r_{n+1},e_{n+1}))). \end{aligned}$$

Thus, we get

$$\psi\left(\nu_b(\Lambda(e_{n+1},r_{n+1}),\Lambda(e_{n+2},r_{n+2})\right) \leq f\left(\psi(\frac{\lambda_n}{2}),\varphi(\frac{\lambda_n}{2})\right).$$

Similarly, we have

$$
\begin{aligned}
& 2^{\psi(s^\sigma \nu_b(\Lambda(r_{n+2},e_{n+2}),\Lambda(r_{n+1},e_{n+1}))} = 2^{\psi(s^\sigma \nu_b(\Upsilon(r_{n+1},e_{n+1}),\Upsilon(r_n,e_n))} \\
\leq \; & (\mu_n+1)^{\psi(s^\sigma \nu_b(\Upsilon(r_{n+1},e_{n+1}),\Upsilon(r_n,e_n))} \\
\leq \; & 2^{f\left(\begin{array}{c} \psi\left(\frac{\nu_b(\Lambda(r_{n+1},e_{n+1}),\Lambda(r_n,e_n))+\nu_b(\Lambda(e_{n+1},r_{n+1}),\Lambda(e_n,r_n))}{2}\right), \\ \varphi\left(\frac{\nu_b(\Lambda(r_{n+1},e_{n+1}),\Lambda(r_n,e_n))+\nu_b(\Lambda(e_{n+1},r_{n+1}),\Lambda(e_n,r_n))}{2}\right)\end{array}\right)} \\
= \; & 2^{f\left(\psi(\frac{\lambda_n}{2}),\varphi(\frac{\lambda_n}{2})\right)}, \qquad (2.14)
\end{aligned}
$$

where

$$
\mu_n = \alpha((\Lambda(r_{n+1},e_{n+1}),\Lambda(e_{n+1},r_{n+1})),(\Upsilon(r_{n+1},e_{n+1}),\Upsilon(e_{n+1},r_{n+1}))) \times
$$
$$
\alpha((\Lambda(r_n,e_n),\Lambda(e_n,r_n)),(\Upsilon(r_n,e_n),\Upsilon(e_n,r_n))).
$$

Thus, we get

$$
\psi\left(s^\sigma \nu_b(\Lambda(r_{n+2},e_{n+2}),\Lambda(r_{n+1},e_{n+1})\right) \leq f\left(\psi(\frac{\lambda_n}{2}),\varphi(\frac{\lambda_n}{2})\right). \qquad (2.15)
$$

Summing both inequalities (2), (2.15), and since ψ is nondecreasing, we have

$$
\psi\left(s^\sigma \lambda_{n+1}\right) \leq f\left(\psi(\frac{\lambda_n}{2}),\varphi(\frac{\lambda_n}{2})\right) \leq \psi(\frac{\lambda_n}{2}), \qquad (2.16)
$$

since ψ is nondecreasing and from inequality (2.16), one can get

$$
\lambda_{n+1} \leq \frac{1}{s^\sigma}\lambda_n.
$$

Hence, by Lemma 1.2, the sequence λ_n is $b-$Cauchy in Γ and then $\{\Lambda(e_n,r_n)\}$ and $\{\Lambda(r_n,e_n)\}$ are also Cauchy sequences in Γ. By the completeness of Γ, there exist $e,r\in\Gamma$ such that

$$
\lim_{n\to\infty} e(e_n,r_n) = \Upsilon(e_n,r_n) = e, \; and \lim_{n\to\infty} e(r_n,e_n) = \Upsilon(r_n,e_n) = r. \qquad (2.17)
$$

Since the pair (Υ,e) satisfies the generalized compatible condition, then by (2.17), one can write

$$
\lim_{n\to\infty} \nu_b(\Upsilon(\Lambda(e_n,r_n),\Lambda(r_n,e_n)),\Lambda(\Upsilon(e_n,r_n),\Upsilon(r_n,e_n))) = 0, \qquad (2.18)
$$

$$
\lim_{n\to\infty} \nu_b(\Upsilon(\Lambda(r_n,e_n),\Lambda(e_n,r_n)),\Lambda(\Upsilon(r_n,e_n),\Upsilon(e_n,r_n))) = 0. \qquad (2.19)
$$

Now, if (i) holds, we apply the same steps of Theorem (2.1), and we get $\Lambda(r,e)=\Upsilon(r,e)$.

Now, assume that (ii) holds. Since the pair Υ,Λ satisfies the generalized compatible and Λ is continuous, we have

$$\begin{aligned}
\lim_{n\to\infty}\Lambda(\Lambda(e_n,r_n),e(r_n,e_n)) &= \Lambda(e,r)\\
&= \lim_{n\to\infty}\Lambda(\Upsilon(e_n,r_n),\Upsilon(r_n,e_n))\\
&= \lim_{n\to\infty}\Upsilon(\Lambda(e_n,r_n),\Lambda e(r_n,e_n)), \qquad (2.20)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\Lambda(\Lambda(r_n,e_n),\Lambda(e_n,r_n)) &= \Lambda(r,e)\\
&= \lim_{n\to\infty}\Lambda(\Upsilon(r_n,e_n),\Upsilon(e_n,r_n))\\
&= \lim_{n\to\infty}\Upsilon(\Lambda(r_n,e_n),\Lambda(e_n,r_n)). \qquad (2.21)
\end{aligned}$$

Then, we can write

$$\alpha((\Lambda(\Lambda(e_n,r_n),\Lambda(r_n,e_n)),\Lambda(\Lambda(r_n,e_n),\Lambda(e_n,r_n))),(\Lambda(e,r),\Lambda(r,e)))\geq 1,$$

and

$$\alpha((\Lambda(\Lambda(r_n,e_n),\Lambda(e_n,r_n)),\Lambda(\Lambda(e_n,r_n),\Lambda(r_n,e_n))),(\Lambda(r,e),\Lambda(e,r)))\geq 1.$$

Applying (2.11), we get

$$\begin{aligned}
2^{\psi(\nu_b(\Upsilon(e,r),\Lambda(e,r)))} &= \lim_{n\to\infty}2^{\psi(\nu_b(\Upsilon(e,r),e(\Upsilon(e_n,r_n),\Upsilon(r_n,e_n))))}\\
&= \lim_{n\to\infty}2^{\psi(\nu_b(\Upsilon(e,r),\Upsilon(e(e_n,r_n),\Lambda(r_n,e_n))))}\\
&\leq \lim_{n\to\infty}(\mu_n+1)^{\psi(s^\sigma\nu_b(\Upsilon(e,r),\Upsilon(e(e_n,r_n),\Lambda(r_n,e_n))))},\\
&\leq \lim_{n\to\infty}2^{f(\psi(\chi_n),\varphi(\chi_n))},
\end{aligned}$$

where

$$\begin{aligned}
\mu_n &= \alpha(\Lambda(e,r),\Lambda(r,e)),(\Upsilon(e,r),\Upsilon(r,e))\times\\
&\quad\alpha\begin{pmatrix}\Lambda(\Lambda(r_n,e_n),\Lambda(e_n,r_n)),\Lambda(\Lambda(e_n,r_n),\Lambda(r_n,e_n))),\\ (\Upsilon(\Lambda(r_n,e_n),\Lambda(e_n,r_n)),\Upsilon(\Lambda(e_n,r_n),\Lambda(r_n,e_n)))\end{pmatrix},
\end{aligned}$$

and

$$\chi_n=\frac{\nu_b(\Lambda(e,r),\Lambda(e(e_n,r_n),\Lambda(r_n,e_n)))+\nu_b(\Lambda(r,e),\Lambda(\Lambda(r_n,e_n),\Lambda(e_nr_n)))}{2},$$

Using (2.20),(2.21), we get $\psi(\nu_b(\Upsilon(e,r),\Lambda(e,r)))=0$ this leads to $\Upsilon(e,r)=\Lambda(e,r)$. Similarly, we can prove that $\Upsilon(r,e)=\Lambda(r,e)$. □

Theorem 2.3. *Let (Γ, ν_b) be a complete (b-ms) (with parameter $s > 1$), and $\Upsilon, \Lambda : \Gamma^2 \to \Gamma$ be two generalized compatible mappings such that Υ is a generalized$\alpha-$admissible mapping w.r.t. Λ and Λ is continuous. Let there is $f \in C$, $\psi \in \Psi, \varphi \in \Phi$ so that the stipulation below holds:*

$$\alpha \begin{pmatrix} (\Lambda(e,r), \Lambda(r,e)), \\ (\Upsilon(e,r), \Upsilon(r,e)) \end{pmatrix} \left[\alpha \begin{pmatrix} (\Lambda(\mu,\kappa), \Lambda(\kappa,\mu)), \\ (\Upsilon(\mu,\kappa), \Upsilon(\kappa,\mu))) \end{pmatrix} \psi(s^{\sigma} \nu_b(\Upsilon(e,r), \Upsilon(\mu,\kappa))) \right]$$
$$\leq f \begin{pmatrix} \psi(\frac{\nu_b(\Lambda(e,r),\Lambda(\mu,\kappa))+\nu_b(\Lambda(r,e),\Lambda(\kappa,\mu))}{2}), \\ \varphi(\frac{\nu_b(\Lambda(e,r),\Lambda(\mu,\kappa))+\nu_b(\Lambda(r,e),\Lambda(\kappa,\mu))}{2}) \end{pmatrix}, \tag{2.22}$$

for all $e, r, \mu, \kappa \in \Gamma, \rho, \sigma > 0,$, and $\alpha : (\Gamma^2 \times \Gamma^2) \to [0, \infty)$. Assume that

(i) $\Upsilon(\Gamma^2) \subseteq \Lambda(\Gamma^2)$

(ii) there is $e_0, r_0 \in \Gamma$ so that

$$\alpha((\Lambda(e_0,r_0), \Lambda(r_0,e_0)), (\Upsilon(e_0,r_0), \Upsilon(r_0,e_0))) \geq 1,$$
$$\alpha((\Lambda(r_0,e_0), \Lambda(e_0,r_0)), (\Upsilon(r_0,e_0), \Upsilon(e_0,r_0))) \geq 1.$$

Also, suppose either

(iv) Υ is continuous, or

(v) $\{e_n\}, \{r_n\}$ are two sequences in Γ so that

$$\alpha((e_{n+1}, r_{n+1}), (e_n, r_n)) \geq 1,$$
$$\alpha((r_n, e_n), (r_{n+1}, e_{n+1})) \geq 1.$$

for all $n \in \mathbb{N} \cup \{0\}$, and $e_n \to e, r_n \to r$ as $n \to \infty$, $e, r \in \Gamma$, we have

$$\alpha((e,r), (e_n, r_n)) \geq 1,$$
$$\alpha((r_n, e_n), (r, e)) \geq 1.$$

Then Υ and Λ have a (ccp).

Proof. Again, as in Theorem (2.1), we can conclude that for all $n \in \mathbb{N} \cup \{0\}$,

$$\alpha((\Lambda(e_{n+1}, r_{n+1}), \Lambda(r_{n+1}, e_{n+1})), (\Upsilon(e_{n+1}, r_{n+1}), \Upsilon(r_{n+1}, e_{n+1})))$$
$$= \alpha((\Upsilon(e_n, r_n), \Upsilon(r_n, e_n)), (\Upsilon(e_{n+1}, r_{n+1}), \Upsilon(r_{n+1}, e_{n+1}))) \geq 1, and$$
$$\alpha((\Lambda(r_{n+1}, e_{n+1}), \Lambda(e_{n+1}, r_{n+1})), (\Upsilon(r_{n+1}, e_{n+1}), \Upsilon(e_{n+1}, r_{n+1})))$$
$$= \alpha((\Upsilon(r_n, e_n), \Upsilon(e_n, r_n)), (\Upsilon(r_{n+1}, e_{n+1}), \Upsilon(e_{n+1}, r_{n+1}))) \geq 1.$$

(2.23)

Denote

$$\lambda_n = \nu_b(\Lambda(e_n, r_n), \Lambda(e_{n+1}, r_{n+1})) + \nu_b(\Lambda(r_n, e_n), \Lambda(r_{n+1}, e_{n+1})), n \in \mathbb{N} \cup \{0\}$$

.

We suppose that $\lambda_n > 0, \forall n \in \mathbb{N}$ because if not, (e_n, r_n) will be a (ccp) and the proof is finished. We claim that $\psi(s^\sigma \lambda_{n+1}) \leq \psi(\lambda_n)$. Using (2.12), letting $e = e_n, r = r_n, \mu = e_{n+1}$, and $\kappa = r_{n+1}$ in (2.22), we have

$$\begin{aligned}
& \psi\left(s^\sigma \nu_b(\Lambda(e_{n+1}, r_{n+1}), \Lambda(e_{n+2}, r_{n+2})\right) \\
= \; & \psi\left(s^\sigma \nu_b(\Upsilon(e_n, r_n), \Upsilon(e_{n+1}, r_{n+1})\right) \\
\leq \; & \mu_n \psi\left(s^\sigma \nu_b(\Upsilon(e_n, r_n), \Upsilon(e_{n+1}, r_{n+1})\right) \\
\leq \; & f\begin{pmatrix} \psi\left(\frac{\nu_b(\Lambda(e_n,r_n),\Lambda(e_{n+1},r_{n+1}))+\nu_b(\Lambda(r_n,e_n),\Lambda(r_{n+1},e_{n+1}))}{2}\right), \\ \varphi\left(\frac{\nu_b(\Lambda(e_n,r_n),\Lambda(e_{n+1},r_{n+1}))+\nu_b(\Lambda(r_n,e_n),\Lambda(r_{n+1},e_{n+1}))}{2}\right) \end{pmatrix} \\
= \; & f\left(\psi(\frac{\lambda_n}{2}), \varphi(\frac{\lambda_n}{2})\right), \qquad (2.24)
\end{aligned}$$

where

$$\mu_n = \alpha((\Lambda(e_n, r_n), \Lambda(r_n, e_n)), (\Upsilon(e_n, r_n), \Upsilon(r_n, e_n))) \times$$
$$\alpha((\Lambda(e_{n+1}, r_{n+1}), \Lambda(r_{n+1}, e_{n+1})), (\Upsilon(e_{n+1}, r_{n+1}), \Upsilon(r_{n+1}, e_{n+1}))).$$

Thus, we get

$$\psi\left(s^\sigma \nu_b(\Lambda(e_{n+1}, r_{n+1}), \Lambda(e_{n+2}, r_{n+2})\right) \leq f\left(\psi(\frac{\lambda_n}{2}), \varphi(\frac{\lambda_n}{2})\right). \qquad (2.25)$$

Similarly, we have

$$\begin{aligned}
& \psi\left(s^\sigma \nu_b(\Lambda(r_{n+2}, e_{n+2}), \Lambda(r_{n+1}, e_{n+1})\right) \\
= \; & \psi\left(s^\sigma \nu_b(\Upsilon(r_{n+1}, e_{n+1}), \Upsilon(r_n, e_n)\right) \\
\leq \; & \mu_n \psi\left(s^\sigma \nu_b(\Upsilon(r_{n+1}, e_{n+1}), \Upsilon(r_n, e_n)\right) \\
\leq \; & f\begin{pmatrix} \psi\left(\frac{\nu_b(\Lambda(r_{n+1},e_{n+1}),\Lambda(r_n,e_n))+\nu_b(\Lambda(e_{n+1},r_{n+1}),\Lambda(e_n,r_n))}{2}\right), \\ \varphi\left(\frac{\nu_b(\Lambda(r_{n+1},e_{n+1}),\Lambda(r_n,e_n))+\nu_b(\Lambda(e_{n+1},r_{n+1}),\Lambda(e_n,r_n))}{2}\right) \end{pmatrix} \\
= \; & f\left(\psi(\frac{\lambda_n}{2}), \varphi(\frac{\lambda_n}{2})\right), \qquad (2.26)
\end{aligned}$$

where

$$\mu_n = \alpha((\Lambda(r_{n+1}, e_{n+1}), \Lambda(e_{n+1}, r_{n+1})), (\Upsilon(r_{n+1}, e_{n+1}), \Upsilon(e_{n+1}, r_{n+1}))) \times$$
$$\alpha((\Lambda(r_n, e_n), \Lambda(e_n, r_n)), (\Upsilon(r_n, e_n), \Upsilon(e_n, r_n))).$$

Thus, we get

$$\psi\left(s^\sigma \nu_b(\Lambda(r_{n+2},e_{n+2}),\Lambda(r_{n+1},e_{n+1})\right)\leq f\left(\psi(\frac{\lambda_n}{2}),\varphi(\frac{\lambda_n}{2})\right). \tag{2.27}$$

By inequalities (2.25), (2.27), and since ψ is non-decreasing, we have

$$\psi\left(s^\sigma \lambda_{n+1}\right)\leq f\left(\psi(\frac{\lambda_n}{2}),\varphi(\frac{\lambda_n}{2})\right)\leq \psi(\frac{\lambda_n}{2}). \tag{2.28}$$

since ψ is nondecreasing and by inequality (2.28), we get

$$\lambda_{n+1}\leq \frac{1}{s^\sigma}\lambda_n.$$

Thus, by Lemma 1.2, the sequence λ_n is b-Cauchy in Γ and then $\{\Lambda(e_n,r_n)\}$ and $\{\Lambda(r_n,e_n)\}$ are also Cauchy sequences in Γ. By the completeness of Γ, there exist $e,r\in\Gamma$ such that

$$\lim_{n\to\infty}\Lambda(e_n,r_n)=\Upsilon(e_n,r_n)=e,\ and\ \lim_{n\to\infty}\Lambda(r_n,e_n)=\Upsilon(r_n,e_n)=r. \tag{2.29}$$

Since the pair (Υ,e) satisfies the generalized compatible condition, then by (2.29), one can write

$$\lim_{n\to\infty}\nu_b(\Upsilon(\Lambda(e_n,r_n),\Lambda(r_n,e_n)),\Lambda(\Upsilon(e_n,r_n),\Upsilon(r_n,e_n)))=0, \tag{2.30}$$

$$\lim_{n\to\infty}\nu_b(\Upsilon(\Lambda(r_n,e_n),\Lambda(e_n,r_n)),\Lambda(\Upsilon(r_n,e_n),\Upsilon(e_n,r_n)))=0. \tag{2.31}$$

Now, if (i) holds, we apply the same steps of Theorem (2.1), and we get $\Lambda(r,e)=\Upsilon(r,e)$.

Now, assume that (ii) holds. Since the pair Υ,Λ satisfies the generalized compatible condition and Λ is continuous, we have

$$\begin{aligned}\lim_{n\to\infty}\Lambda(\Lambda(e_n,r_n),\Lambda(r_n,e_n)) &= \Lambda(e,r)\\ &= \lim_{n\to\infty}\Lambda(\Upsilon(e_n,r_n),\Upsilon(r_n,e_n))\\ &= \lim_{n\to\infty}\Upsilon(\Lambda(e_n,r_n),\Lambda(r_n,e_n)),\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}\lim_{n\to\infty}\Lambda(\Lambda(r_n,e_n),\Lambda(e_n,r_n)) &= \Lambda(r,e)\\ &= \lim_{n\to\infty}\Lambda(\Upsilon(r_n,e_n),\Upsilon(e_n,r_n))\\ &= \lim_{n\to\infty}\Upsilon(\Lambda(r_n,e_n),\Lambda(e_n,r_n)).\end{aligned} \tag{2.33}$$

Then, we have

$$\alpha((\Lambda(\Lambda(e_n,r_n),\Lambda(r_n,e_n)),\Lambda(\Lambda(r_n,e_n),\Lambda(e_n,r_n))),(\Lambda(e,r),\Lambda(r,e)))\geq 1,$$

and

$$\alpha((\Lambda(\Lambda(r_n,e_n),\Lambda(e_n,r_n)),\Lambda(\Lambda(e_n,r_n),\Lambda(r_n,e_n))),(\Lambda(r,e),\Lambda(e,r)))\geq 1.$$

Applying (2.22), we get

$$\begin{aligned}
&\psi(\nu_b(\Upsilon(e,r),\Lambda(e,r))) = \lim_{n\to\infty}\psi(\nu_b(\Upsilon(e,r),\Lambda(\Upsilon(e_n,r_n),\Upsilon(r_n,e_n))))\\
=&\lim_{n\to\infty}\psi(\nu_b(\Upsilon(e,r),\Upsilon(\Lambda(e_n,r_n),\Lambda(r_n,e_n))))\\
\leq&\lim_{n\to\infty}\mu_n\psi(\nu_b(\Upsilon(e,r),\Upsilon(\Lambda(e_n,r_n),\Lambda(r_n,e_n)))),\\
\leq&\lim_{n\to\infty}f\left(\psi(\chi_n),\varphi(\chi_n)\right),
\end{aligned}$$

where

$$\begin{aligned}
\mu_n &= \alpha(\Lambda(e,r),\Lambda(r,e)),(\Upsilon(e,r),\Upsilon(r,e))\times\\
&\quad\alpha\begin{pmatrix}(\Lambda(\Lambda(e_n,r_n),\Lambda(r_n,e_n)),\Lambda(\Lambda(r_n,e_n),\Lambda(e_n,r_n))),\\ (\Upsilon(\Lambda(e_n,r_n),\Lambda(r_n,e_n)),\Upsilon(\Lambda(r_n,e_n),\Lambda(e_n,r_n)))\end{pmatrix}.
\end{aligned}$$

and

$$\chi_n = \frac{\nu_b(\Lambda(e,r),\Lambda(\Lambda(e_n,r_n),\Lambda(r_n,e_n))) + \nu_b(\Lambda(r,e),\Lambda(\Lambda(r_n,e_n),\Lambda(e_n r_n)))}{2}.$$

Using (2.32),(2.33), we get $\psi(\nu_b(\Upsilon(e,r),\Lambda(e,r))) = 0$ implies that $\Upsilon(e,r) = \Lambda(e,r)$. Similarly, we can prove that $\Upsilon(r,e) = \Lambda(r,e)$. □

Theorem 2.4. *Suppose that all requirements of Theorems (2.1) or (2.2) or (2.3) are fulfilled. In addition, let the stipulation below holds:*

(vi) If $\Lambda(e,r) = \Upsilon(e,r)$ *and* $\Lambda(r,e) = \Upsilon(r,e)$

then

$$\alpha((\Lambda(e,r),\Lambda(r,e)),(\Upsilon(e,r),\Upsilon(r,e))) \geq 1,$$

and

$$\alpha((\Lambda(r,e),\Lambda(e,r)),(\Upsilon(r,e),\Upsilon(e,r))) \geq 1.$$

Then Λ *and* Υ *have a unique (ccp).*

Proof. From Theorem (2.1) or (2.2) or (2.3), we know that the set of (ccp) of Λ and Υ is nonempty. Let (e,r) and (e^*,r^*) are (ccp) of Λ and Υ, this yields

$$\Lambda(e,r) = \Upsilon(e,r),\ \ \Lambda(r,e) = \Upsilon(r,e),$$

and

$$\Lambda(e^*,r^*) = \Upsilon(e^*,r^*),\ \ \Lambda(r^*,e^*) = \Upsilon(r^*,e^*).$$

We will prove that $\Lambda(e,r)=\Lambda(e^*,r^*)$ and $\Lambda(r,e)=\Lambda(r^*,e^*)$.

It follows from (vi) that

$$\alpha((\Lambda(e,r),\Lambda(r,e)),(\Upsilon(e,r),\Upsilon(r,e)))\geq 1$$
$$\alpha((\Lambda(r,e),\Lambda(e,r)),(\Upsilon(r,e),\Upsilon(e,r)))\geq 1,$$

and

$$\alpha\left((\Lambda(e^*,r^*),\Lambda(r^*,e^*)),(\Upsilon(e^*,r^*),\Upsilon(r^*,e^*))\right)\geq 1$$
$$\alpha((\Lambda(r^*,e^*),\Lambda(e^*,r^*)),(\Upsilon(r^*,e^*),\Upsilon(e^*,r^*)))\geq 1,$$

From Theorem 2.1, using above inequalities, we have

$$\begin{aligned} & \psi\left(\nu_b(\Lambda(e,r),\Lambda(e^*,r^*)\right)+\rho \\ \leq\ & \psi\left(s^\sigma\nu_b(\Upsilon(e,r),\Upsilon(e^*,r^*)\right)+\rho \\ \leq\ & \left(\psi\left(s^\sigma\nu_b(\Upsilon(e,r),\Upsilon(e^*,r^*)\right)+\rho\right)^{\mu_n} \\ \leq\ & f\begin{pmatrix} \psi(\frac{\nu_b(\Lambda(e,r),\Lambda(e^*,r^*))+\nu_b(\Lambda(r,e),\Lambda(r^*,e^*))}{2}), \\ \varphi(\frac{\nu_b(\Lambda(e,r),\Lambda(e^*,r^*))+\nu_b(\Lambda(r,e),\Lambda(r^*,e^*))}{2}) \end{pmatrix}+\rho, \end{aligned} \tag{2.34}$$

where

$$\mu_n=\alpha((\Lambda(e,r),\Lambda(r,e)),(\Upsilon(e,r),\Upsilon(r,e))),\alpha((\Lambda(e^*,r^*),\Lambda(r^*,e^*)),(\Upsilon(e^*,r^*),\Upsilon(r^*,e^*))).$$

From Theorem 2.2, we have

$$\begin{aligned} 2^{\psi(\nu_b(\Lambda(e,r),\Lambda(e^*,r^*)))} &\leq 2^{\psi(s^\sigma\nu_b(\Upsilon(e,r),\Upsilon(e^*,r^*)))} \\ &\leq (\mu_n+1)^{\psi(s^\sigma\nu_b(\Upsilon(e,r),\Upsilon(e^*,r^*)))} \\ &\leq 2^{f\left(\psi(\frac{\nu_b(\Lambda(e,r),\Lambda(e^*,r^*))+\nu_b(\Lambda(r,e),\Lambda(r^*,e^*))}{2}),\varphi(\frac{\nu_b(\Lambda(e,r),\Lambda(e^*,r^*))+\nu_b(\Lambda(r,e),\Lambda(r^*,e^*))}{2})\right)}. \end{aligned} \tag{2.35}$$

From Theorem 2.3, we have

$$\begin{aligned} \psi(\nu_b(\Lambda(e,r),\Lambda(e^*,r^*))) &\leq \psi(s^\sigma\nu_b(\Upsilon(e,r),\Upsilon(e^*,r^*))) \\ &\leq (\mu_n+1)\psi(s^\sigma\nu_b(\Upsilon(e,r),\Upsilon(e^*,r^*))) \\ &\leq f\begin{pmatrix} \psi(\frac{\nu_b(\Lambda(e,r),\Lambda(e^*,r^*))+\nu_b(\Lambda(r,e),\Lambda(r^*,\Lambda^*))}{2}), \\ \varphi(\frac{\nu_b(\Lambda(e,r),\Lambda(e^*,r^*))+\nu_b(\Lambda(r,e),\Lambda(r^*,e^*))}{2}) \end{pmatrix}. \end{aligned} \tag{2.36}$$

From (2.34), (2.35) and (2.36), we have

$$f\left(\psi(\nu_b(\Lambda(e,r),\Lambda(e^*,r^*)),\varphi(\nu_b(\Lambda(e,r),\Lambda(e^*,r^*)))\right)=\psi(\nu_b(\Lambda(e,r),\Lambda(e^*,r^*)).$$

By the hypotheses of f, ψ, φ, we get either $\psi(\nu_b(\Lambda(e,r), \Lambda(e^*, r^*)) = 0$ or $\varphi(\nu_b(\Lambda(e,r), \Lambda(e^*, r^*))) = 0$. Thus we have $\Lambda(e,r) = \Lambda(e^*, r^*)$. Similarly, we can prove that $\Lambda(r,e) = \Lambda(r^*, e^*)$. □

An example below support Theorems 2.1, 2.2, 2.3, and 2.4.

Example 2.1. *Let $f(e,r) = \tau e$, $0 < \tau < 1$ and $\Gamma = [0, \infty)$ endowed with*

$$\nu_b(e,r) = (e-r)^2$$

for all $e, r \in \Gamma$. It is clear that (Γ, ν_b) is a complete $b-$ms with a coefficient $s = 2$. Assume that $\Upsilon, \Lambda : \Gamma \times \Gamma \to \Gamma$ by

$$\Upsilon(e,r) = \begin{cases} e-r & e \geq r \\ 0 & otherwise \end{cases} \quad and \ \Lambda(e,r) = \begin{cases} e+r & e \geq r \\ 0 & otherwise \end{cases},$$

It is obvious that $\Upsilon(\Gamma^2) \subseteq \Lambda(\Gamma^2)$ and the mapping Υ is continuous. Define $\alpha : \Gamma^2 \times \Gamma^2 \to [0, +\infty)$ by

$$\alpha((e,r),(\mu,\kappa)) = \begin{cases} 2, & e \geq \mu, \ r \leq \kappa \\ 0, & otherwise \end{cases}.$$

Then for each $e_\circ, r_\circ \in \Gamma$, we find that

$$\begin{aligned} \alpha\left[\left(\Lambda\left(e_\circ, r_\circ\right), \Lambda\left(r_\circ, e_\circ\right)\right), \left(\Upsilon\left(e_\circ, r_\circ\right), \Upsilon\left(r_\circ, e_\circ\right)\right)\right] &= 2 > 1, \\ \alpha\left[\left(\Lambda\left(r_\circ, e_\circ\right), \Lambda\left(e_\circ, r_\circ\right)\right), \left(\Upsilon\left(r_\circ, e_\circ\right), \Upsilon\left(e_\circ, r_\circ\right)\right)\right] &= 2 > 1. \end{aligned}$$

For all $n \in \mathbb{N}$, let $e_n = \frac{n}{n+1}$ and $r_n = \frac{1}{n}$ be two sequences such that

$$\begin{aligned} \alpha\left[\left(\Lambda\left(e_{n+1}, r_{n+1}\right), \Lambda\left(r_n, e_n\right)\right)\right] &\geq 1, \\ \alpha\left[\left(\Lambda\left(r_n, e_n\right), \Lambda\left(e_{n+1}, r_{n+1}\right)\right)\right] &\geq 1. \end{aligned}$$

Then, $\lim_{n\to\infty} e_n = 1$ and $\lim_{n\to\infty} r_n = 0$. Certainly, $0, 1 \in \Gamma$ and

$$\begin{aligned} \alpha\left[\left(\Lambda\left(0,1\right), \Lambda\left(r_n, e_n\right)\right)\right] &= 2 > 1, \\ \alpha\left[\left(\Lambda\left(1,0\right), \Lambda\left(e_n, r_n\right)\right)\right] &= 2 > 1. \end{aligned}$$

Under this sequences, we can write

$$\begin{aligned} &\lim_{n\to\infty} \nu_b\left(\Upsilon\left[\Lambda\left(e_n, r_n\right), \Lambda\left(r_n, e_n\right)\right], \Lambda\left[\Upsilon\left(e_n, r_n\right), \Upsilon\left(r_n, e_n\right)\right]\right) \\ &= \nu_b\left(\Upsilon\left[\Lambda\left(1,0\right), \Lambda\left(0,1\right)\right], \Lambda\left[\Upsilon\left(1,0\right), \Upsilon\left(0,1\right)\right]\right) \\ &= \nu_b\left(\Upsilon\left(1,0\right), \Lambda\left(1,0\right)\right) = \nu_b\left(1,1\right) = 0, \end{aligned}$$

similarly, one can prove that

$$\lim_{n\to\infty} \nu_b\left(\Upsilon\left[\Lambda\left(r_n, e_n\right), \Lambda\left(e_n, r_n\right)\right], \Lambda\left[\Upsilon\left(r_n, e_n\right), \Upsilon\left(e_n, r_n\right)\right]\right) = 0.$$

whenever $e_n, r_n \in \Gamma$, *such that*

$$\begin{aligned}\lim_{n\to\infty} \Upsilon(e_n, r_n) &= \Upsilon(1,0) = 1 = \Lambda(1,0) = \lim_{n\to\infty} \Lambda(e_n, r_n),\\ \lim_{n\to\infty} \Upsilon(r_n, e_n) &= \Upsilon(0,1) = 0 = \Lambda(0,1) = \lim_{n\to\infty} \Lambda(r_n, e_n).\end{aligned}$$

Therefore, Υ *and* Λ *are generalized compatible.*

Now, for $e = \kappa = 1$ *and* $r = \mu = 0$*, if*

$$\alpha\left[\left(\Lambda(e,r), \Lambda(r,e)\right), \left(\Lambda(\mu,\kappa), \Lambda(\kappa,\mu)\right)\right] > 1,$$

this implies that

$$\begin{aligned}&\alpha\left[\left(\Upsilon(e,r), \Upsilon(r,e)\right), \left(\Upsilon(\mu,\kappa), \Upsilon(\kappa,\mu)\right)\right]\\ =\;&\alpha\left[\left(\Upsilon(1,0), \Upsilon(0,1)\right), \left(\Upsilon(0,1), \Upsilon(1,0)\right)\right] = \alpha\left[(1,0),(0,1)\right] = 2 > 1\end{aligned}$$

for all $(1,0),(0,1) \in \Gamma^2$*. This prove that* Υ *is a generalized* $\alpha-$*admissible w.r.t.* Λ.

Finally, we will try to verify the contractive conditions (2.1),(2.11) and (2.22) of Theorems 2.1, 2.2, and 2.3 respectively. Take $\psi(\theta) = \frac{\theta}{3}$ *and* $\phi(\theta) = \theta$ *for all* $\theta \in [0,\infty)$*. Put* $\sigma = \rho = 1$*,* $\tau = \frac{1}{2}$*,* $e = \kappa = \frac{1}{2}$ *and* $r = \mu = \frac{1}{4}$*, then we have*

$$\begin{aligned}&\alpha\left[\left(\Lambda(e,r), \Lambda(r,e)\right), \left(\Upsilon(e,r), \Upsilon(r,e)\right)\right]\\ &\times\alpha\left[\left(\Lambda(\mu,\kappa), \Lambda(\kappa,\mu)\right), \left(\Upsilon(\mu,\kappa), \Upsilon(\kappa,\mu)\right)\right]\\ =\;&\alpha\left[\left(\Lambda\left(\frac{1}{2},\frac{1}{4}\right), \Lambda\left(\frac{1}{4},\frac{1}{2}\right)\right), \left(\Upsilon\left(\frac{1}{2},\frac{1}{4}\right), \Upsilon\left(\frac{1}{4},\frac{1}{2}\right)\right)\right]\\ &\times\alpha\left[\left(\Lambda\left(\frac{1}{4},\frac{1}{2}\right), \Lambda\left(\frac{1}{2},\frac{1}{4}\right)\right), \left(\Upsilon\left(\frac{1}{4},\frac{1}{2}\right), \Upsilon\left(\frac{1}{2},\frac{1}{4}\right)\right)\right]\\ =\;&\alpha\left[\left(\frac{3}{4},0\right), \left(0,\frac{1}{4}\right)\right] \times \alpha\left[\left(0,\frac{3}{4}\right), \left(0,\frac{1}{4}\right)\right]\\ =\;&2 \times 0 = 0,\end{aligned} \tag{2.37}$$

$$\begin{aligned}&\psi\left(\frac{\nu_b\left(\Lambda(e,r), \Lambda(\mu,\kappa)\right) + \nu_b\left(\Lambda(r,e), \Lambda(\kappa,\mu)\right)}{2}\right)\\ =\;&\psi\left(\frac{\nu_b\left(\Lambda\left(\frac{1}{2},\frac{1}{4}\right), \Lambda\left(\frac{1}{4},\frac{1}{2}\right)\right) + \nu_b\left(\Lambda\left(\frac{1}{4},\frac{1}{2}\right), \Lambda\left(\frac{1}{2},\frac{1}{4}\right)\right)}{2}\right)\\ =\;&\psi\left(\frac{\nu_b\left(\frac{3}{4},0\right) + \nu_b\left(0,\frac{3}{4}\right)}{2}\right) = \psi\left(\frac{9}{16}\right) = \frac{3}{16},\end{aligned} \tag{2.38}$$

$$\phi\left(\frac{\nu_b\left(\Lambda(e,r), \Lambda(\mu,\kappa)\right) + \nu_b\left(\Lambda(r,e), \Lambda(\kappa,\mu)\right)}{2}\right) = \phi\left(\frac{9}{16}\right) = \frac{9}{16}, \tag{2.39}$$

$$s^{\sigma}\nu_b\left(\Upsilon\left(e,r\right),\Upsilon\left(\mu,\kappa\right)\right)=2\nu_b\left(\Upsilon\left(\frac{1}{2},\frac{1}{4}\right),\Upsilon\left(\frac{1}{4},\frac{1}{2}\right)\right)=2\nu_b\left(\frac{1}{4},0\right)=\frac{1}{8}. \tag{2.40}$$

It follows from definition of f and (2.37)-(2.40) that

$$\begin{aligned}
&\left(\psi\left(s^{\sigma}\nu_b\left(\Upsilon\left(e,r\right),\Upsilon\left(\mu,\kappa\right)\right)\right)+\rho\right)^{\left(\begin{array}{c}\alpha\left(\begin{array}{c}\left(\Lambda\left(e,r\right),\Lambda\left(r,e\right)\right),\\ \left(\Upsilon\left(e,r\right),\Upsilon\left(r,e\right)\right)\end{array}\right)\\ \times\alpha\left(\begin{array}{c}\left(\Lambda\left(\mu,\kappa\right),\Lambda\left(\kappa,\mu\right)\right),\\ \left(\Upsilon\left(\mu,\kappa\right),\Upsilon\left(\kappa,\mu\right)\right)\end{array}\right)\end{array}\right)}\\
&= \left(\psi\left(\frac{1}{4}\right)+1\right)^{0}=1\\
&\leq \frac{35}{32}=\frac{1}{2}\times\frac{3}{16}+1\\
&= f\left(\frac{3}{16},\frac{9}{16}\right)+1\\
&= f\left(\begin{array}{c}\psi\left(\frac{\nu_b(\Lambda(e,r),\Lambda(\mu,\kappa))+\nu_b(\Lambda(r,e),\Lambda(\kappa,\mu))}{2}\right),\\ \phi\left(\frac{\nu_b(\Lambda(e,r),\Lambda(\mu,\kappa))+\nu_b(\Lambda(r,e),\Lambda(\kappa,\mu))}{2}\right)\end{array}\right)+\rho. \qquad (2.41)
\end{aligned}$$

Also, we can write

$$\begin{aligned}
&\left(\begin{array}{c}\alpha\left(\begin{array}{c}\left(\Lambda\left(e,r\right),\Lambda\left(r,e\right)\right),\\ \left(\Upsilon\left(e,r\right),\Upsilon\left(r,e\right)\right)\end{array}\right)\\ \times\alpha\left(\begin{array}{c}\left(\Lambda\left(\mu,\kappa\right),\Lambda\left(\kappa,\mu\right)\right),\\ \left(\Upsilon\left(\mu,\kappa\right),\Upsilon\left(\kappa,\mu\right)\right)\end{array}\right)\\ +1\end{array}\right)^{\psi\left(s^{\sigma}\nu_b\left(\Upsilon(e,r),\Upsilon(\mu,\kappa)\right)\right)}\\
&= (0+1)^{\psi\left(\frac{1}{8}\right)}=1\\
&< 2^{\frac{3}{32}}=2^{\frac{3}{16}\times\frac{1}{2}}=2^{f\left(\frac{3}{16},\frac{9}{16}\right)}\\
&= 2^{f\left(\psi\left(\frac{\nu_b(\Lambda(e,r),\Lambda(\mu,\kappa))+\nu_b(\Lambda(r,e),\Lambda(\kappa,\mu))}{2}\right),\phi\left(\frac{\nu_b(\Lambda(e,r),\Lambda(\mu,\kappa))+\nu_b(\Lambda(r,e),\Lambda(\kappa,\mu))}{2}\right)\right)}. \qquad (2.42)
\end{aligned}$$

Additionally, the contractive condition (2.22) of Theorem 2.3 is directly hold. So, by (2.41)-(2.42) all hypotheses of Theorems 2.1, 2.2, and 2.3 are fulfilled, so by Theorem 2.4 the mappings Υ and e have a unique (ccp), here it is $(0,0)\in\Gamma^2$.

If we put $\Lambda = I$, (where I is the identity mapping) in Theorem 2.1, we get the important result below.

Corollary 2.1. *Let (Γ,ν_b) be a complete (b-ms) (with parameter $s>1$), and $\Upsilon, I : \Gamma^2 \to \Gamma$ be two generalized compatible mappings such that Υ is generalized $\alpha-$admissible mapping w.r.t. I. Let there is*

$f \in C$, $\psi \in \Psi, \varphi \in \Phi$ so that the stipulation below holds

$$
\begin{aligned}
&(\psi\left(s^{\sigma}\nu_b(\Upsilon(e,r),\Upsilon(\mu,\kappa)) + \rho\right)^{\alpha\left(\begin{array}{c}((e,r),(r,e)),\\(\Upsilon(e,r),\Upsilon(r,e))\end{array}\right)\alpha\left(\begin{array}{c}((\mu,\kappa),(\kappa,\mu)),\\(\Upsilon(\mu,\kappa),\Upsilon(\kappa,\mu))\end{array}\right)} \\
\leq\ & f\left(\begin{array}{c}\psi\left(\frac{\nu_b((e,r),(\mu,\kappa))+\nu_b((r,e),(\kappa,\mu))}{2}\right),\\ \varphi\left(\frac{\nu_b((e,r),(\mu,\kappa))+\nu_b((r,e),(\kappa,\mu))}{2}\right)\end{array}\right) \\
&+\rho, \qquad (2.43)
\end{aligned}
$$

for all $e, r, \mu, \kappa \in \Gamma, \rho, \sigma > 0$, and $\alpha : (\Gamma^2 \times \Gamma^2) \to [0, \infty)$. Assume that

(i) $\Upsilon(\Gamma^2) \subseteq \Gamma^2$,

(ii) there is $e_0, r_0 \in \Gamma$ so that

$$
\begin{aligned}
\alpha(((e_0,r_0),(r_0,e_0)),(\Upsilon(e_0,r_0),\Upsilon(r_0,e_0))) &\geq 1,\\
\alpha(((r_0,e_0),(e_0,r_0)),(\Upsilon(r_0,e_0),\Upsilon(e_0,r_0))) &\geq 1.
\end{aligned}
$$

Also, suppose either

(iv) Υ is continuous, or

(v) $\{e_n\}, \{r_n\}$ are two sequences in Γ so that

$$
\begin{aligned}
\alpha((e_{n+1},r_{n+1}),(e_n,r_n)) &\geq 1,\\
\alpha((r_n,e_n),(r_{n+1},e_{n+1})) &\geq 1.
\end{aligned}
$$

for all $n \in \mathbb{N} \cup \{0\}$, and $e_n \to e, r_n \to r$ as $n \to \infty$, $e, r \in \Gamma$, we have

$$
\begin{aligned}
\alpha((e,r),(e_n,r_n)) &\geq 1,\\
\alpha((r_n,e_n),(r,e)) &\geq 1.
\end{aligned}
$$

Then Υ has a (ccp).

3. An important application

This part is very important in this paper, where the existence solution to a (nie) using Corollary 2.1 is presented.

Here, we refers to χ by the class functions $\varkappa : [0, \infty) \to [0, \infty)$ so that $\varkappa$ is an increasing function and there is $\psi \in \Psi$, $\phi \in \Phi$, and $f \in C$ such that $\varkappa(\kappa) = \frac{1}{2} f(\psi(\kappa), \phi(\kappa))$ for all $\kappa \in [0, \infty)$. Assume the problem below:

$$
j(\varpi) = \int_u^v (\eth_1(\varpi,\ell) + \eth_2(\varpi,\ell))\,(\daleth_1(\ell,j(\ell)) + \daleth_2(\ell,j(\ell)))\,d\ell, \qquad (3.1)
$$

for all $\varpi \in [u, v]$. Suppose that $\eth_1, \eth_2, \daleth_1, \daleth_2$ are continuous functions which satisfy the hypotheses below:

(i) for all $\varpi, \ell \in [u,v]$, $\eth_1(\varpi,\ell), \eth_2(\varpi,\ell) \geq 0$,

(ii) for all $y, z \in \mathbb{R}$ with $y \geq z$, there is $\mho, ¥$ so that

$$\begin{aligned} 0 &\leq \daleth_1(\ell, y) - \daleth_1(\ell, z) \leq \mho\varkappa(y-z), \\ 0 &\leq \daleth_2(\ell, y) - \daleth_2(\ell, z) \leq ¥\varkappa(y-z), \end{aligned}$$

(iii) we get

$$\max\{\mho^2, ¥^2\}\left\{\sup_{\varpi\in[u,v]} \int_u^v (\eth_1(\varpi,\ell) + \eth_2(\varpi,\ell))d\ell\right\}^2 \leq 1.$$

To discuss the existence of a unique solution of the problem (3.1), we formulate the theorem below:

Theorem 3.1. *Under the assumptions (i)-(iii) with* $\eth_1, \eth_2 \in C([u,v]\times[u,v],\mathbb{R})$ *and* $\daleth_1, \daleth_2 \in C([u,v]\times\mathbb{R}\times\mathbb{R})$, *the problem (3.1) has a solution in* $C([u,v],\mathbb{R})$.

Proof. Let $\Gamma = C([u,v],\mathbb{R})$ be the set of real continuous functions on $[u,v]$ endowed with the distance

$$\nu_b(e,r) = \sup_{\varpi\in[u,v]} \left(|e(\varpi) - r(\varpi)|\right)^2, \ \forall e, r \in \Gamma.$$

It's obvious that, the pair (Γ, ν_b) is a complete $b-$ms with a coefficient $s = 2$.

Define mappings $\Upsilon : \Gamma\times\Gamma \to \Gamma$ and $\alpha : \Gamma^2\times\Gamma^2 \to \mathbb{R}^+$ by

$$\begin{aligned} \Upsilon(e,r)(\varpi) &= \int_u^v \eth_1(\varpi,\ell)\left(\daleth_1(\ell, e(\ell)) + \daleth_2(\ell, r(\ell))\right)d\ell \\ &\quad + \int_p^q \eth_2(\varpi,\ell)\left(\daleth_1(\ell, e(\ell)) + \daleth_2(\ell, r(\ell))\right)d\ell, \end{aligned}$$

and

$$\alpha\left((e,r),(\mu,\kappa)\right) = \begin{cases} 1, & e \geq \mu, \ r \leq \kappa \\ 0, & \text{otherwise} \end{cases}.$$

for all $\varpi \in [u,v]$, $(e,r),(\mu,\kappa) \in \Gamma^2$. If the mapping Υ has a (ccp) in Γ, then it is a solution of the problem (3.1).

Since, for each $e, r, \mu, \kappa \in \Gamma$, $\alpha\left[\left((e,r),(r,e)\right),\left(I(\mu,\kappa), I(\kappa,\mu)\right)\right] = 1$ and

$$\alpha\left[\left(\Upsilon(e,r), \Upsilon(r,e)\right),\left(\Upsilon(\mu,\kappa), \Upsilon(\kappa,\mu)\right)\right] = 1,$$

we conclude that Υ is a generalized $\alpha-$admissible w.r.t. I and by the continuity of $\eth_1, \eth_2, \daleth_1$, and $\daleth_2$, we have Υ is a continuous mapping. Also, for any two sequences $\{e_n\}$ and $\{r_n\}$ in Γ, suppose that

$$\begin{aligned} \lim_{n\to\infty} \nu_b\left(\Upsilon\left[I(e_n), I(r_n)\right], I\left[\Upsilon(e_n, r_n), \Upsilon(r_n, e_n)\right]\right) &= 0, \\ \lim_{n\to\infty} \nu_b\left(\Upsilon\left[I(r_n), I(e_n)\right], I\left[\Upsilon(r_n, e_n), \Upsilon(e_n, r_n)\right]\right) &= 0. \end{aligned}$$

Thus, we have

$$\lim_{n\to\infty}\Upsilon\left(e_n,r_n\right)=\lim_{n\to\infty}Ie_n=\lim_{n\to\infty}e_n,\ \ \lim_{n\to\infty}\Upsilon\left(r_n,e_n\right)=\lim_{n\to\infty}r_n.$$

Therefore, the pair (Υ, I) is generalized compatible. Again, it follows from the definition of α that if

$$\alpha\left[\left(I\left(e\right),I\left(r\right)\right),\left(I\left(\mu\right),I\left(\kappa\right)\right)\right]=\alpha\left(e,r,\mu,\kappa\right)=1,$$

this implies that

$$\alpha\left[\left(\Upsilon\left(e,r\right),\Upsilon\left(r,e\right)\right),\left(\Upsilon\left(\mu,\kappa\right),\Upsilon\left(\kappa,\mu\right)\right)\right]=1,$$

and

$$\alpha\left[\left(e,r\right),\left(\Upsilon\left(e,r\right),\Upsilon\left(r,e\right)\right)\right]\times\alpha\left[\left(\mu,\kappa\right),\left(\Upsilon\left(\mu,\kappa\right),\Upsilon\left(\kappa,\mu\right)\right)\right]=1. \tag{3.2}$$

Now we are going to verify the hypothesis (2.43) of Corollary 2.1, for all $e, r, \mu, \kappa \in \Gamma$,

$$\begin{aligned}
&\nu_b\left(\Upsilon(e,r),\Upsilon(\mu,\kappa)\right)\\
=\ &\sup_{\varpi\in[u,v]}\left(\left|\Upsilon(e,r)(\varpi)-\Upsilon(\mu,\kappa)(\varpi)\right|\right)^2\\
=\ &\sup_{\varpi\in[u,v]}\left(\begin{array}{c}\left|\int_u^v \eth_1(\varpi,\ell)\left(\daleth_1(\ell,e(\ell))+\daleth_2(\ell,r(\ell))\right)d\ell+\int_u^v \eth_2(\varpi,\ell)\left(\daleth_1(\ell,e(\ell))+\daleth_2(\ell,r(\ell))\right)d\ell\right.\\ \left.-\int_u^v \eth_1(\varpi,\ell)\left(\daleth_1(\ell,\mu(\ell))+\daleth_2(\ell,\kappa(\ell))\right)d\ell-\int_u^v \eth_2(\varpi,\ell)\left(\daleth_1(\ell,\mu(\ell))+\daleth_2(\ell,\kappa(\ell))\right)d\ell\right|\end{array}\right)^2\\
=\ &\sup_{\varpi\in[u,v]}\left(\begin{array}{c}\left|\int_u^v \eth_1(\varpi,\ell)\left[\left(\daleth_1(\ell,e(\ell))-\daleth_1(\ell,\mu(\ell))\right)+\left(\daleth_2(\ell,r(\ell))-\daleth_2(\ell,\kappa(\ell))\right)\right]d\ell\right.\\ \left.+\int_u^v \eth_2(\varpi,\ell)\left[\left(\daleth_1(\ell,e(\ell))-\daleth_1(\ell,\mu(\ell))\right)+\left(\daleth_2(\ell,r(\ell))-\daleth_2(\ell,\kappa(\ell))\right)\right]d\ell\right|\end{array}\right)^2.
\end{aligned}$$

Applying assumption (ii), one can get

$$\begin{aligned}
&\nu_b\left(\Upsilon(e,r),\Upsilon(\mu,\kappa)\right)\\
\leq\ &\sup_{\varpi\in[u,v]}\left(\begin{array}{c}\left|\int_u^v \eth_1(\varpi,\ell)\left[\mho\varkappa(e(\ell)-\mu(\ell)))+\yen\varkappa(r(\ell)-\kappa(\ell))\right]d\ell\right.\\ \left.+\int_u^v \eth_2(\varpi,\ell)\left[\mho\varkappa(e(\ell)-\mu(\ell)))+\yen\varkappa(r(\ell)-\kappa(\ell))\right]d\ell\right|\end{array}\right)^2\\
\leq\ &\max\{\mho^2,\yen^2\}\times\\
&\sup_{\varpi\in[u,v]}\left(\left|\int_u^v\left(\eth_1(\varpi,\ell)+\eth_2(\varpi,\ell)\right)\left[\varkappa(|e(\ell)-\mu(\ell)|)+\varkappa(|r(\ell)-\kappa(\ell)|)\right]d\ell\right|\right)^2.
\end{aligned} \tag{3.3}$$

By using the definition of $\varkappa$ and the distance ν_b, we have

$$\varkappa\left|e(\ell)-\mu(\ell)\right|^2\leq\varkappa\nu_b(e,\mu)\text{ and }\varkappa\left|r(\ell)-\kappa(\ell)\right|^2\leq\varkappa\nu_b(r,\kappa),\ \forall\varpi\in[u,v]. \tag{3.4}$$

It follows from (3.3), (3.4) and assumption (iii) that

$$\begin{aligned}
& \nu_b\left(\Upsilon(e,r),\Upsilon(\mu,\kappa)\right) \\
\leq\ & \max\{\mho^2, ¥^2\} \times [\varkappa^2\nu_b(e,\mu) + \varkappa^2\nu_b(r,\kappa)] \times \left\{\sup_{\varpi\in[u,v]} \int_u^v \left(\eth_1(\varpi,\ell) + \eth_2(\varpi,\ell)\right) d\ell\right\}^2 \\
\leq\ & \varkappa^2\nu_b(e,\mu) + \varkappa^2\nu_b(r,\kappa) \\
=\ & 2\pi^2\left(\frac{\nu_b(e,\mu)+\nu_b(r,\kappa)}{2}\right) \\
\leq\ & 2\times\frac{1}{4} f\left(\psi\left(\frac{\nu_b(e,\mu)+\nu_b(r,\kappa)}{2}\right), \phi\left(\frac{\nu_b(e,\mu)+\nu_b(r,\kappa)}{2}\right)\right) \\
=\ & \frac{1}{2} f\left(\psi\left(\frac{\nu_b(e,\mu)+\nu_b(r,\kappa)}{2}\right), \phi\left(\frac{\nu_b(e,\mu)+\nu_b(r,\kappa)}{2}\right)\right).
\end{aligned}$$

Thus, for all $e, r, \mu, \kappa \in \Gamma$, we

$$2^1\nu_b\left(\Upsilon(e,r),\Upsilon(\mu,\kappa)\right) \leq f\left(\psi\left(\frac{\nu_b(e,\mu)+\nu_b(r,\kappa)}{2}\right), \phi\left(\frac{\nu_b(e,\mu)+\nu_b(r,\kappa)}{2}\right)\right).$$

Add $\rho > 0$ to the both sides, we have

$$\left(2^1\nu_b\left(\Upsilon(e,r),\Upsilon(\mu,\kappa)\right) + \rho\right) \leq f\left(\psi\left(\frac{\nu_b(e,\mu)+\nu_b(r,\kappa)}{2}\right), \phi\left(\frac{\nu_b(e,\mu)+\nu_b(r,\kappa)}{2}\right)\right) + \rho.$$

Put $\psi(\kappa) = \kappa$, for all $\kappa \in [0,\infty)$, $s = 2$, $\sigma = 1$, and using (3.2), we get

$$\begin{aligned}
& \psi(s^\sigma\nu_b\left(\Upsilon(e,r),\Upsilon(\mu,\kappa)\right) + \rho)^{\alpha[(e,r),(\Upsilon(e,r),\Upsilon(r,e))]\times\alpha[(\mu,\kappa),(\Upsilon(\mu,\kappa),\Upsilon(\kappa,\mu))]} \\
\leq\ & f\left(\psi\left(\frac{\nu_b(e,\mu)+\nu_b(r,\kappa)}{2}\right), \phi\left(\frac{\nu_b(e,\mu)+\nu_b(r,\kappa)}{2}\right)\right) + \rho.
\end{aligned}$$

Therefore all stipulations of Corollary 2.1 are fulfilled. Then the mapping Υ has a (ccp) which is a solution of the system (3.1) in Γ. □

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References

[1] M. Berzig, S. Chandok, M.S. Khan, Generalized Krasnoselskii fixed point theorem involving auxiliary functions in bimetric spaces and application to two-point boundary value problem, Appl. Math. Comput. 248 (2014), 323–327.

[2] D. Gopal, M. Abbas, C. Vetro, Some new fixed point theorems in manger PM-spaces with application to Volterra type integral equation, Appl. Math. Comput. 232 (2014), 955–967.

[3] Z. Liu, X. Li, S.M. Kang, S.Y. Cho, Fixed point theorems for mappings satisfying contractive condition of integral type and applications, Fixed Point Theory Appl. 2011 (2011), 64.

[4] H.K. Pathak, M.S. Khan, R. Tiwari, A common fixed point theorem and its application to nonlinear integral equations, Comput. Math. Appl. 53 (2007), 961–971.

[5] H.A. Hammad, M. De la Sen, A Solution of Fredholm integral equation by using the cyclic r_s^q-rational contractive mappings technique in $b-$metric-like spaces, Symmetry, 11 (2019), 1184.

[6] H. A. Hammad, M. De la Sen, Solution of nonlinear integral equation via fixed point of cyclic α_s^q-rational contraction mappings in metric-like spaces, Bull. Braz. Math. Soc. New Ser. 51 (2020), 81-105.

[7] N. Shahzad, O. Valero, M.A. Alghamdi, A fixed point theorem in partial quasi-metric spaces and an application to software engineering, Appl. Math. Comput. 268 (2015), 1292–1301.

[8] N.Hussain, M.A. Taoudi, Krasnoselskii-type fixed point theorems with applications to Volterra integral equations, Fixed Point Theory Appl. 2013, (2013), 196.

[9] I.A. Bakhtin, The contraction principle in quasimetric spaces, Funct. Anal. 30 (1989) 26-37.

[10] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inf. Univ. Ostrav. 1 (1993), 5-11.

[11] A. Aghajani, M. Abbas, J.R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, Math. Slovaca, 64 (4) (2014), 941-960.

[12] R. Miculescu, A. Mihail, New fixed point theorems for set-valued contractions in b-metric spaces, J. Fixed Point Theory Appl. 19 (2017), 2153-2163.

[13] M. Shah, S. Simic, N. Hussain, A. Sretenovic, S. Radenović, Common fixed points theorems for occasionally weakly compatible pairs on cone metric type spaces, J. Comput. Anal. Appl. 14 (2012), 290-297.

[14] N. Hussain, M. Shah, KKM mappings in cone b-metric spaces, Comput. Math. Appl. 62 (2011), 1677-1684.

[15] M. Khamsi and N. Hussain, KKM mappings in metric type spaces, Nonlinear Anal., Theory Meth. Appl. 73 (2010), 3123-3129.

[16] S. Radenović, K. Zoto, N. Dedović, V. Šešum-Cavic, A. Ansari, Bhaskar-Guo-Lakshmikantam-Ćirić type results via new functions with applications to integral equations, Appl. Math. Comput. 357 (2019), 75-87.

[17] D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, Nonlinear Anal., Theory Meth. Appl. 11 (1987), 623-632.

[18] T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal., Theory Meth. Appl. 65 (2006), 1379-1393.

[19] M. Abbas, M.A. Khan, S. Radenović, Common coupled fixed point theorems in cone metric spaces for w-compatible mappings, Appl. Math. Comput. 217 (2010), 195-202.

[20] H. Aydi, M. Postolache, W. Shatanawi, Coupled fixed point results for $(\psi,\phi)-$weakly contractive mappings in ordered G-metric spaces, Comput. Math. Appl. 63 (2012), 298-309.

[21] V. Berinde, Coupled fixed point theorems for contractive mixed monotone mappings in partially ordered metric spaces, Nonlinear Anal., Theory Meth. Appl. 75 (2012), 3218-3228.

[22] B.S. Choudhury, P. Maity, Coupled fixed point results in generalized metric spaces, Math. Comput. Model. 54 (2011), 73-79.

[23] H.A. Hammad, M. De la Sen, A coupled fixed point technique for solving coupled systems of functional and nonlinear integral equations, Mathematics, 7 (2019), 634.

[24] H.A. Hammad, D.M. Albaqeri, R.A. Rashwan, Coupled coincidence point technique and its application for solving nonlinear integral equations in RPOCbML spaces, J. Egypt. Math. Soc. 28 (2020), 8.

[25] N. Hussain, M. Abbas, A. Azam, J. Ahmad, Coupled coincidence point results for a generalized compatible pair with applications, Fixed Point Theory Appl. 2014 (2014), 62.

[26] H.A. Hammad, H. Aydi, M. De la Sen, Generalized dynamic process for an extended multi-valued F-contraction in metric-like spaces with applications, Alex. Eng. J. 59 (2020), 3817–3825.

[27] V. Lakshmikantham, Lj. B. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. Theory Methods Appl. 70, (2009), 4341-4349.

[28] H. Alsulami, S. Gulyaz, E. Karapinar, I.M. Erhan, Fixed point theorems for a class of $\alpha-$ admissible contractions and applications to boundary value problem, Abstr. Appl. Anal. 2014 (2014), Article ID 187031.

[29] A.H. Ansari, S. Chandok, C. Ionescu, Fixed point theorems on b-metric spaces for weak contractions with auxiliary functions, J. Inequal. Appl. 2014 (2014), 429.